Relativistic Effects on Entanglement

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http://cs.jpl.nasa.gov/qct/qat.html

This talk is based on

"Quantum Entanglment of Moving Bodies" R. Gingrich, C. Adami quant-ph/0205179

"Quantum Entropy and Special Relativity" A. Peres, P. Scudo, D. Terno Phys. Rev. Lett. 88 (2002) 230402

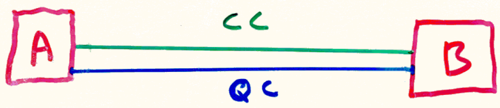
Overview of talk

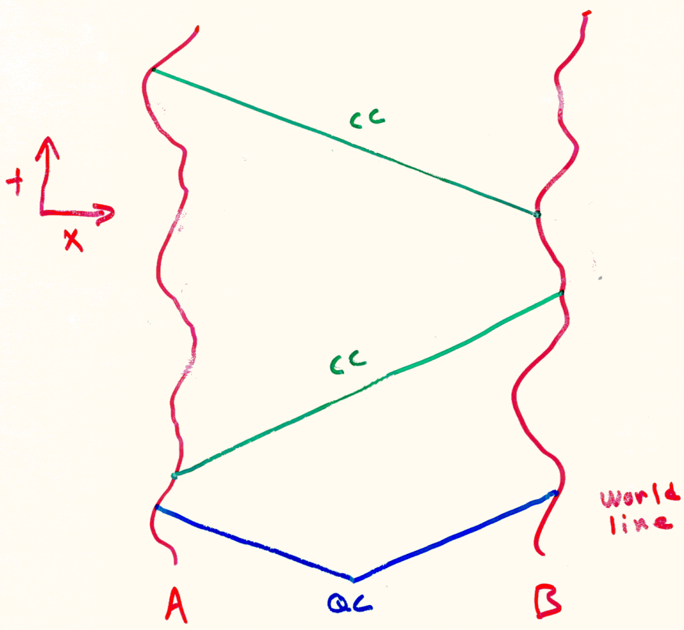
- what is entanglement
- measures of entanglement
- spin and relativity
- Lorentz boosts
- boosting a Bell state
- the large boost limit

uses for this research

- quantum algorithms for GPS
- distributed quantum computing
- quantum computing with relativistic particles

the Big Picture





what is entanglement?

A quantum state of two systems is entangled if it cannot be created by local operations and classical communication alone.

the spin of an electron is often used as the quantum system. A pure quantum state of two spins looks like

$$|\psi\rangle_{AB} = \sum_{\lambda,\sigma=\downarrow}^{\uparrow} \alpha_{\lambda\sigma} |\lambda\rangle_A |\sigma\rangle_B$$

where $\sum_{\lambda\sigma} |\alpha_{\lambda\sigma}|^2 = 1$

Spins A and B are entangled iff

$$|\psi\rangle_{AB} \neq \left(\sum_\lambda \beta_\lambda |\lambda\rangle_A\right)\left(\sum_\sigma \gamma_\sigma |\sigma\rangle_B\right)$$

for any β_λ and γ_σ

what is entanglement?

A mixed state of two Spins is described by

$$\rho_{AB} = (\rho_{AB})_{\lambda\sigma,\lambda'\sigma'} |\lambda\rangle_A |\sigma\rangle_B {}_B\langle\sigma'| {}_A\langle\lambda'|$$

where $(\rho_{AB})_{\lambda\sigma,\lambda'\sigma'}$ is a trace 1 positive semi-definite operator

A mixed state is entangled iff

$$\rho_{AB} \neq \sum_i w_i \rho_A^{(i)} \otimes \rho_B^{(i)}$$

where $w_i > 0$ and $\sum_i w_i = 1$

A measure of entanglement

the Bell states

$$|\phi^\pm\rangle = |\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle$$
$$|\psi^\pm\rangle = |\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$$

are the most entangled two-spin states.

They can be used to do

- teleportation
- super dense coding
- quantum key distribution

Any measure of entanglement should be 1 for a Bell state and 0 for an unentangled state.

A measure of entanglement

The entanglement of formation, E_f, is a measure of entanglement. It is the asymptotic minimum number of Bell states needed to create a state.

for a pure state

$$E_f(|\psi\rangle_{AB}) = -\mathrm{Tr}\left[\rho_A \log_2 \rho_A\right]$$

where $\rho_A = \mathrm{Tr}_B\left[|\psi\rangle_{AB}\langle\psi|\right]$

for a mixed state ρ_{AB}

$$E_f(\rho_{AB}) = \text{entropy}\left(\frac{1}{2} - \frac{\sqrt{1-cc^2}}{2}\right)$$

$$cc = \max\left[\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\right]$$

where the λ_i are the square roots of the eigenvalues of

$$\sigma_y \otimes \sigma_y \rho_{AB}^T \sigma_y \otimes \sigma_y \rho_{AB}$$

in decreasing order

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How do we describe spin in a relativistic framework?

for a particle at rest just use $\vec{J}$. Define $|\vec{0}\lambda\rangle$ as

$$P^\mu |\vec{0}\lambda\rangle = \begin{cases} |\vec{0}\lambda\rangle m & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

$$\vec{J}^2 |\vec{0}\lambda\rangle = |\vec{0}\lambda\rangle \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)$$

$$J_z |\vec{0}\lambda\rangle = |\vec{0}\lambda\rangle \lambda \qquad \lambda = \uparrow, \downarrow$$

Define $L(\vec{\xi}_{\vec{p}})$ as the Lorentz boost with the following property

$$L(\vec{\xi}_{\vec{p}}) \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{m^2+p^2} \\ p_x \\ p_y \\ p_z \end{bmatrix}$$

The rapidity $\vec{\xi}_{\vec{p}}$ is defined by

$$\sinh|\vec{\xi}_{\vec{p}}| = \frac{|\vec{p}|}{m}$$

$$\frac{\vec{\xi}_{\vec{p}}}{|\vec{\xi}_{\vec{p}}|} = \frac{\vec{p}}{|\vec{p}|}$$

How do we describe spin in a relativistic framework?

Define

$$|\vec{p}\lambda\rangle = L(\vec{\xi}_{\vec{p}})|\vec{0}\lambda\rangle$$

Note: $|\vec{p}\uparrow\rangle \longleftrightarrow u(\vec{p},\uparrow) = \sqrt{E+m}\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix}$

where $u(\vec{p},\lambda)$ are solutions to the Dirac equation

$$(\gamma^\mu p_\mu - m)u(\vec{p},\lambda) = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$H = \gamma^0\left(\sum_{i=1}^{3}\gamma^i p_i + m\right)$$

How do we describe spin in a relativistic framework

the spin operator

$$S_j = \frac{i}{2}\gamma^k\gamma^l = \frac{1}{2}\begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix} \quad \text{in Pauli Rep.}$$

(j,k,l) cyclic

and angular momentum operator

$$\vec{L} = (\vec{x} \times \vec{p})$$

do not separatly commute with H

$$[S_j, H] = -i\sum \epsilon_{jkl}\gamma^0\gamma^k p^l$$

$$[L_j, H] = i\sum \epsilon_{jkl}\gamma^0\gamma^k p^l$$

so they are not conserved quantities as they are in non-relativistic Q.M.

but $\vec{J} = \vec{S} + \vec{L}$ is.

The eigenvalue λ is called the "rest frame spin"

Notation and review of Lorentz transformations

Any Lorentz transf. Λ can be written

$$\Lambda = L(\vec{\xi}) R(\vec{\theta})$$

where $L(\vec{\xi})$ is a pure boost of rapidity $\vec{\xi}$

e.g. $L(\xi \hat{x}) = \begin{bmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

in the 4-vector representation.

$R(\vec{\theta})$ is a rotation by $|\vec{\theta}|$ around $\hat{\theta}$

e.g. $R(\theta \hat{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$

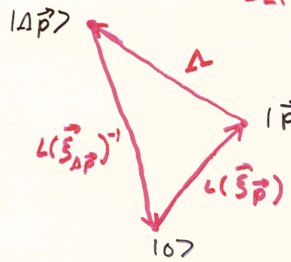
in 4-vector notation.

Boosting a Bell state

To apply a Lorentz transf. Λ to $|\vec{p}, \lambda\rangle$ we write

$$\Lambda |\vec{p}, \lambda\rangle = \Lambda L(\vec{\xi}_{\vec{p}}) |\vec{0}, \lambda\rangle$$

$$= L(\vec{\xi}_{\Lambda\vec{p}}) \underbrace{\left(L^{-1}(\vec{\xi}_{\Lambda\vec{p}}) \Lambda L(\vec{\xi}_{\vec{p}}) \right)}_{R(\vec{\theta}(\Lambda, \vec{p}))} |\vec{0}, \lambda\rangle$$



$$= |\Lambda\vec{p}, \lambda'\rangle D^{(\frac{1}{2})}_{\lambda'\lambda}[R(\Lambda, \vec{p})]$$

where $D^{(\frac{1}{2})}_{\lambda'\lambda}$ is the representation of $SO(3)$ corresponding to $j = \frac{1}{2}$.

Boosting a Bell state

In order to make this representation of the Lorentz group unitary we must have

$$\langle \lambda' \vec{p}' | \vec{p} \lambda \rangle = \langle \lambda' \vec{p}' | \Lambda^{\dagger} \Lambda | \vec{p} \lambda \rangle$$

using $\langle \lambda' \vec{p}' | \vec{p} \lambda \rangle = N(\vec{p}) \delta^3(\vec{p} - \vec{p}') \delta_{\lambda\lambda'}$

we get $N(\vec{p}) \delta^3(\vec{p} - \vec{p}') = N(\Lambda\vec{p}) \delta^3(\Lambda\vec{p} - \Lambda\vec{p}')$

solving for $N(\vec{p})$ gives us

$$N(\vec{p}) = 2\sqrt{m^2 + p^2}$$

A general state of 1 particle can be written

$$|\psi\rangle = \sum_{\lambda} \int g_{\lambda}(\vec{p}) |\vec{p} \lambda\rangle \, \tilde{d}\vec{p}$$

in order to have $\langle \lambda' \vec{p}' | \psi \rangle = g_{\lambda'}(\vec{p}')$

we set $\tilde{d}\vec{p} = \dfrac{d^3\vec{p}}{2\sqrt{m^2 + p^2}}$

Boosting a Bell State

In non-relativistic Q.M., when we say we have two spins in a $|\Phi^+\rangle$ state we really mean that we have

$$|\psi\rangle_{AA'BB'} = \iint h(\vec{p}, \vec{q}) |\vec{p}\rangle_{A'} |\vec{q}\rangle_{B'} \, d^3\vec{p} \, d^3\vec{q} \times \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle_{AB} + |\downarrow\downarrow\rangle_{AB} \right)$$

but the momentum wavefunction can be ignored. Not so in rel. Q.M.

In rel. Q.M. a general state can be written

$$|\psi\rangle_{AA'BB'} = \sum_{\lambda\sigma} \iint g_{\lambda\sigma}(\vec{p}, \vec{q}) |\vec{p} \lambda\rangle |\vec{q} \sigma\rangle \, \tilde{d}\vec{p} \, \tilde{d}\vec{q}$$

where

$$\sum_{\lambda\sigma} \iint |g_{\lambda\sigma}(\vec{p}, \vec{q})|^2 \, \tilde{d}\vec{p} \, \tilde{d}\vec{q} = 1$$

Boosting a Bell state

To an observer in a frame transformed by Λ^{-1} the state looks like

$$\Lambda\otimes\Lambda|\Psi\rangle_{AA'BB'}=\sum_{\substack{\lambda\sigma\\ \lambda'\sigma'}}\iint g_{\lambda\sigma}(\vec{p},\vec{q})\,|\Lambda\vec{p}\,\lambda'\rangle|\Lambda\vec{q}\,\sigma'\rangle \times D_{\lambda'\lambda}(\vec{p},\Lambda)D_{\sigma'\sigma}(\vec{q},\Lambda)\,\tilde{d}\vec{p}\,\tilde{d}\vec{q}$$

or, by making a change of varibles and a slight change of notation

$$\begin{bmatrix} g_{\uparrow\uparrow}\\ g_{\uparrow\downarrow}\\ g_{\downarrow\uparrow}\\ g_{\downarrow\downarrow}\end{bmatrix}(\vec{p},\vec{q})\rightarrow D(\Lambda^{-1}\vec{p},\Lambda)\otimes D(\Lambda^{-1}\vec{q},\Lambda)\begin{bmatrix} g_{\uparrow\uparrow}\\ g_{\uparrow\downarrow}\\ g_{\downarrow\uparrow}\\ g_{\downarrow\downarrow}\end{bmatrix}(\Lambda^{-1}\vec{p},\Lambda^{-1}\vec{q})$$

Boosting a Bell state

to calculate $D(\vec{p},\Lambda)$ we first observe that since $\Lambda=L(\vec{\xi})R(\vec{\theta})$,

$$D(\vec{p},\Lambda)=D(\vec{p},L(\vec{\xi}))\,D(\vec{p},R(\vec{\theta}))$$

and $D(\vec{p},R(\vec{\theta}))=D(R(\vec{\theta}))$.

So, we only worry about boosts an w.l.o.g. we can take $\vec{\xi}=\xi\hat{z}$.

Take

$$\vec{p}=(\sqrt{m^2+p^2},\,p\cos\theta\sin\phi,\,p\sin\theta\sin\phi,\,p\cos\phi)$$

then

$$D(\vec{p},\xi)=\begin{bmatrix}\alpha & \beta e^{i\theta}\\ -\beta e^{-i\theta} & \alpha\end{bmatrix}$$

Boosting a Bell state

where

$$\alpha=\sqrt{\frac{E+m}{E'+m}}\left(\cosh\left(\frac{\xi}{2}\right)+\frac{p\cos\phi}{E+m}\sinh\left(\frac{\xi}{2}\right)\right)$$

$$\beta=\frac{p\sin\phi}{\sqrt{(E+m)(E'+m)}}\sinh\left(\frac{\xi}{2}\right)$$

$$E=\sqrt{m^2+p^2}$$

$$E'=E\cos(\xi)+p\cos\phi\sinh(\xi)$$

we can represent $|\phi^+\rangle$ with

$$g_{\lambda\sigma}(\vec{p},\vec{q})=\frac{1}{\sqrt{2}}\delta_{\lambda\sigma}\,f(\vec{p})\,f(\vec{q})$$

where

$$f(\vec{p})=\sqrt{\frac{1}{N(\sigma_r,m)}e^{-\frac{p^2}{2\sigma_r^2}}}$$

is a "relativistic Gaussian" and $N(\sigma_r,m)$ is the normalization.

Boosting a Bell state

After applying the boost $L(\xi\hat{z})$, tracing out momentum and integrating over θ we get

$$\rho_{AB}=\begin{bmatrix}\frac{a^2+b^2}{2} & 0 & 0 & \frac{a^2}{2}\\ 0 & ab & 0 & 0\\ 0 & 0 & ab & 0\\ \frac{a^2}{2} & 0 & 0 & \frac{a^2+b^2}{2}\end{bmatrix}$$

where

$$a=\int\alpha^2|f(\vec{p})|^2\frac{p^2\sin\phi}{2\pi}\,\tilde{d}\vec{p}$$

$$b=\int\beta^2|f(\vec{p})|^2\frac{p^2\sin\phi}{2\pi}\,\tilde{d}\vec{p}$$

the next slide shows a plot of concurrence vs. rapidity for different values of $\frac{\sigma_r}{m}$.

The large boost limit

For a given $\vec{p}$ there is a maximal amount of spin rotation that a Lorentz boost can give

$$\lim_{\xi \to \infty} \begin{bmatrix} \alpha \\ \beta e^{i\theta} \end{bmatrix} = \sqrt{\frac{E+m}{2(E+p\cos\phi}} \begin{bmatrix} 1 + \frac{p\cos\phi}{E+m} \\ \frac{p\sin\phi}{E+m} e^{i\theta} \end{bmatrix}$$

at $\phi = \arccos\left(\frac{1 - \sqrt{1-(\frac{p}{m})^2}}{\frac{p}{m}} \right)$

β is maximized

$$\beta \to \frac{\frac{p}{m}}{1 + \sqrt{1+(\frac{p}{m})^2}}$$

which increases monotonically from 0 to 1 and represents the maximal amount of rotation by a boost on a momentum $\vec{p}$ with $|\vec{p}| = p$.

Conclusions

- entanglement is a resource
- spins behave differently in relativistic QM.
- Spin and momentum degrees of freedom can exchange entanglement when boosted.
- depending on the initial state there is a maximal amount of entanglement exchange