Quantum Entanglement: An Introduction

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Introduction

- Entanglement is the quamtum mechanical property that Schördinger singled out many decades ago (1935) as:

  "the characteristic trait of quantum mechanics"
- Extensively studied in connection with Bell's inequality (60s). The Bell state
$$\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle),$$
which violates the Bell's inequality, a test for quantum mechanics, is an (maximally) entangled state.
- Recently, entanglement has begun to be viewed as potentially useful resources in quantum computing and quantum information.
- Quantum teleportation.
- Quantum cryptography.
- Superdense coding.
- Quantum key distribution
- Telecloning.
- .....

What is entanglement?

General definition of entanglement

Let Q denote a global quantum system consisting of some subsystems $Q_1, Q_2, \cdots, Q_n$. The Hilbert space of subsystem Q_i is H_i and the Hilbert space for Q is then

$$H = \bigotimes_{j=1}^{n} H_j. \tag{1}$$

Let ρ be a density operator of the global system Q on H.

Then ρ is said to be entangled with respect to the Hilbert space decomposition (1) if it can not be written in the form

$$\rho = \sum_{k=1}^{K} p_k \left(\bigotimes_{j=1}^{n} \rho_{j,k} \right),$$

for some positive integer K, where the p_k's are positive real numbers such that

$$\sum_{k=1}^{K} p_k = 1,$$

and where each $\rho_{j,k}$ is a density operator on the Hilbert space H_j.

If ρ is a pure state, then ρ is entangled if it can not be written in the form

$$\rho = \bigotimes_{j=1}^{n} \rho_j,$$

where ρ_j is a (pure state) density operator on the Hilbert space H_j.

Pure State

A pure state can be described by a wave function. Suppose that the system is in the state $|\Psi\rangle$, then the corresponding density operator can be written as

$$\rho = |\Psi\rangle\langle\Psi|. \tag{2}$$

Now we prove that

Theorem 1 *A pure state $|\Psi\rangle$ is nonentangled iff the wave function is factorized*

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle,$$

where $|\Psi_i\rangle$ is wave vector describing the subsystem i $(i = 1, 2)$.

Proof:

$\longleftarrow$: It is easy to see that

$$\rho = (|\Psi_1\rangle \otimes |\Psi_2\rangle)(\langle\Psi_1| \otimes \langle\Psi_2|) = (|\Psi_1\rangle\langle\Psi_1|) \otimes (|\Psi_2\rangle\langle\Psi_2|) \equiv \rho_1 \otimes \rho_2,$$

where $\rho_1 \equiv |\Psi_1\rangle\langle\Psi_1|$, $\rho_2 \equiv |\Psi_2\rangle\langle\Psi_2|$.

$\longrightarrow$: We first prove that ρ_1 and ρ_2 are all pure if $\rho_1 \otimes \rho_2$ is pure. From

$$\mathrm{tr}\rho^2 = \mathrm{tr}\left(\rho_1^2 \otimes \rho_2^2\right) = \mathrm{tr}\left(\rho_1^2\right)\mathrm{tr}\left(\rho_2^2\right) = 1$$

and the fact

$$\mathrm{tr}\left(\rho_1^2\right) \le 1, \qquad \mathrm{tr}\left(\rho_2^2\right) \le 1,$$

we have

$$\mathrm{tr}\left(\rho_1^2\right) = \mathrm{tr}\left(\rho_2^2\right) = 1,$$

namely, ρ_1 and ρ_2 are pure states. This means that there must exist two states Ψ_i, $i = 1, 2$, such that

$$\rho_i = |\Psi_i\rangle\langle\Psi_i|.$$

Then the state ρ can be written as

$$\rho = (|\Psi_1\rangle\langle\Psi_1|) \otimes (|\Psi_2\rangle\langle\Psi_2|) = (|\Psi_1\rangle \otimes |\Psi_2\rangle)(\langle\Psi_1| \otimes \langle\Psi_2|)$$

namely, the system is in the state $|\Psi_1\rangle \otimes |\Psi_2\rangle$.

Mixed state

A general mixed state takes the form

$$\rho = \sum_{i=1}^{K} p_i |\Psi^{(i)}\rangle\langle\Psi^{(i)}|,$$

where

- $|\Psi^{(i)}\rangle$ is a pure state of the global system.
- p_i is the probability of the PURE state $|\Psi^{(i)}\rangle$:

$$\sum_{i=1}^{K} p_i = 1.$$

Comparing this with the general definition of entangled state, we find that

- ρ is nonentangled iff every pure state $|\Psi^{(i)}\rangle\langle\Psi^{(i)}|$ can be written as
  $$|\Psi^{(i)}\rangle\langle\Psi^{(i)}| = \rho_1^{(i)} \otimes \rho_2^{(i)}$$
  namely, every $|\Psi^{(i)}\rangle\langle\Psi^{(i)}|$ is nonentangled.
- If a $|\Psi^{(i)}\rangle\langle\Psi^{(i)}|$ is not entangled, then ρ is not entangled.

Measure of Entanglement: Pure state case

Entropy

The Von Neumann entropy of a quantum state ρ can be written as

$$S = -\mathrm{tr}\left(\rho \log_2 \rho\right),$$

We have

- For pure states: $S = 0$
- For mixed states: $0 < S \le 1$

How to calculate entropy:

Suppose the eigenvalues of ρ are λ_i. Then the entropy can be written as

$$S = -\sum_i \lambda_i \log_2(\lambda_i)$$

Example: 2-dim case.

As ρ is hermitian and $\mathrm{tr}\rho = 1$, we can write ρ as

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta^* & 1-\alpha \end{pmatrix},$$

Two eigenvalues

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1-x^2}, \qquad \lambda' = 1 - \lambda, \qquad x^2 = 4|\rho|.$$

Entropy

$$S = -\left[\lambda \log_2 \lambda + (1-\lambda)\log_2(1-\lambda)\right]$$

Pure state case:

$$\lambda = 1 \longrightarrow S = 0$$

Relation between S and x:

- $\mathrm{tr}\rho^2 \le 1$ leads to $0 \le \lambda \le 1$.

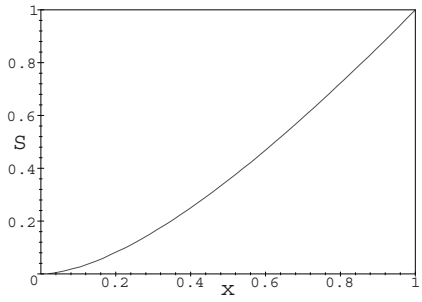

Figure 1: Entropy as a function of x.

- $0 \le |\rho| = \lambda(1-\lambda) \le 1/4$ leads to $0 \le x \le 1$.
- When $x = 0$, $S = 0$, pure state.
- Plot of S against x.
- x itself can be regarded as a measure of entropy.

Entanglement of Formation

The entanglement of a bipartite pure state can be measured by its entropy of either subsystem A (Alice) or B (Bob)

$$\mathcal{E}(\rho) = S(\rho_A) = S(\rho_B)$$

$\mathcal{E}(\rho)$ is called the Entanglement of Formation.

Remark: In above definition, we have used the fact that

$$S(\rho_A) = S(\rho_B)$$

which is true for a pure state with two subsystems (See S. M. Barnett and S. J. D. Phoenix, Phys. Rev. A. 40(1989)2042.)

How to calculate the Entanglement of Formation:

- **Find the reduced density operator ρ_A or ρ_B;**

$$\rho_A = \mathrm{tr}_B \rho.$$

- **Calculate the entropy of ρ_A.**

Example 1: Calculate the entanglement of Bell state

$$\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$$

Density operator

$$\rho = \frac{1}{2}\left(|0,1\rangle\langle 0,1| - |0,1\rangle\langle 1,0| - |1,0\rangle\langle 0,1| + |1,0\rangle\langle 1,0|\right),$$

Reduced density operator

First term: $\mathrm{tr}_B(|01\rangle\langle 0,1|) = \mathrm{tr}_B(|0\rangle\langle 0| \otimes |1\rangle\langle 1|) = |0\rangle\langle 0|.$

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate entropy of ρ_A

$$S_A = -\frac{1}{2}\log_2\frac{1}{2} - -\frac{1}{2}\log_2\frac{1}{2} = 1,$$

which is the Entanglement of Formation.

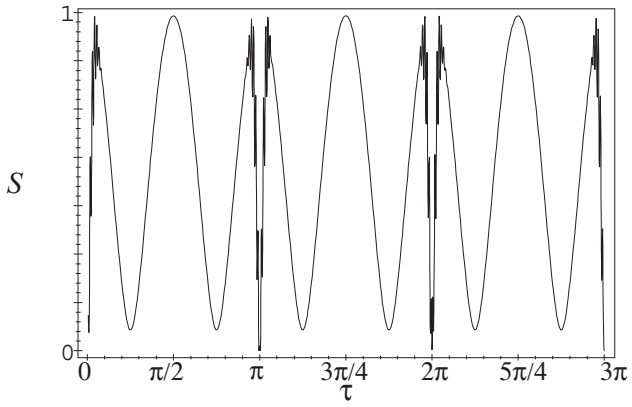

Figure 2: Entanglement of Jaynes-Cumming model.

Example 2: Jaynes-Cummings Model

- **Calculate the entropy of ATOM, not field !**

Concurrence

Essentially the determinant of ρ_A.

Definition:

$$0 \le C \equiv |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle| \le 1$$

where

- **σ_y is the *spin-flip* operator and $|\Psi^*\rangle$ is the complex conjugate of $|\Psi\rangle$.**
- **Concurrence itself can also be regarded as a measure of entanglement which ranges from 0 to 1.**
- **Maximal entanglement: $C = 1$.**

Theorem 2 *The entanglement of formation of a 2-dim bipartite pure state Ψ can be written as*

$$E(\Psi) = S(C(\Psi)),$$

where

- *$C(\Psi)$ is the Concurrence*
- *S is defined as*

$$S(x) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2}\right) \qquad \text{for } 0 \le x \le 1$$

- *and H is the entropy function*

$$H(x) = -[x\log_2 x + (1-x)\log_2(1-x)].$$

- *Concurrence is essentially the 4 times the determinant of ρ_A.*

Magic basis:

$$|e_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle),$$
$$|e_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle),$$

$$|e_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle),$$
$$|e_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle).$$

Expand the pure state as

$$|\psi\rangle = \sum_{i=1}^{4} \alpha_i |e_i\rangle,$$

Then the concurrence is

$$C(\psi) = \left|\sum_{i=1}^{4} \alpha_i^2\right|.$$

Measure of Entanglement: Mixed states

Definition 1: The entanglement of formation $E(\mathcal{Y})$ of an ensemble of bipartite pure states $\mathcal{E} = \{p_i, \mathcal{Y}\}$ is the ensemble average

$$E(\mathcal{Y}) = \sum_i p_i E(\mathcal{Y})$$

of the entanglement of formation of the pure states in the ensemble.

Definition 2: The entanglement of formation $E(M)$ of a bipartite mixed state M is the minimum of $E(\mathcal{Y})$ over ensembles $\mathcal{E} = \{p_i, \mathcal{Y}_i\}$ realizing the mixed state $M = \Sigma_i\, p_i |\mathcal{Y}_i\rangle\langle\mathcal{Y}_i|$.

Difficult to calculate!!!

Fortunately, the definition of concurrence still works for bipartite 2-dim systems.

Maximal Entanglement of Pure states

Problem

Maximal entanglement of bipartite nonorthogonal pure states

$$|\psi\rangle = \mu|\alpha\rangle \otimes |\beta\rangle + \nu|\gamma\rangle \otimes |\delta\rangle, \tag{3}$$

where

- **μ, ν are two complex numbers**
- **$|\alpha\rangle$ and $|\gamma\rangle$ are linearly independent normalized states of the systems 1, and $|\beta\rangle$ and $|\delta\rangle$ are linearly independent normalized states of the system 2.**
- **Here we are interested in the nonorthogonal case**

$$\langle\alpha|\gamma\rangle \neq 0, \qquad \langle\beta|\delta\rangle \neq 0$$

- **Note that the normalization constant is not $|\mu|^2 + |\nu|^2$, but**

$$\left[|\mu^2| + |\nu|^2 + \mu\nu^*\langle\gamma|\alpha\rangle\langle\delta|\beta\rangle + \mu^*\nu\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle\right]^{-1/2}. \tag{4}$$

To do:

- **Give a classification of maximal entanglement for the state (3).**
- **We then apply the condition to coherent states and find a simple condition for maximally entangled coherent states in terms of the coherence parameters.**
- **Some new classes of maximally entangled coherent states and their limit to maximally entangled Bell-like states.**

Concurrence

Concurrence

$$0 \leq C \equiv |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle| \leq 1$$

where

- **σ_y is the *spin-flip* operator and $|\Psi^*\rangle$ is the complex conjugate of $|\Psi\rangle$.**
- **Concurrence itself can also be regarded as a measure of entanglement which ranges from 0 to 1.**
- **Maximal entanglement: $C = 1$.**

Trick: Introduce an orthogonal normalized basis in the subspace spanned by $|\alpha\rangle$ and $|\gamma\rangle$ and by $|\beta\rangle$ and $|\delta\rangle$:

$$|0\rangle = |\alpha\rangle, \qquad |1\rangle = \frac{|\gamma\rangle - \langle\alpha|\gamma\rangle|\alpha\rangle}{\sqrt{1 - \langle\alpha|\gamma\rangle^2}}$$

Result:

$$C = \frac{2|\mu\nu|\sqrt{(1 - |\langle\alpha|\gamma\rangle|^2)(1 - |\langle\beta|\delta\rangle|^2)}}{|\mu|^2 + |\nu|^2 + \mu\nu^*\langle\gamma|\alpha\rangle\langle\delta|\beta\rangle + \mu^*\nu\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle}. \tag{5}$$

Entanglement Condition

$C = 0$ iff one the following is true

- **$\mu = 0$;**
- **$\nu = 0$;**
- **$|\langle\alpha|\gamma\rangle|^2 = 1$, namely, $|\alpha\rangle = \pm|\gamma\rangle$;**
- **$|\langle\beta|\delta\rangle|^2 = 1$, namely, $|\beta\rangle = \pm|\delta\rangle$.**

Maximal Entanglement: Classification

Let

- **$\mu = k\nu e^{i\theta}$, where k and θ: real parameters, $k > 0$.**

- **Noting that $|\langle\alpha|\gamma\rangle| \leq 1$ and $|\langle\beta|\delta\rangle| \leq 1$, we write**

$$\langle\alpha|\gamma\rangle = \sin a\, e^{i\theta_1}, \qquad \langle\beta|\delta\rangle = \sin b\, e^{i\theta_2},$$

where a, b and θ_1, θ_2 are all real parameters and $0 \leq a, b \leq \pi/2$.

Then $C = 1$ implies

$$k' = 2\cos a \cos b - 2\sin a \sin b \cos(\theta - \theta_1 - \theta_2), \tag{6}$$

where $k' \equiv (k^2 + 1)/k$.

Case 1: $-1 \leq \cos(\theta - \theta_1 - \theta_2) \leq 0$. In this case we have

$$k' \leq 2\cos a \cos b + 2\sin a \sin b = 2\cos(a - b) \leq 2, \tag{7}$$

namely $(k-1)^2 \leq 0$, or $k = 1$, $k' = 2$. Inserting $k' = 2$ into relation (7) we also have $\cos(a - b) = 1$, namely $a = b + 2m\pi$ (m an integer). Then from Eq.(6) we have

$$\cos(\theta - \theta_1 - \theta_2) = -1, \tag{8}$$

namely $\theta - \theta_1 - \theta_2 = \pi$. So in this case the MES condition is obtained as

$$\mu = \nu e^{i\theta}, \qquad \langle\alpha|\gamma\rangle = -\langle\beta|\delta\rangle^* e^{i\theta}. \tag{9}$$

Case 2: $0 \leq \cos(\theta - \theta_1 - \theta_2) \leq 1$. In this case the second term in (6) is always non-negative and thus

$$k' \leq 2\cos a \cos b \leq 2, \tag{10}$$

which leads to $k = 1$ and $k' = 2$ as in case 1. Then the relation (10) is valid only when

$$\cos a = \cos b = 1 \text{ or } \sin a = \sin b = 0. \tag{11}$$

So the MES condition in this case is

$$|\mu| = |\nu|, \quad \langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle = 0, \tag{12}$$

which is clearly the orthogonal case.

In summary we obtain the following theorem

Theorem 3 *The states (3) are MES if and only if one of the following conditions is satisfied*

1. $\mu = \nu e^{i\theta}$ *and* $\langle\alpha|\gamma\rangle = -\langle\beta|\delta\rangle^* e^{i\theta}$ *(θ is a real parameter) for the nonorthogonal case;*
2. $|\mu| = |\nu|$ *for the orthogonal case* $\langle\alpha|\gamma\rangle = \langle\beta|\delta\rangle = 0$.

MES: Antisymmetric MES

Now we turn to the explicit construction of MES.

Choose (ϑ is an arbitrary phase)

$$|\gamma\rangle = e^{i\vartheta}|\beta\rangle, \qquad |\delta\rangle = e^{i(\theta+\pi-\vartheta)}|\alpha\rangle, \tag{13}$$

Normalized MES:

$$|\psi\rangle = \frac{\mu}{|\mu|\sqrt{2(1-|\langle\alpha|\beta\rangle|^2)}}(|\alpha\rangle\otimes|\beta\rangle - |\beta\rangle\otimes|\alpha\rangle), \tag{14}$$

Note: the phase $e^{i\theta}$ does not enter the anti-symmetric state.

Question: Are there MES other than the antisymmetric MES? YES!

MES: Maximally entangled coherent states

All four states in (3) are coherent states (α: complex number)

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{15}$$

Overlap between two coherent states $|\alpha\rangle$ and $|\gamma\rangle$:

$$\langle\alpha|\gamma\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\gamma|^2-2\alpha^*\gamma\right)\right]. \tag{16}$$

Then the MES condition (9) simplifies to

$$|\alpha^2+|\gamma|^2-2\alpha^*\gamma = |\beta|^2+|\delta|^2-2\beta\delta^*-2i(\theta+\pi), \tag{17}$$

from which we conclude that

Theorem 4 *Coherent states are maximally entangled if and only if* $\mu = \nu\exp(i\theta)$ *and both sides of Eq.(17) have the same* real *part*

$$|\alpha^2+|\gamma|^2-2\mathrm{Re}(\alpha^*\gamma) = |\beta|^2+|\delta|^2-2\mathrm{Re}(\beta\delta^*). \tag{18}$$

The difference of their imaginary parts gives rise to a relative phase θ

$$\theta = \mathrm{Im}(\alpha^*\gamma)-\mathrm{Im}(\beta\delta^*)-\pi. \tag{19}$$

Some examples:

- $|\alpha\rangle\otimes|\beta\rangle - |(1-\lambda'/|\alpha|)\alpha\rangle\otimes|(1\mp\lambda'/|\beta|)\beta\rangle$
- $|\alpha\rangle\otimes|-\alpha\rangle - |-\alpha\rangle\otimes|\alpha\rangle$ **when** $\beta=-\alpha$ **and** $\lambda'=2|\alpha|$
- $|\alpha\rangle\otimes|-\alpha\rangle - |-\alpha\rangle\otimes|-3\alpha\rangle$ **when** $\beta=-\alpha$ **and** $\lambda'=2|\alpha|$

- $|\alpha\rangle\otimes|\beta\rangle - e^{-i(|\alpha\gamma|\pm|\beta\delta|)}\,|i|\gamma|\alpha/|\alpha|\rangle\otimes|\pm i|\delta|\beta/|\beta|\rangle$.
- $|\alpha\rangle\otimes|\beta\rangle - e^{-i(|\alpha|^2\pm|\alpha|^2)}\,|i\alpha\rangle\otimes|\pm i\beta\rangle$ **when** $|\alpha|=|\beta|=|\gamma|=|\delta|$.
- $|\alpha\rangle\otimes|-\alpha\rangle - |i\alpha\rangle\otimes|i\alpha\rangle$, $\quad(\beta=-\alpha)$

- $|\alpha\rangle\otimes|-\alpha\rangle - e^{-i2|\alpha|^2}\,|i\alpha\rangle\otimes|-i\alpha\rangle$, $\quad(\beta=-\alpha)$
- $|\alpha\rangle\otimes|\alpha\rangle - |i\alpha\rangle\otimes|-i\alpha\rangle$, $\quad(\beta=\alpha)$
- $|\alpha\rangle\otimes|\alpha\rangle - e^{-i2|\alpha|^2}\,|i\alpha\rangle\otimes|i\alpha\rangle$. $\quad(\beta=\alpha)$

Remark: Last two states can be obtained from the first two states, respectively, by the local transformation $1\otimes(-1)^{a_2a_2}$.

Limit to Bell states

Consider, for example,

$$|\alpha\rangle\otimes|-\alpha\rangle - |i\alpha\rangle\otimes|i\alpha\rangle,$$

Expand it in Fock space as

$$\frac{e^{-|\alpha|^2}}{\sqrt{2(1-e^{-2|\alpha|^2})}}\sum_{m,n=0}^{\infty}\frac{\alpha^{m+n}}{\sqrt{m!\,n!}}\left[(-1)^n - i^{m+n}\right]|m\rangle\otimes|n\rangle,$$

in which only terms with $m+n=1$ survive in the limit $|\alpha|\to 0$ (there is *no* term $|0\rangle\otimes|0\rangle$). The limiting state is readily obtained as

$$\frac{1}{\sqrt{2}}\left(e^{i\pi/4}|0\rangle\otimes|1\rangle - e^{-i\pi/4}|1\rangle\otimes|0\rangle\right), \tag{20}$$

which is clearly a Bell-like MES.

Conclusion

- **Brief introduction entanglement.**
- **General definition of entanglement and its meaning for pure and mixed states.**
- **Measures of entanglement**
  - **Entanglement of formation**
  - **Concurrence**
  - **Different measures for mixed states**
- **Maximal entanglement for pure states**
  - **Classification**
  - **Antisymmetric maximally entangled states**
  - **Maximally entangled coherent states**