Relativistic Electrons

(Choonkyu Lee, SNU)

PART I :

Relativistic Quantum Mechanics (Rel. QM)

- Formulate rel. wave eq. for electron (Dirac eq.)
- Plane wave sol.'s
- Nonrelativistic limit
- Klein paradox

"Birth and then death"

PART II :

Quantum Field Theory (QFT) Description

- What is quantum field ?
- Continuum limit of fermion chain
- Dirac fields in (1+1)-, (2+1)- and (3+1)-dimensions

"Rebirth"

PART I : $\boxed{\text{Rel. QM}}$

※ Simplify eqs. by setting $c = \hbar = 1$

Nonrelativistic QM

- (Free particle) Make replacements $H \to i\frac{\partial}{\partial t}$, $\vec{p} \to \frac{1}{i}\vec{\nabla}$ in $H = \frac{1}{2m}\vec{p}^{\,2}$:

$$i\frac{\partial}{\partial t}\psi = \underbrace{-\frac{1}{2m}\vec{\nabla}^2}_{\shortparallel\ \mathcal{H}}\psi \qquad \text{(free Schrödinger equation)} \qquad (1)$$

- (In the presence of e.m. potentials) Make the same replacements in $H = \frac{1}{2m}(\vec{p}-e\vec{A})^2 + e\Phi$ ← electric charge

$\vec{E} = -\partial_t\vec{A} - \vec{\nabla}\Phi$
$\vec{B} = \vec{\nabla}\times\vec{A}$

$\vec{\nabla} \to \vec{\nabla} - ie\vec{A}$
$\partial_t \to \partial_t + ie\Phi$
in (1)

$$\longrightarrow \quad i\frac{\partial}{\partial t}\psi = \Big[\underbrace{-\frac{1}{2m}(\vec{\nabla}-ie\vec{A})^2 + e\Phi}_{\shortparallel\ \mathcal{H}}\Big]\psi \qquad (2)$$

- (For a spin-½ electron) Use 2-column wave function $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$. ← spinor

Noting that $\vec{\nabla}^2$ has the square root $\vec{\sigma}\cdot\vec{\nabla}$ (i.e., $(\vec{\sigma}\cdot\vec{\nabla})^2 = \vec{\nabla}^2$, since $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$), may generalize (2) as

※ $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$i\frac{\partial}{\partial t}\psi = \Big[\underbrace{-\frac{1}{2m}[\vec{\sigma}\cdot(\vec{\nabla}-ie\vec{A})]^2 + e\Phi}_{\shortparallel\ \mathcal{H}}\Big]\psi \qquad (3)$$

(∴ Pauli equation)

$$※\ -\frac{1}{2m}[\vec{\sigma}\cdot(\vec{\nabla}-ie\vec{A})]^2 = -\frac{1}{2m}\underbrace{\sigma_i\sigma_j}_{\shortparallel\ \delta_{ij}+i\epsilon_{ijk}\sigma_k}(\nabla_i - ieA_i)(\nabla_j - ieA_j)$$

$$= -\frac{1}{2m}(\vec{\nabla}-ie\vec{A})^2 - \frac{i}{4m}\epsilon_{ijk}\sigma_k \overset{-ie(\partial_iA_j-\partial_jA_i)}{\overset{\shortparallel}{[\nabla_i - ieA_i, \nabla_j - ieA_j]}}$$

$$= -\frac{1}{2m}(\vec{\nabla}-ie\vec{A})^2 - \frac{e}{2m}\vec{\sigma}\cdot\vec{B}$$

$\epsilon_{ijk}(\partial_iA_j - \partial_jA_i) = 2B_k$

$H_{mag} = -\vec{\mu}\cdot\vec{B}$, with $\vec{\mu} = \frac{e}{2m}g\underbrace{\vec{S}}_{\shortparallel\ \vec{\sigma}/2}\cdot\vec{B}$ → $g = 2$ ← gyromagnetic ratio

Remarks :

(i) of the (good) form $i\frac{\partial}{\partial t}\psi = \mathcal{H}\psi$, with hermitian $\mathcal{H}$ (usually bounded from below) ← generates unitary time-evolution of state vectors

← "the ground state exists"

(ii) ∃ positive-definite conserved norm in the form

$$\|\psi\|^2 = \int d^3r\,\psi^\dagger\psi. \qquad (4)$$

(allow probability interpretation)

Relativistic wave eq.

$$P^\mu = (P^0 = H, \vec{p}) \longrightarrow \frac{1}{i}\partial^\mu$$

with $\partial^\mu = \eta^{\mu\nu} \boxed{\partial_\nu}$, $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$

$\partial_\nu \equiv \frac{\partial}{\partial x^\nu}$ $\qquad X^\mu = (X^0 = t, \vec{r})$

- (Free, spinless) Make replacements

$$H \to i\frac{\partial}{\partial t}, \quad \vec{p} \to \frac{1}{i}\vec{\nabla}$$

in $H = \sqrt{\vec{p}^2 + m^2}$ or $H^2 - \vec{p}^2 - m^2 = 0$:

Lorentz-covariant eq.

$$(-\partial^\mu\partial_\mu + m^2)\psi = 0, \quad \text{(Klein-Gordon equation)} \qquad (5)$$

where

$$\partial^\mu\partial_\mu \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + \vec{\nabla}^2.$$

← d'Alembertian (=Minkowski Laplacian)

But, (5) is
- ✓ not of the form $i\frac{\partial}{\partial t}\psi = \mathcal{H}\psi$ (actually, more like Maxwell wave eqs. for e.m. potentials),
- ✓ No positive-definite conserved norm.

- (For spin-½) Expect multi-component wave function. As Dirac wanted to find a 'good' form (i.e., 1st-order eq. in ∂_t), tried to factor $-\partial^\mu\partial_\mu + m^2$ as

$$\begin{pmatrix}\text{linear} \\ \text{in } \partial\end{pmatrix} \cdot \begin{pmatrix}\text{linear} \\ \text{in } \partial\end{pmatrix}.$$

← some $n \times n$ matrices

— OK to take $n=2$ (2-spinor) in the massless case, i.e., $m=0$. How? Look

$$-\partial^\mu\partial_\mu = (-\partial_t \pm \vec{\sigma}\cdot\vec{\nabla})(-\partial_t \mp \vec{\sigma}\cdot\vec{\nabla}),$$

and so (5) may be replaced by

Lorentz-covariant eqs.

$$\left.\begin{aligned} 0 &= \frac{1}{i}(-\partial_t - \vec{\sigma}\cdot\vec{\nabla})\psi_1 \equiv \frac{1}{i}\sigma_\mu\partial^\mu\psi_1 \\ &\text{or} \\ 0 &= \frac{1}{i}(-\partial_t + \vec{\sigma}\cdot\vec{\nabla})\psi_2 \equiv \frac{1}{i}\bar{\sigma}_\mu\partial^\mu\psi_2 \end{aligned}\right\} \to i\partial_t\psi_{1,2} = \underbrace{\pm\vec{\sigma}\cdot\overbrace{(-i\vec{\nabla})}^{\vec{p}}}_{\mathcal{H}_{\text{Weyl}}}\psi_{1,2} \qquad (6)$$

— With $m \neq 0$, require $n=4$ (4-spinor/Dirac spinor or <u>two</u> 2-spinors). How is it done?

✓ Possible to take the 'square root' of $\partial^\mu\partial_\mu$ to be $\gamma^\mu\frac{1}{i}\partial_\mu$, with a set of four <u>4×4 γ-matrices</u> γ^μ ($\mu = 0,1,2,3$) satisfying

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu} I. \qquad (7)$$

※ $(\gamma^\mu\frac{1}{i}\partial_\mu)(\gamma^\nu\frac{1}{i}\partial_\nu) = -\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu$

Explicitly, four γ-matrices can be chosen

※ $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (\text{: Weyl basis}). \qquad (8)$$

✓ Because we have

$$(-\partial^\mu \partial_\mu + m^2) = -(\gamma^\nu \tfrac{1}{i}\partial_\nu - m)(\gamma^\mu \tfrac{1}{i}\partial_\mu + m),$$

(5) may be replaced by the <u>Dirac equation</u>

(Lorentz-covariant eq.)

$$\boxed{(\underbrace{\gamma^\mu \tfrac{1}{i}\partial_\mu}_{\gamma_\mu \frac{1}{i}\partial^\mu,\ (\gamma_0 = -\gamma^0,\ \gamma_i = \gamma^i)} + m)\overset{\leftarrow \text{4-spinor}}{\Psi} = 0} \rightarrow i\partial_t \Psi = [\underbrace{m\overset{\beta}{\gamma^0} + \overset{\vec{\alpha}}{\gamma^0\vec{\gamma}}\cdot(-i\vec{\nabla})}_{\mathcal{H}_D}]\Psi. \qquad (9)$$

✓ Note that, writing $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ (ψ_1, ψ_2 : <u>2-spinors</u>), (9) is equivalent to

spin matrix : $\frac{1}{2}\vec{\sigma}$

$$\frac{1}{i}(-\partial_t - \vec{\sigma}\cdot\vec{\nabla})\psi_1 + m\psi_2 = 0, \qquad (\text{cf. } (6)) \qquad (10)$$
$$\frac{1}{i}(-\partial_t + \vec{\sigma}\cdot\vec{\nabla})\psi_2 + m\psi_1 = 0.$$

<u>Remarks:</u>

(i) The Dirac eq. has a <u>good</u> form, with a hermitian Hamiltonian

$$\mathcal{H}_D = \vec{\alpha}\cdot\vec{p} + m\overset{=\gamma^0}{\beta}$$

$\partial_\mu(\underbrace{\Psi^\dagger\gamma^0}_{\bar{\Psi}\text{ (: Dirac adjoint)}}\gamma^\mu\Psi) = 0$

Admits also the positive definite <u>conserved</u> norm

$$\|\Psi\|^2 = \int d^3\vec{r}\ \Psi^\dagger\Psi \qquad (11)$$

← probability density?

(But, the spectrum of $\mathcal{H}_D$ is unbounded from below).

(ii) The choice of γ-matrices is not unique (: perform unitary transf. $\Psi \rightarrow \tilde{\Psi} = U\Psi$, $\gamma^\mu \rightarrow \tilde{\gamma}^\mu = U\gamma^\mu U^{-1}$). still satisfy $\tilde{\gamma}^\mu\tilde{\gamma}^\nu + \tilde{\gamma}^\nu\tilde{\gamma}^\mu = -2\eta^{\mu\nu}I$

Another popular γ-matrix choice :

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (\text{Dirac repre.}) \qquad (12)$$

- (In the presence of e.m. potentials) Note that (2) is obtained from (1) by so-called minimal coupling replacements

$$\partial_t \rightarrow \partial_t + ie\underset{A^0 = -A_0}{\Phi}, \quad \vec{\nabla} \rightarrow \vec{\nabla} - ie\vec{A} \longrightarrow \boxed{\partial_\mu \rightarrow \partial_\mu - ieA_\mu}, \text{ with } A^\mu = (A^0 = \Phi, \vec{A})$$

(※ With $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $(\vec{E})_i \equiv F^{0i}$, $(\vec{B})_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$)

So, the desired Dirac eq. may take the form

(very beautiful)

$$\boxed{[\gamma^\mu \tfrac{1}{i}(\partial_\mu - ieA_\mu) + m]\Psi = 0} \rightarrow i\partial_t\Psi = [\underbrace{\vec{\alpha}\cdot(\vec{p} - e\vec{A}) + m\beta + e\Phi}_{\mathcal{H}_D}]\Psi. \qquad (13)$$

Using the Dirac repre. (12), write $\Psi = \begin{pmatrix} \psi_U \\ \psi_L \end{pmatrix}$ ← 2-spinors, to have

$$(13) \rightarrow \left[\begin{array}{l} i\frac{\partial}{\partial t}\psi_U = \overbrace{\vec{\sigma}\cdot(\vec{p}-e\vec{A})}^{\equiv}\psi_L + (e\Phi + m)\psi_U \\ i\frac{\partial}{\partial t}\psi_L = \vec{\sigma}\cdot(\vec{p}-e\vec{A})\psi_U + (e\Phi - m)\psi_L \end{array}\right. \quad \text{(useful later)} \qquad (14)$$

Plane wave solutions of the free Dirac eq.

- (Plane wave solutions) The wave form — an eigenfunction of $\mathcal{H}_D = m\gamma^0 + \gamma^0\vec{\gamma}\cdot(\frac{1}{i}\vec{\nabla})$ with eigenvalue E

$$\Psi(x) = e^{-iEt}\, e^{i\vec{p}\cdot\vec{r}}\, \underbrace{\omega(\vec{p})}_{\text{certain 4-column}} \qquad (15)$$

becomes a sol. of the Dirac eq. if $\omega(\vec{p})$ satisfies

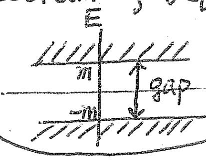
$$[-\gamma^0 E + \vec{\gamma}\cdot\vec{p} + m]\,\omega(\vec{p}) = 0\,. \qquad (16)$$

Since $(\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m)(-\gamma^0 E + \vec{\gamma}\cdot\vec{p} + m) = -E^2 + \vec{p}^2 + m^2$, (16) admits a nontrivial sol. only when

$$E = +\sqrt{\vec{p}^2+m^2} \quad \text{or} \quad -\sqrt{\vec{p}^2+m^2}$$

both define 2-dim. sol. space ← spin-½

Spectrum of $\mathcal{H}_D$:



Often express these plane waves by (here $p^\mu \equiv (\underbrace{+\sqrt{\vec{p}^2+m^2}}_{= p^0}, \vec{p})$) ← particle people

$$\Psi(x) = \left\{\begin{array}{l} e^{ip\cdot x}\,\omega_+^i(\vec{p}) \\ e^{-ip\cdot x}\,\omega_-^i(\vec{p}) \end{array}\right. , \quad (i=1,2) \qquad (17)$$

$\omega_-^i(\vec{p})$ ← really for $\omega(-\vec{p})$ with $E = -\sqrt{\vec{p}^2+m^2}$

and then, from (16),

$$(\gamma_\mu p^\mu + m)\,\omega_+(p) = 0\,, \quad (-\gamma_\mu p^\mu + m)\,\omega_-(\vec{p}) = 0\,. \qquad (18)$$

- (Explicit forms of $\omega_\pm(\vec{p})$)

—In the case $\vec{p}=0$ ($\rightarrow p^0 = m$), (18) reduces to

$$\gamma^0\,\omega_\pm^i(\vec{p}=0) = \pm\,\omega_\pm^i(\vec{p}=0)\,. \qquad (19)$$

So, in the Dirac basis where $\gamma^0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$, we can choose

$$\omega_+^1(\vec{p}=0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \omega_+^2(\vec{p}=0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \omega_-^1(\vec{p}=0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \omega_-^2(\vec{p}=0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \qquad (20)$$

which are <u>eigenstates of the z-comp. of spin</u> $\Sigma_3 = \left(\begin{array}{c|c} \frac{1}{2}\sigma_3 & 0 \\ \hline 0 & \frac{1}{2}\sigma_3 \end{array}\right)$. [Note that, in the Dirac basis, both ψ_U and ψ_L behave as spinors under spatial rotations. Why? See (14).] More generally, using 2-spinors $\chi_\pm$ satisfying $\frac{\vec{\sigma}}{2}\cdot\vec{s}\,\chi_\pm = \pm\frac{1}{2}\chi_\pm$ ($\vec{s}$: chosen polarization direction) ← unit vector

$\vec{\Sigma} = \left(\begin{array}{c|c} \frac{1}{2}\vec{\sigma} & 0 \\ \hline 0 & \frac{1}{2}\vec{\sigma} \end{array}\right)$

$$\omega_+^1(\vec{p}=0) = \left(\begin{array}{c}\chi_+\\ \hline 0\end{array}\right),\quad \omega_+^2(\vec{p}=0) = \left(\begin{array}{c}\chi_-\\ \hline 0\end{array}\right),\quad \omega_-^1(\vec{p}=0) = \left(\begin{array}{c}0\\ \hline \chi_+\end{array}\right),\quad \omega_-^2(\vec{p}=0) = \left(\begin{array}{c}0\\ \hline \chi_-\end{array}\right). \qquad (21)$$

$(\vec{\Sigma}\cdot\vec{s})' = \frac{1}{2}$ $(\vec{\Sigma}\cdot\vec{s})' = -\frac{1}{2}$

convenient to write in the form (14)

— For general $\vec{p} \neq 0$, one sees from (18) that $\omega_{\pm}^{i}(\vec{p})$ may be chosen

normalization

$$\omega_{+}^{1,2}(\vec{p}) = \sqrt{\frac{m+p^0}{2m}} \begin{pmatrix} \chi_{\pm} \\ \hdashline \frac{\vec{p}\cdot\vec{\sigma}}{m+p^0}\chi_{\pm} \end{pmatrix}, \qquad \omega_{-}^{1,2}(\vec{p}) = \sqrt{\frac{m+p^0}{2m}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{m+p^0}\chi_{\pm} \\ \hdashline \chi_{\pm} \end{pmatrix}. \qquad (22)$$

- (General sol. of the free Dirac equation)

positive-energy　　negative energy part

$$\psi(x) = \sum_{i=1,2} \int d^3\vec{p} \left\{ \omega_{+}^{i}(\vec{p}) e^{ip\cdot x} a^{i}(\vec{p}) + \omega_{-}^{i}(\vec{p}) e^{-ip\cdot x} b^{i*}(\vec{p}) \right\} \qquad (23)$$

($a^{i}(\vec{p}), b^{*}(\vec{p})$: arbitrary complex numbers)

Nonrelativistic limit

In (14), insert　　"concentrate on positive-energy sol.'s"

$$\Psi(\vec{x},t) = \begin{pmatrix} \psi_U(\vec{x}) \\ \boxed{\psi_L(\vec{x})} \end{pmatrix} e^{-i(m+\boxed{\epsilon})t} \qquad (24)$$

small　small

to obtain (here, $\vec{\pi} \equiv \vec{p} - e\vec{A}$)

$$(\epsilon - e\Phi)\psi_U = \vec{\sigma}\cdot\vec{\pi}\,\boxed{\psi_L}, \qquad (25)$$

insert

$$(2m+\epsilon - e\Phi)\psi_L = \vec{\sigma}\cdot\vec{\pi}\,\psi_U \rightarrow \psi_L = \frac{1}{2m+\epsilon-e\Phi}\vec{\sigma}\cdot\vec{\pi}\,\psi_U$$

From these,

$$(\epsilon - e\Phi)\psi_U = \vec{\sigma}\cdot\vec{\pi}\,\frac{1}{2m+\cancel{\epsilon-e\Phi}}\,\vec{\sigma}\cdot\vec{\pi}\,\psi_U$$

ignore

$$\rightarrow \left[\frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})^2 + e\Phi\right]\psi_U = \epsilon\,\psi_U \quad (\text{: Pauli equation!}) \qquad (26)$$

Klein paradox

✓ If we concentrate only on positive-energy sol.'s of the Dirac eq., its prediction, say, for the hydrogen atom spectrum appears to be superior than the prediction based on the Pauli equation.

✓ But, how about negative-energy sol.'s? Why are they there? If we accept their existence, we can conceive of the radiative decay of the usual hydrogen ground state into the "negative-energy sea" — the hydrogen atom is unstable!

Use the Fermi Golden rule

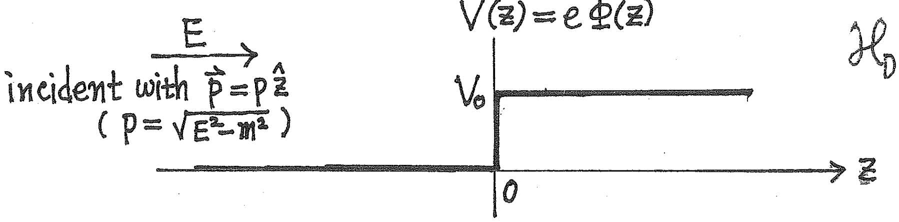
✓ The Dirac wave equation does not make sense if the interaction becomes sufficiently strong (← Klein paradox)

compared to the electron rest energy

- (The Klein paradox problem)

Q. For a spin-½ Dirac particle (mass m; charge e) incident with energy $E\,(>m)$ on an electrostatic barrier potential

has kinetic energy



$$\mathcal{H}_D=\gamma^0\vec{\gamma}\cdot\vec{p}+\gamma^0 m+V(z)\ ,$$

calculate the reflection and transmission coefficients.

A. In the Dirac γ-matrix basis, write the stationary wave function as

$$z<0:\ \psi(z)=\sqrt{\frac{E+m}{p}}\,e^{ipz}\begin{pmatrix}\chi_+\\ \cdots\cdots\\ \frac{p}{E+m}\chi_+\end{pmatrix}+\underset{\substack{\uparrow\\ \text{reflection coeffi.}}}{\underline{\underline{r}}}\sqrt{\frac{E+m}{p}}\,e^{-ipz}\begin{pmatrix}\chi_+\\ \cdots\cdots\\ -\frac{p}{E+m}\chi_+\end{pmatrix},\quad\begin{pmatrix}\text{assume}\\ \sigma_z\chi_+=\chi_+\end{pmatrix}\qquad(27)$$

Dirac adjoint

※ $\bar{\Psi}=\Psi^\dagger\gamma^0$

$$z>0:\ \psi(z)=\underset{\substack{\uparrow\\ \text{transmission coeffi.}}}{\underline{\underline{t}}}\sqrt{\frac{E-V_0+m}{q}}\,e^{iqz}\begin{pmatrix}\chi_+\\ \cdots\cdots\\ \frac{q}{E-V_0+m}\chi_+\end{pmatrix},\quad\left(q\equiv\sqrt{(E-V_0)^2-m^2}\right)$$

"play the role of the prob. current density"

(※ Waves normalized so that $j_z=\bar{\Psi}\gamma^3\Psi$ may be given by an expression independent of momentum). r and t fixed by the <u>continuity of the sol. at $z=0$</u> (as required by current conservation $\partial_z j_z=0$):

$$\underset{\text{(upper)}}{\sqrt{\frac{E+m}{p}}(1+r)=\sqrt{\frac{E-V_0+m}{q}}\,t}\ ,\quad\underset{\text{(lower)}}{\sqrt{\frac{p}{E+m}}(1-r)=\sqrt{\frac{q}{E-V_0+m}}\,t}\qquad(28)$$

$$\rightarrow\ t=\frac{2\sqrt{\frac{q}{p}}\sqrt{\frac{E+m}{E-V_0+m}}}{1+\frac{q}{p}\frac{E+m}{E-V_0+m}}\ ,\qquad r=\frac{1-\frac{q}{p}\frac{E+m}{E-V_0+m}}{1+\frac{q}{p}\frac{E+m}{E-V_0+m}}$$

for <u>real</u> q

$$\therefore\ T=|t|^2=\frac{4\rho}{(1+\rho)^2}\ ,\quad R=1-T=|r|^2=\left(\frac{1-\rho}{1+\rho}\right)^2\ .\qquad(29)$$

$$\left(\rho\equiv\frac{q}{p}\,\frac{E+m}{E-V_0+m}\right)$$

"Easy"

※ For $(m<)\,E<m+V_0$, q is imaginary; then $T=0$, $R=1$

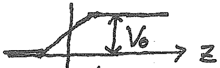
What is the problem? Note

✓ For E slightly above $m+V_0$, the expression (29) is consistent with the nonrelativistic result.

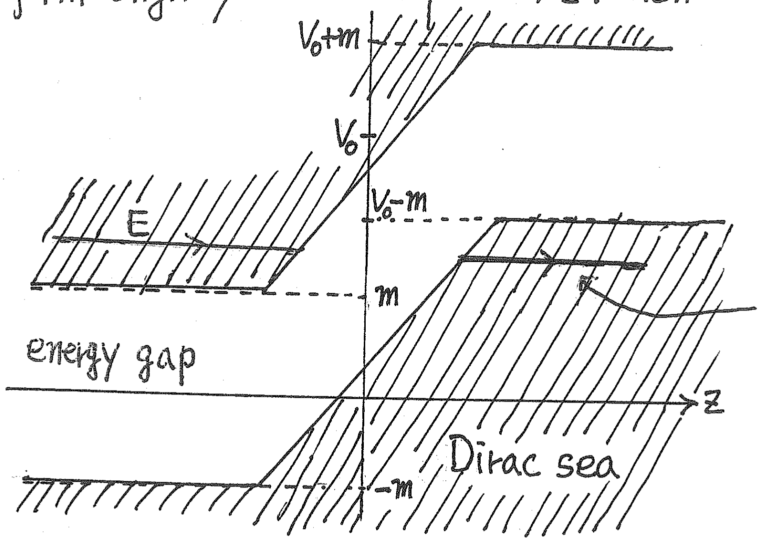
✓ But, for very large $V_0\,(>2m)$ and $E<V_0-m$, we have $\rho<0$ and as a result $\boxed{T<0\ \text{and}\ R>1}$!

"q becomes real again"

paradox!!

— How did this happen? To see the situation somewhat better, alter the potential form slightly : to 

. Then



"In the shaded region, energetically accessible states exist."

Tunnelling may be possible!

To get out of the atom stability prob. and Klein paradox, Dirac proposed the hole 'theory'. Since electrons are fermions, if all negative energy states (in the Dirac sea) are <u>filled</u>, electrons will not fall into the sea due to the Pauli principle (and neither enter the region occupied by the sea energy level). In this picture, we can instead have a particle corresponding to the vacancy in the sea levels — (oppositely charged) $\boxed{\text{hole}}$ or $\boxed{\text{antiparticle}}$.

create a hole in the Dirac sea

"Klein paradox can be related to electron-positron creation."

antiparticle of electron

— But this means that we now have to turn to a many-particle theory (giving up such a beautiful Dirac eq.). Any way to <u>save the Dirac equation</u> while, at the same time,

as a theory for particle <u>and</u> antiparticle

✓ incorporate many-particle aspects mentioned above,

✓ somehow, manage to have the 'good' quantum evolution law $i\frac{d}{dt}|\ \rangle = H|\ \rangle$, with a well-defined hermitian, positive-definite, Hamiltonian H.

➡ Change the qualification of ψ : from the one-particle wave function to a $\boxed{\text{quantized field operator}}$. This leads to

"Quantum Field Theory (QFT)"

PART II : Quantum Field Theory (QFT) Description

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What is quantum field?

Quantum field theory (QFT in short) — the quantum mechanics of continuous systems — arose at the beginning of the quantum era, i.e., in the problem of blackbody radiation. It became fully developed in quantum electrodynamics, the QFT describing the entire dynamics involving photons and electrons, and this is the theory now recognized by many as the most successful edifice in physics. It was then further generalized to incorporate other interactions of non-electromagnetic origin, yielding the Standard Model of present days. All these confirm that QFT is a framework which is agile enough to accommodate both Einstein's Special Relativity and general quantum principle in a consistent way. But what we want to discuss here is not this fundamental interaction side the cloak of QFT has much use for, but QFT as an effective tool to study certain emergent phenomena in condensed matter physics.

Dynamical freedoms in QFT are of course assumed by quantum fields, which may loosely be characterized as *spatial networks of (quantum) oscillators* that can sustain fluctuation in micro-variables of the system. Quantum fields are abstract machines capable of having various particles created here and there from the vacuum (the ground state of the system) and having some of them disappear as well. The way they do these things as time develops is not completely random; quantum fields satisfy field equations of certain form. In fact, one can associate suitable Hamiltonians to these systems. The field equations (or related Hamiltonians) dictate also how the given fields interact with other fields or between themselves. It is this language of quantum fields that theoretical physicists have found most fruitful in describing diverse phenomena of quantum many-body systems. If the system exhibits certain symmetries, quantum fields make representations of those in the group theoretical sense.

Let us be more explicit on the construction of quantum fields. We will start from their building blocks — oscillators.[1] We are here referring to the very oscillators when one says that the electromagnetic field consists of numerous 'oscillators'. As it turns out, for the description of the microscopic world, we require two different types of oscillator (which are mathematically 'operators' in some linear space), viz.,

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[1] In this lecture, oscillators are defined to be any objects satisfying commutation relations (1) (if bosonic) or anticommutation relations (2) (if fermionic). There are two key notions encoded in these algebraic structures of oscillators: the first is the creation and annihilation algorithm of certain basic things, and the second is the statistical nature as for a collection of them.

(i) bose oscillators $a_i, a_i^\dagger$ (i labels independent oscillators) satisfying commutation relations

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \ , \quad [a_i, a_j^\dagger] = \delta_{ij} \ , \tag{1}$$

(ii) fermi oscillators $c_i, c_j^\dagger$ satisfying anticommutation relations

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \ , \quad \{c_i, c_j^\dagger\} = \delta_{ij} \ , \tag{2}$$

where $[A, B] \equiv AB - BA$ and $\{A, B\} \equiv AB + BA$. (If both types are present simultaneously, assume that $[a_i, c_j] = [a_i, c_j^\dagger] = 0$, etc.). If we define the (naive) vacuum $|0\rangle$ by the conditions like $a_i|0\rangle = 0$ (or $c_i|0\rangle = 0$), the basis for the state space can be identified with

$$\left\{ |n_1, n_2, \cdots\rangle \equiv \prod_i \frac{1}{\sqrt{n_i!}} (a_i^\dagger)^{n_i} |0\rangle ; n_i = 0, 1, 2, \cdots, \infty \right\} \tag{3}$$

for a system of bose oscillators, and for a system of fermi oscillators we can consider the basis

$$\left\{ |n_1, n_2, \cdots\rangle \equiv \prod_i (c_i^\dagger)^{n_i} |0\rangle ; n_i = 0, 1 \right\} \ . \tag{4}$$

The states $|n_1, n_2, \cdots\rangle$ here are called the occupation-number basis, since n_i refers to the eigenvalue of the number operator $N_i \equiv a_i^\dagger a_i$ or $c_i^\dagger c_i$ (i not summed). Note that we have an infinite-dimensional state space in association with each bose oscillator, but the state space related to each fermi oscillator is only two-dimensional, i.e., occupied ($n_i = 1$) or unoccupied ($n_i = 0$). With several different bose or/and fermi oscillators in the system, its state space is the direct product of those spaces associated with independent oscillators present.

If the Hamiltonian is not given, the above oscillators will not know how to oscillate in time. Suppose for instance that it is the decoupled quadratic Hamiltonian of the form ($\mathcal{E}_i$ are some positive numbers)

$$H^{(b)} = \sum_i \mathcal{E}_i \, a_i^\dagger a_i \tag{5}$$

or

$$H^{(f)} = \sum_i \mathcal{E}_i \, c_i^\dagger c_i \ . \tag{6}$$

Then the related time-dependent oscillator variables (the so-called Heisenberg-picture oscillator variables)

$$a_i(t) \equiv e^{iH^{(b)}t} a_i e^{-iH^{(b)}t} \ , \quad c_i(t) \equiv e^{iH^{(f)}t} c_i e^{-iH^{(f)}t} \tag{7}$$

satisfy the 'oscillator' equations (see the remark 1 below)

$$\dot{a}_i(t) = i[H^{(b)}(t), a_i(t)] = -i\mathcal{E}_i\, a_i(t)\ , \tag{8a}$$

$$\dot{c}_i(t) = i[H^{(f)}(t), c_i(t)] = -i\mathcal{E}_i\, c_i(t)\ , \tag{8b}$$

where $H^{(b)}(t) \equiv e^{iH^{(b)}t}H^{(b)}e^{-iH^{(b)}t} = \sum_i \mathcal{E}_i a_i^\dagger(t)a_i(t)$ ($= H^{(b)}$) and $H^{(f)}(t) \equiv e^{iH^{(f)}t}H^{(f)}e^{-iH^{(f)}t} = \sum_i \mathcal{E}_i\, c_i^\dagger(t)c_i(t)$ ($= H^{(f)}$). Planck's constant is equal to 1 in this lecture. Note that (8a) and (8b) are the results of using the equal-time (anti-)commutation relations

$$\begin{aligned}&[a_i(t), a_j(t)] = [a_i^\dagger(t), a_j^\dagger(t)] = \{c_i(t), c_j(t)\} = \{c_i^\dagger(t), c_j^\dagger(t)\} = 0\ ,\\ &[a_i(t), a_j^\dagger(t)] = \{c_i(t), c_j^\dagger(t)\} = \delta_{ij}\ .\end{aligned} \tag{9}$$

By positing the Hamiltonian as above, our oscillators become subject to well-defined quantum dynamics. But we cannot say that this defines *a* QFT, since we have not yet made clear the role of our oscillators in *space*.

■ (*Remark* 1) The c-number (i.e., non-quantum) counterpart of the oscillator equation (8a)

$$\dot{z}_i = -i\mathcal{E}_i\, z_i \tag{10}$$

represent classical equations of motion for the system governed by the classical Hamiltonian $H_{cl}^{(b)} = \sum_i \mathcal{E}_i\, z_i^* z_i$, where the complex (phase-space) variables z_i, z_i^* satisfy the Poisson bracket relations

$$[z_i, z_j]_{\text{P.B.}} = [z_i^*, z_j^*]_{\text{P.B.}} = 0\ , \quad [z_i, iz_j^*]_{\text{P.B.}} = \delta_{ij}\ . \tag{11}$$

One may encode this canonical structure and also the equations of motion by writing the first-order Lagrangian

$$L_{(\text{1st})}^{(b)} = \sum_i (iz_i^*)\dot{z}_i - H_{cl}^{(b)} = \sum_i iz_i^*(\dot{z}_i + i\mathcal{E}_i\, z_i)\ . \tag{12}$$

Here note that, if one writes $z_i = \frac{1}{\sqrt{2}}(\sqrt{\mathcal{E}_i}Q_i + i\frac{P_i}{\sqrt{\mathcal{E}_i}})$ and $z_i^* = \frac{1}{\sqrt{2}}(\sqrt{\mathcal{E}_i}Q_i - i\frac{P_i}{\sqrt{\mathcal{E}_i}})$ (with $[Q_i, Q_j]_{\text{P.B.}} = [P_i, P_j]_{\text{P.B.}} = 0$ and $[Q_i, P_j]_{\text{P.B.}} = \delta_{ij}$), one can recover more familiar expressions pertaining to a system of harmonic oscillators, viz.,

$$H_{cl}^{(b)} = \sum_i \left(\frac{1}{2}P_i^2 + \frac{1}{2}\mathcal{E}_i^2 Q_i^2\right)\ , \tag{13}$$

$$\begin{aligned} L^{(b)}_{(1\mathrm{st})} &= \sum_i P_i\dot{Q}_i - \sum_i \left(\frac{1}{2}P_i^2 + \frac{1}{2}\mathcal{E}_i^2 Q_i^2\right) + (\text{total derivative}) \\ &\sim \sum_i P_i\dot{Q}_i - H^{(b)}_{cl} \,. \end{aligned} \tag{14}$$

(Here '$\sim$' means the equality up to an unimportant total derivative term). The second order Lagrangian, $L^{(b)}_{(2\mathrm{nd})} = \sum_i \left(\frac{1}{2}\dot{Q}_i^2 - \frac{1}{2}\mathcal{E}_i^2 Q_i^2\right)$,is obtained from (14) if one eliminates all P's by using their equations of motion $P_i = \dot{Q}_i$. To consider the c-number theory corresponding to the fermionic oscillator system, one has to introduce *Grassmannian variables* (and the *graded* Poisson brackets).

■ (*Remark* 2) A quantum theory can be obtained by 'quantizing' the corresponding classical theory according to Dirac's quantization rule[2]

(classical theory)	$\longrightarrow$	(quantum theory)
classical observables $A(p(t), q(t))$	$\longrightarrow$	quantum operators $A(\hat{p}(t), \hat{q}(t))$
$[A(t), B(t)]_{\mathrm{P.B.}}$	$\longrightarrow$	$-i[\hat{A}(t), \hat{B}(t)]$

For example, $z = \frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{E}}Q + i\frac{P}{\sqrt{\mathcal{E}}}\right)$ and $z^* = \frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{E}}Q - i\frac{P}{\sqrt{\mathcal{E}}}\right)$ go over after quantization to $\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{E}}\hat{Q} + i\frac{\hat{P}}{\sqrt{\mathcal{E}}}\right)$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{E}}\hat{Q} - i\frac{\hat{P}}{\sqrt{\mathcal{E}}}\right)$; also, the standard commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ or $[\hat{Q}, \hat{P}] = i$ can be inferred from the classical Poisson bracket relation $i[z, z^*]_{\mathrm{P.B.}} = 1$ or $[Q, P]_{\mathrm{P.B.}} = 1$. This classical-quantum correspondence can be extended to theories involving fermionic variables, with due care paid to such things as Grassmannian property. Note also that one needs to know the classical theory to give the path integral description for the given quantum theory.

Having introduced oscillators, we can now install a suitably-built oscillator network in space to obtain a QFT. The oscillator here had better be arranged in such a way that appropriate information/disturbance can be transferred from one place in space to another through the network. This will especially be the case if the oscillators are coupled by nearest-neighbor-type interactions, say, as in a one-dimensional

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[2] This quantization procedure does not specify the correct operator ordering, however. For simple cases this ordering ambiguity can be resolved through posteriori considerations like the imposition of hermiticity. We also remark that there exists the quantum counterpart (i.e., operator) to the classical Lagrangian or action. (The action S is simply the Lagrangian integrated over time, i.e., $\int dt\, L$, and the significance of the quantum action operator can be found in Schwinger's action principle).

chain of them. Adopting the canonical variables[3] Q_i and P_i, $i = 1, 2, \cdots, N$ in association with the i-th bose oscillator (i here indicates the lattice site successively numbered along the chain), we can take the Hamiltonian of this chain as

$$H = \sum_{i=1}^{N} \left\{ \frac{1}{\tilde{\mu}} P_i^2 + \frac{\tilde{T}}{2} (Q_i - Q_{i-1})^2 \right\} \ , \quad (\tilde{\mu}, \tilde{T} > 0) \tag{15}$$

so that Q_i may well represent the instantaneous displacement of the i-th atom from its equilibrium position. This system has the second-order Lagrangian

$$L_{(2\text{nd})} = \sum_{i=1}^{N} \left[\frac{\tilde{\mu}}{2} \dot{Q}_i^2 - \frac{\tilde{T}}{2} (Q_i - Q_{i-1})^2 \right] \ . \tag{16}$$

Denoting the equilibrium position of the i-th atom as $x_i = \Delta i$ (Δ is the lattice spacing), we will now consider the *continuum limit*: $\Delta \to 0$, $N \to \infty$ for $N\Delta = L$(:fixed). At the same time, parameters $\tilde{\mu}$ and $\tilde{T}$ may be adjusted such that $\mu \equiv \frac{\tilde{\mu}}{\Delta}$ and $T \equiv \tilde{T}\Delta$ have finite limits as $\Delta \to 0$. We are only interested in the long-range (i.e., $r \gg \Delta$) fluctuation dynamics of the network here. We may then introduce bosonic (real) quantum fields $\phi(x, t)$ and $\Pi(x, t)$ according to

$$\phi(x = \Delta i, t) = Q_i(t) \ , \quad \Pi(x = \Delta i, t) = \frac{1}{\Delta} P_i(t) \tag{17}$$

and consider the corresponding limit theory.(Fields are said to be *real* if they satisfy 'operator hermiticity', and *complex* if not). That is a QFT based on the Hamiltonian

$$H = \int_0^L dx \left\{ \frac{1}{2\mu} \Pi^2(x, t) + \frac{T}{2} \left(\frac{\partial \phi(x, t)}{\partial x} \right)^2 \right\} \tag{18}$$

with the field variables satisfying the equal-time commutation relations

$$[\phi(x, t), \phi(x', t)] = [\Pi(x, t), \Pi(x', t)] = 0 \ , \quad [\phi(x, t), \Pi(x', t)] = i\delta(x - x') \ . \tag{19}$$

The related field theory action is

$$S_{(2\text{nd})} \equiv \int dt \ L_{(2\text{nd})} = \int dt \int_0^L dx \left\{ \frac{1}{2} \mu \left(\frac{\partial \phi(x, t)}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial \phi(x, t)}{\partial x} \right)^2 \right\} \ , \tag{20}$$

and so we expect by invoking the stationary action principle that $\phi(x, t)$ satisfy the field equation of the form

$$\left(\mu \frac{\partial^2}{\partial t^2} - T \frac{\partial^2}{\partial x^2} \right) \phi(x, t) = 0 \ . \tag{21}$$

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[3] Although we here take these variables to be quantum (Heisenberg-picture) operators, it is possible to look upon them as classical variables in view of the classical-quantum correspondence mentioned in the Remark 2 above.

This equation indeed follows upon evaluating the Heisenberg equations of motion

$$\dot{\phi}(x,t) = i[H(t), \phi(x,t)] \; , \quad \dot{\Pi}(x,t) = i[H(t), \Pi(x,t)] \tag{22}$$

with our Hamiltonian (18).

■ (*Remark* 3) As is discussed in standard textbooks, the above continuum limit can be given a rigorous justification. This should be expected since the discrete Hamiltonian (15) is exactly diagonalizable by suitable canonical transformations for arbitrary N; that is, using the oscillator variables related to the normal modes of the system, the Hamiltonian can be recast into the decoupled form (13) or (5). In the continuum-limit theory, i.e., in the QFT with the Hamiltonian (18), the diagonalization of the Hamiltonian is achieved by considering the Fourier transform of the field variables, i.e., in terms of momentum-space operators a_k, $a_k^\dagger$ related to $\phi(x,t)$ schematically by

$$\phi(x,t) \sim \sum_k \left[e^{ikx} a_k^\dagger(t) + e^{-ikx} a_k(t) \right] . \tag{23}$$

Using these variables, the Hamiltonian (18) can be expressed in the form $\sum_k \mathcal{E}(k) a_k^\dagger(t) a_k(t)$+(const.) with suitable energy dispersion $\mathcal{E}(k)$. The operators $a_k(t), a_k^\dagger(t)$ (with *momentum* index k) are oscillator variables pertaining to true 'particles' of this QFT.

The above construction may be repeated with a two- or three-dimensional network, to obtain a two- or three-dimensional QFT where the bosonic field $\phi(\vec{x},t)$ (here $\vec{x} = (x,y)$ or (x,y,z)) may satisfy the equation like

$$\left(\mu \frac{\partial^2}{\partial t^2} - T\vec{\nabla}^2 \right) \phi(\vec{x},t) = 0 \; . \tag{24}$$

This equation is linear. But, if our oscillators have been subject to more complicated interactions beyond the strictly harmonic types considered in (15), a nonlinear field equation for $\phi(\vec{x},t)$ could have been obtained. We can also contemplate lattice Hamiltonians having very different structures from the form (15); then, QFTs with Hamiltonians not at all like the one in (18) can emerge in the continuum limit. For instance, if our lattice Hamiltonian of bose oscillators contains terms like

$\frac{1}{2\tilde{m}}\sum_i(a_i^\dagger - a_{i-1}^\dagger)(a_i - a_{i-1})$, $\sum_i \tilde{U}_i a_i^\dagger a_i$ and $\sum_{i\neq j} \tilde{V}_{ij} a_i^\dagger a_i a_j^\dagger a_j$ (with $\tilde{V}_{ij} = \tilde{V}_{ji}$), then the Hamiltonian of the resulting QFT in the continuum limit will acquire the form

$$\begin{aligned} H &= \int dx \left\{ \frac{1}{2m} \frac{\partial \psi^\dagger(x,t)}{\partial x} \frac{\partial \psi(x,t)}{\partial x} + \psi^\dagger(x,t) U(x) \psi(x,t) \right\} \\ &\quad + \int dx dy \; \psi^\dagger(x,t)\psi^\dagger(y,t) V(x,y) \psi(x,t) \psi(y,t) \; , \end{aligned} \tag{25}$$

with the (complex) quantum fields $\psi(x,t)$ and $\psi^\dagger(x,t)$, constructed according to $\psi(x = \Delta i, t) = \frac{1}{\sqrt{\Delta}} a_i(t)$ and $\psi^\dagger(x = \Delta i, t) = \frac{1}{\sqrt{\Delta}} a_i^\dagger(t)$, satisfying the equal-time commutation relations

$$[\psi(x,t), \psi(x',t)] = [\psi^\dagger(x,t), \psi^\dagger(x',t)] = 0 \; , \quad [\psi(x,t), \psi^\dagger(x',t)] = \delta(x - x') \; . \tag{26}$$

For this QFT we can write the first-order action as

$$\begin{aligned} S_{(1\mathrm{st})} &= \int dt \Bigg[\int dx \; \psi^\dagger(x,t) \Big\{ i \frac{\partial}{\partial t} + \frac{1}{2m} \frac{\partial^2}{\partial x^2} - U(x) \Big\} \psi(x,t) \\ &\quad - \int dx dy \; \psi^\dagger(x,t) \psi^\dagger(y,t) V(x,y) \psi(x,t) \psi(y,t) \Bigg] \; . \end{aligned} \tag{27}$$

This theory — the one-dimensional Schrödinger QFT — has a noteworthy feature in that we have the simple *number operator* $N(t) = \int dx \; \psi^\dagger(x,t)\psi(x,t)$ which is conserved (i.e., time-independent) since $[N(t), H(t)] = 0$. Because of the latter property, one may well study the given number sector of this QFT; then, one obtains a many-body Schrödinger equation for interacting nonrelativistic *bosons* with the two-body potential $V(x,y)$ under the influence of the external potential $U(x)$.[4] Two- or three-dimensional Schrödinger QFTs, based on the field variables $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$ (with $\vec{x} = (x,y)$ or (x,y,z)), are straightforward extensions of this, and it is this framework that forms the basis of present-day quantum many-body theory.

QFTs can be built also using a network of fermi oscillators. But it should be noted that, because of their anticommuting nature, a specific form of Hamiltonian which made a good sense with bose variables may not define any sensible physical system with fermi variables instead. Such is the case with the Hamiltonian (18). If the fields there, i.e., $\Pi(x,t)$ and $\phi(x,t)$ were derived using fermi oscillators, they would have to satisfy the equal-time anticommutation relations like

$$\{\phi(x,t), \phi(x',t)\} = \{\Pi(x,t), \Pi(x',t)\} = 0 \; , \quad \{\phi(x,t), \Pi(x',t)\} = i\delta(x - x') \; . \tag{28}$$

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[4] See any book on many-body theory for the demonstration of this.

But, as one can easily convince oneself, the Hamiltonian (18) then becomes trivial. The Schrödinger QFT Hamiltonian (25), on the other hand, leads to a nontrivial physical system even if the fields there, $\psi(x,t)$ and $\psi^\dagger(x,t)$, happen to satisfy the equal-time anticommutation relations (instead of (26))

$$\{\psi(x,t),\psi(x',t)\} = \{\psi^\dagger(x,t),\psi^\dagger(x',t)\} = 0 \ , \quad \{\psi(x,t),\psi^\dagger(x',t)\} = \delta(x-x') \ . \qquad (29)$$

This QFT corresponds to a system of interacting nonrelativistic fermions, described by a fully antisymmetric wave function (for a given total particle number, which can be arbitrary). That is, by a certain dubious conspiracy or by accident, the Schrödinger QFT in any spatial dimension makes sense for both bose and fermi fields. In case the particle come in several flavors[5] distinguished by some label 'a', one may well describe the corresponding system by considering the generalized Schrödinger QFT with multi-component bose or fermi fields $\psi_a(x,t)$ and $\psi_a^\dagger(x,t)$.

Quantum fields in the Minkowski spacetime — the ones particle theorists use in their daily business — should have the nature not very different from those discussed above[6], but for the fact that these fields should make suitable irreducible representations of the Poincaré group. By demanding the latter, it becomes very simple to have the Special Relativity principle of Einstein implemented on the dynamics of these fields. The Poincaré algebra has two Casimir operators the eigenvalues of which can be identified with *mass* and *spin* of the particle. So, given the mass and spin for a fundamental particle, we know immediately what kind of quantum field should be used for the description of the particle and can also tell the *free* dynamics of the field chosen.[7] For instance, for a spin-zero particle of mass m, we must use the (real or complex) Lorentz scalar field $\phi(\vec{x},t)$ whose free field dynamics is the well-known Klein-Gordon equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2c^2\right)\phi \equiv (-\partial^\mu\partial_\mu + m^2c^2)\phi = 0 \ . \qquad (30)$$

Here, c is the speed of light, $\partial_\mu = \left(\frac{\partial}{\partial x^0} \equiv \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right)$ is the four-vector derivative, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$ where $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$ is the Minkowski metric. For a spin-half particle (like an electron) of mass m it is the (multi-component) Dirac field

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[5] This may represent ordinary spin degrees of freedom.

[6] That is, they are also objects made up of (bose or fermi) oscillators, with the statistical property of created particles determined by the equal-time commutation or anticommutation relations of the fields.

[7] *Free* dynamics of a given field is important since it contains the information on the *asymptotic states* of the associated particle. The free field equation thus assumes the role similar to Newton's first law in classical mechanics. [Note that, in special relativity, Newton's first law is tantamount to the equations $\frac{d^2x^\mu}{d\tau^2} = 0$ and $\frac{ds^\mu}{d\tau} = 0$, if $x^\mu(\tau)$ and $s^\mu(\tau)$ denote the world trajectory and the polarization four-vector of a (spinning) free particle.]

$\psi_\alpha(\vec{x},t)$ that should be used, and for the corresponding free field dynamics we have a first-order matrix differential equation in the name of the *Dirac equation*

$$\left(\frac{1}{i}\gamma^\mu\partial_\mu + mc\right)\psi = 0\ , \tag{31}$$

where $\gamma^\mu = (\gamma^0, \gamma^i)$ denote a set of 4×4 matrices satisfying the Dirac-Clifford algebraic relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}\ . \tag{32}$$

For detailed forms of the γ-matrices, etc., readers may consult the Appendix. To describe a spin-one particle a Lorentz four-vector field $A^\mu(\vec{x},t)$ is required and, if massless (as in the case of light quanta), we have the Maxwellian equation for the corresponding free field dynamics:

$$\partial^\mu F_{\mu\nu} = 0\ , \quad (F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu)\ . \tag{33}$$

■ (*Remark* 4) For the Lorentz-covariant field equations discussed above, one can find the corresponding Hamiltonian descriptions[8] so that these field equations may result as the Heisenberg equations of motion with the suitable field

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[8] Here, for the Dirac equation (31) which is *of first order in the time derivative*, one might wonder how this can be recast using the Lagrangian and Hamiltonian formalism. Well, looking at (27) (with $V(x,y)$ set to zero), we see that the first-order Schrödinger field equation

$$\left\{i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial x^2} - U(x)\right\}\psi(x,t) = 0$$

results from considering the stationary condition for the first-order action

$$\begin{aligned} S_{(1\text{st})} &= \int dt \int dx\ \psi^\dagger(x,t)\left\{i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial x^2} - U(x)\right\}\psi(x,t) \\ &= \int dt\ \left[\int dx\ \psi^\dagger(x,t) i\frac{\partial}{\partial t}\psi(x,t) - H\right]\ , \quad \left(H = \int dx\ \psi^\dagger(x,t)\left\{-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right\}\psi(x,t)\right)\ . \end{aligned}$$

Then, applying the analogy to the Dirac equation (31) which can be rewritten as (here set $c = 1$ also)

$$\left(i\frac{\partial}{\partial t} - \frac{1}{i}\gamma^0\gamma^i\partial_i - \gamma^0 m\right)\psi = 0\ ,$$

we may consider the first-order action of the form

$$\begin{aligned} S_{(1\text{st})} &= \int dt \int d\vec{x}\ \psi^\dagger(\vec{x},t)\left\{i\frac{\partial}{\partial t} - \frac{1}{i}\gamma^0\gamma^i\partial_i - \gamma^0 m\right\}\psi(\vec{x},t) \\ &= -\int dt \int d\vec{x}\ \bar{\psi}(\vec{x},t)\left(\frac{1}{i}\gamma^\mu\partial_\mu + m\right)\psi(\vec{x},t)\ , \quad (\bar{\psi}(\vec{x},t) \equiv \psi^\dagger(\vec{x},t)\gamma^0)\ . \end{aligned}$$

Hamiltonians. Of course, this presupposes one's knowledge on what to choose between equal-time commutation or anticommutation relations for the fields. A remarkable thing, which emerges in the context of relativistic QFTs, is the *spin-statistics connection*: one does not obtain a consistent quantum system, unless the field pertaining to integer-spin particles satisfy equal-time commutation relations (i.e., follow bose statistics) and those pertaining to half-integer-spin particles satisfy equal-time anticommutation relations (i.e., follow fermi statistics). Thus the Dirac field $\psi_\alpha(\vec{x},t)$ must be assumed to satisfy the equal-time anticommutation relations

$$\{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{x}',t)\} = \{\psi^\dagger_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{x}',t)\} = 0\,,\ \ \{\psi_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{x}',t)\} = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}')\,. \tag{34}$$

Only when these relations (and not equal-time commutation relations) are assumed, one can produce a sensible quantum system with the Dirac Hamiltonian $H=\int d\vec{x}\ \psi^\dagger\left(\vec{\alpha}\cdot\frac{1}{i}\vec{\nabla}+m\gamma^0\right)\psi$ (see the footnote 8). Here, $\vec{\alpha}\equiv\gamma^0\vec{\gamma}$.

■ (*Remark* 5) For these free-field systems the Hamiltonians can always be diagonalized by making appropriate Fourier transforms, i.e., by expressing our configuration-space field variables in terms of momentum-space (bose or fermi) oscillators. Denoting the latter by $a_{\vec{p}(i)}(t)$ and $a^\dagger_{\vec{p}(i)}(t)$ ($\vec{p}$ is the momentum index, and (i) for independent polarizations and for other quantum numbers if exist[9]), the Hamiltonians then become (with $c\equiv 1$)

$$H\sim\sum_{\vec{p},i}\sqrt{\vec{p}^2+m^2}\,a^\dagger_{\vec{p}(i)}a_{\vec{p}(i)}\ , \tag{35}$$

showing that a particle of momentum $\vec{p}$ has energy $\mathcal{E}(\vec{p})=\sqrt{\vec{p}^2+m^2}$. This is in accord with the Special Relativity. If we restrict our attention to *nonrelativistic* excitations, i.e., to particles with $|\vec{p}|\ll m$, the system can be described by the effective Hamiltonian $H_{\text{eff}}\sim\sum_{\vec{p},i}\frac{\vec{p}^2}{2m}\,a^\dagger_{\vec{p}(i)}a_{\vec{p}(i)}$ (for the given total particle number), regardless of the spin of the particle. This effective theory is the free Schrödinger QFT, formulated using momentum-space, bose or fermi, oscillators. This explains partly why the Schrödinger QFT makes sense for both bose and fermi fields.

From this the Hamiltonian of the system may also be identified to be

$$H=\int d\vec{x}\ \bar{\psi}(\vec{x},t)\left(\frac{1}{i}\vec{\gamma}\cdot\vec{\nabla}+m\right)\psi(\vec{x},t)=\int d\vec{x}\ \psi^\dagger(\vec{x},t)\left(\vec{\alpha}\cdot\frac{1}{i}\vec{\nabla}+m\gamma^0\right)\psi(\vec{x},t)\ ,$$

where $\vec{\alpha}\equiv\gamma^0\vec{\gamma}$.

[9] There can be distinct particle and anti-particle excitations, for instance. See the Appendix.

Since Lorentz symmetry is largely responsible for the relativistic field dynamics discussed above, one might suppose that, in condensed matter physics (where one's interest is centered on nonrelativistic many-body phenomena), this structure becomes totally obsolete. Well, it is mistaken. Physics is like one organic body — a structure found in one corner of the body may appear again in another, very distinct, part of the body (possibly in a slightly adjusted form)[10]. Relativistic field structure, especially its ultrarelativistic or massless limit form, can prove quite relevant even in the domain of condensed matter physics (although Lorentz symmetry no longer plays role here). Look at the phonon field equation (21) or (24), which is supposed to describe collective excitations in lattice systems: this is really the massless Klein-Gordon equation, but for the fact that the speed of light in the latter gets replaced by the speed of sound $c = \sqrt{\frac{T}{\mu}}$. Considering the specially simple nature of this equation, one might say that this is merely an accidental thing and this kind of 'theory analogue' cannot be expected, say, with the Dirac field equation (because it appears to have a highly nontrivial structure of Lorentz symmetry built in). That is wrong. Even the Dirac field structure — especially the massless case — does show up effectively, albeit in lower-dimensional setting, as one considers collective dynamics of certain condensed matter systems. This condensed-matter emergence of the Dirac field structure will be exemplified in the lectures that follow. See also the Appendix for more elaborations on the structure of the Dirac field equations in $(1+1)$-, $(2+1)$-, and $(3+1)$- spacetime dimensions.

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[10] One might say that Nature is an economical animal with good memory: if it finds one stable (or robust) structure, it likes to utilize it repeatedly.

From fermion chains to Dirac field theory

- 1D tight-binding chain with the Hamiltonian

$$H = \sum_{i=1}^{N} \varepsilon_i c_i^\dagger c_i + \underbrace{\sum_{i=1}^{N} T_i}_{} \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right)$$

↖ hopping matrix elements

$c_i, c_i^\dagger$: fermionic annihilation/creation operators of a 'particle' on site i satisfying anticommutation relations

$$\{ c_i^\dagger, c_j \} = \delta_{ij} \ , \quad \{ c_i, c_j \} = \{ c_i^\dagger, c_j^\dagger \} = 0 .$$

Dirac field theory in (1+1)-, (2+1)- and (3+1) dimensions

We will here describe some of the salient features of relativistic fermionic QFTs in spacetime dimensions $D = 1+1$, $2+1$, and $3+1$ in particular. First note that, in the context of relativistic quantum mechanics, the *free* Dirac equation for a particle of mass m reads (here we assume $(D-1)+1$-dimensional spacetime)[11]

$$\left(\frac{1}{i} \gamma^\mu \partial_\mu + m \right) \psi(\vec{x}, t) = 0 \ , \tag{36}$$

where $\mu (= 0, 1, \cdots, D-1)$ is the Lorentz index, $\psi(\vec{x}, t) = \{ \psi_\alpha(\vec{x}, t); \alpha = 1, 2, \cdots \}$ is a certain column-vector wave function, $\gamma^\mu \partial_\mu \equiv \gamma^0 \partial_0 + \gamma^i \partial_i = \gamma^0 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \frac{\partial}{\partial \vec{x}}$ ($i = 1, \cdots, D-1$ used for the spatial index only), and the γ-matrices satisfy the Dirac-Clifford algebra

$$\{ \gamma^\mu, \gamma^\nu \} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \eta^{\mu\nu} \ , \tag{37}$$

$$(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} . \tag{38}$$

For conveniences in interpretation, we will further assume that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. Detailed forms of the γ-matrices will be presented later. If we multiply (36) from the left by the conjugate Dirac operator $\left(-\frac{1}{i} \gamma^\mu \partial_\mu + m \right)$, we obtain the Klein-Gordon-type equation

$$(-\partial^\mu \partial_\mu + m^2) \psi(\vec{x}, t) = 0 \ , \quad (\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu) \tag{39}$$

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[11] We have here followed condensed-matter theorist's practice of taking $c = \hbar = 1$.

since $(\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu) = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = -\eta^{\mu\nu}\partial_\mu \partial_\nu = \partial_t^2 - \vec{\nabla}^2$.

The wave equation (39) is solved by plane waves of the form $\psi(\vec{x}, t) \propto e^{-i\mathcal{E}(\vec{p})t + i\vec{p}\cdot\vec{x}}$, where the dispersion relation is

$$\mathcal{E}(\vec{p}) = \pm\sqrt{\vec{p}^2 + m^2}\ . \tag{40}$$

The full Dirac spinor, i.e., the solution of (36) is given by the form

$$\psi(\vec{x}, t) = \omega(\vec{p}) e^{-i\mathcal{E}(\vec{p})t + i\vec{p}\cdot\vec{x}}\ , \quad (\omega(\vec{p}) = \omega_+(\vec{p}) \text{ or } \omega_-(\vec{p})) \tag{41}$$

where $\psi_>(\vec{p})$ or $\psi_<(\vec{p})$ denotes a column vector satisfying

$$\begin{aligned} &(-\gamma^0\sqrt{\vec{p}^2 + m^2} + \vec{\gamma}\cdot\vec{p} + m)\,\omega_+(\vec{p}) = 0\ , \quad (\text{ for } \mathcal{E}(\vec{p}) = +\sqrt{\vec{p}^2 + m^2})\ , \\ &(+\gamma^0\sqrt{\vec{p}^2 + m^2} + \vec{\gamma}\cdot\vec{p} + m)\,\omega_-(\vec{p}) = 0\ , \quad (\text{ for } \mathcal{E}(\vec{p}) = -\sqrt{\vec{p}^2 + m^2})\ . \end{aligned} \tag{42}$$

There are typically several linearly independent solutions for $\omega_+(\vec{p})$ or $\omega_-(\vec{p})$; in $D = 3 + 1$ dimensions, for example, there are two (corresponding to the two spin states). The Dirac spinor wave function should also carry this label, but we will suppress it here. The Hamiltonian whose eigenvalues are the allowed energy values, i.e., $\mathcal{E}(\vec{p}) = \pm\sqrt{\vec{p}^2 + m^2}$, may be deduced by multiplying the Dirac equation (36) by γ^0 (from the left) and comparing the result with the Schrödinger equation $i\frac{\partial}{\partial t}\psi = H\psi$. One then finds the Dirac Hamiltonian related to the quantum mechanical equation (36)

$$\begin{aligned} H_{\mathrm{D}} &= \frac{1}{i}\gamma^0\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 m \qquad (43) \\ &\equiv \vec{\alpha}\cdot\vec{P} + \beta m\ , \quad \left(\vec{P} = \frac{1}{i}\vec{\nabla}\ ,\ \vec{\alpha} \equiv \gamma^0\vec{\gamma}\ ,\ \beta \equiv \gamma^0\right)\ , \end{aligned}$$

which is a hermitian operator (with the eigenvalues $\pm\sqrt{\vec{p}^2 + m^2}$ on wave functions of the form $\psi(\vec{x}) = \omega(\vec{p})e^{i\vec{p}\cdot\vec{x}}$).

The spectrum of the above Dirac Hamiltonian is clearly unbounded from below, with both positive and negative energy states. We thus have no ground state, and hence no consistent quantum theory to speak of. To save the theory from this rather unfortunate demise, Dirac introduced the *hole theory interpretation.* In this interpretation of Dirac, the ground state is identified with the many-body state where all negative energy states are occupied. The stability of this state is assured if the particles obey *Fermi statistics* and so, by the Pauli-exclusion principle, only one particle is allowed to occupy each state. [This ground state is similar to that of a degenerate Fermi gas in nonrelativistic many-body physics (with the Fermi surface at $\mathcal{E} = -m$)]. The necessity of identifying the ground state in this way leads to the existence of two kinds of excitations — *particles*, which occur when a positive energy state is occupied, and *antiparticles* (the analog of *holes* in the case of the degenerate Fermi gas), which occur when a negative energy state is unoccupied. Both particle and antiparticle excitations lead to energy larger than the Dirac ground state. It was this consideration that led Dirac to conjecture the existence of the positron.

Can we put what we just said in words into a precise mathematical framework? The answer is yes — the above picture results if (36) is not regarded as a (one-particle) Schrödinger wave equation but as a field equation for the Dirac *field operator* $\psi(\vec{x},t)$. We then have the solution to the operator equation (36) written as

$$\psi(\vec{x},t) = \int d\vec{p}\ \left\{ \omega_+(\vec{p}) e^{-i\mathcal{E}(\vec{p})t+i\vec{p}\cdot\vec{x}} a(\vec{p}) + \omega_-(\vec{p}) e^{i\mathcal{E}(\vec{p})t+i\vec{p}\cdot\vec{x}} b(-\vec{p}) \right\} \ , \qquad (44)$$

where $a(\vec{p}), a^\dagger(\vec{p})$ and $b(\vec{p}), b^\dagger(\vec{p})$ are pairs of fermi oscillator variables (or creation and annihilation operators) satisfying

$$\begin{aligned} &\{a(\vec{p}),a(\vec{q})\} = \{a^\dagger(\vec{p}),a^\dagger(\vec{q})\} = 0 \ , \quad \{a(\vec{p}),a^\dagger(\vec{q})\} = \delta(\vec{p}-\vec{q}) \ , \\ &\{b(\vec{p}),b(\vec{q})\} = \{b^\dagger(\vec{p}),b^\dagger(\vec{q})\} = 0 \ , \quad \{b(\vec{p}),b^\dagger(\vec{q})\} = \delta(\vec{p}-\vec{q}) \ , \\ &\{a(\vec{p}),b(\vec{q})\} = \{a(\vec{p}),b^\dagger(\vec{q})\} = 0 \ . \end{aligned} \qquad (45)$$

These anticommutation relations, which take care of Fermi statistics, also imply the equal-time anticommutation relations for our field operators $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$:

$$\{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{x}',t)\} = \{\psi^\dagger_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{x}',t)\} = 0 \ , \ \{\psi_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{x}',t)\} = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}') \ . \qquad (46)$$

The ground state or vacuum, $|0\rangle$, is the state which is annihilated by all operators $a(\vec{p})$, $b(\vec{p})$, i.e.,

$$a(\vec{p})|0\rangle = 0 \quad \text{and} \quad b(\vec{p})|0\rangle = 0 \ . \qquad (47)$$

Then many particle and antiparticle states are formed by operating creation operators on this vacuum, i.e., are represented by

$$a^\dagger(\vec{p}_1)\cdots a^\dagger(\vec{p}_m) b^\dagger(\vec{q}_1)\cdots b^\dagger(\vec{q}_n)|0\rangle \ . \qquad (48)$$

They correspond to eigenstates of the Dirac field Hamiltonian, which can be written as (see the footnote 8 of Sec. 1)

$$H = \int d\vec{x}\ \psi^\dagger(\vec{x},t) \left(\frac{1}{i}\gamma^0\gamma^i\partial_i + \gamma^0 m \right) \psi(\vec{x},t) \qquad (49)$$

or

$$H = \int d\vec{p}\ \sqrt{\vec{p}^2+m^2} \left\{ a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p}) \right\} \qquad (50)$$

with eigenvalue $E = \sqrt{\vec{p}_1^{\,2} + m^2} + \cdots + \sqrt{\vec{q}_n^{\,2} + m^2}$. All eigenvalues are positive, and the particle and antiparticle excitations have identical spectra. These excitations are distinguished by their *charges*, which are eigenvalues of the operator

$$Q = \int d\vec{x}\ \psi^\dagger(\vec{x},t)\psi(\vec{x},t) = \int d\vec{p}\ \{a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p})\}\ . \tag{51}$$

In what follows, we will discuss more specific aspects relevant to Dirac equations in $D = 1+1$, $2+1$, and $3+1$ dimensions.

1. $1+1$ Dimensions

The minimal representations of the Dirac-Clifford algebra in $1+1$ dimensions are obtained using the Pauli matrices as, for example,

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \quad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\ . \tag{52}$$

Another choice
$\gamma^0 = \sigma_z$, $\gamma^1 = i\sigma_y$
(: Dirac basis)

With this choice the two dimensional Dirac operator

$$\frac{1}{i}\gamma^\mu\partial_\mu + m = \begin{pmatrix} m & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & m \end{pmatrix} \tag{53}$$

is real. This means that the field operator can be chosen to be real (in the sense of operator hermiticity); in this case, the field ψ is called a *Majorana spinor.* A complex spinor which obeys the same equation, which is thought of as a combination of two Majorana spinors, is called a *Dirac spinor.* (Note that the electromagnetic field cannot be coupled to a Majorana spinor, as the coupling term in the equation brings in the imaginary unit i and thus leads to an inconsistency[12]). The action from which the Dirac equation can be derived is

$$S_{(1\mathrm{st})} = -\int d^2x\bar{\psi}(x,t)\left(\frac{1}{i}\gamma^\mu\partial_\mu + m\right)\psi(x,t)\ , \quad (\bar{\psi}(x,t) \equiv \psi^\dagger(x,t)\gamma^0)\ . \tag{54}$$

If the spinor is complex (i.e., for a Dirac spinor), this action has a symmetry under changing the phase of the spinor, i.e., $\psi(x,t) \to e^{i\lambda}\psi(x,t)$; as a consequence of this $U(1)$ symmetry, the current is given by[13]

$$J^\mu(x,t) = \bar{\psi}(x,t)\gamma^\mu\psi(x,t) \tag{55}$$

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[12] With several Majorana spinors, it is possible to have a coupling to the gauge fields which are purely imaginary matrices. This occurs for example if the Majorana spinors make the adjoint representation of the (Yang-Mills) gauge group.

[13] In QFT the current $J^\mu(x,t)$, being given by the product of two field operators *at the same point,* is generally ill-defined. Even in free QFT, we had better take the antisymmetric product $J^\mu(x,t) = \frac{1}{2}\{\bar{\psi}_\alpha(x,t), \gamma^\mu_{\alpha\beta}\psi_\beta(x,t)\}$ (rather than (55)), to obtain a quantity with regular behavior.

is conserved.

If the mass of the fermion were zero, then the Dirac equation would have a simpler form

$$\frac{1}{i}\gamma^\mu\partial_\mu\psi(x,t)=0 \quad \text{or} \quad \begin{pmatrix} 0 & -\partial_0+\partial_1 \\ \partial_0+\partial_1 & 0 \end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}=0\ , \tag{56}$$

which decomposes into two equations

$$(\partial_0-\partial_1)\psi_L = 0\ , \tag{57a}$$
$$(\partial_0+\partial_1)\psi_R = 0\ . \tag{57b}$$

The two independent fermion fields described by ψ_L and ψ_R are called *Weyl spinors.* Since the equations are real, the spinors can also be chosen to be real. In this case, they are called *Majorana-Weyl spinors.* (Weyl spinors can be coupled to electromagnetic fields, but Majorana-Weyl spinors cannot). In this massless case we have the action

$$S_{(1\mathrm{st})}=i\int d^2x\ \left\{\psi_R^\dagger(\partial_0+\partial_1)\psi_R+\psi_L^\dagger(\partial_0-\partial_1)\psi_L\right\}\ , \tag{58}$$

which has a higher symmetry compared to the massive case: it is invariant under separate phase changes of the left and right moving spinors, $\psi_L(x,t)\to e^{i\chi_L}\psi_L(x,t)$ and $\psi_R(x,t)\to e^{i\chi_R}\psi_R(x,t)$. This enhanced symmetry is called *chiral symmetry*, with the related conserved currents given by

$$J_R^\mu=\left(\psi_R^\dagger\quad 0\right)\gamma^0\gamma^\mu\begin{pmatrix}\psi_R\\ 0\end{pmatrix}=\left(J_R^0=\psi_R^\dagger\psi_R\ ,\ J_R^1=\psi_R^\dagger\psi_R\right)\ , \tag{59a}$$

$$J_L^\mu=\left(0\quad \psi_L^\dagger\right)\gamma^0\gamma^\mu\begin{pmatrix}0\\ \psi_L\end{pmatrix}=\left(J_L^0=\psi_L^\dagger\psi_L\ ,\ J_L^1=-\psi_L^\dagger\psi_L\right)\ . \tag{59b}$$

This is a $U(1)\times U(1)$ symmetry.

2. $2+1$ Dimensions

In 2+1 dimensions, we can still find a representation of the Dirac-Clifford algebra in terms of Pauli matrices, say

$$\gamma^0=\sigma^2\ ,\quad \gamma^1=i\sigma^3\ ,\quad \gamma^2=i\sigma^1\ . \tag{60}$$

With this choice the Dirac equation is again real, i.e.,

$$\begin{pmatrix}\partial_1+m & -\partial_0+\partial_2\\ \partial_0+\partial_2 & -\partial_1+m\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0\ , \tag{61}$$

and hence the spinor field that solves this equation can still be taken to be either real (Majorana) or complex (Dirac). This equation has the further feature that it is not invariant under parity. A parity transformation can be implemented by reflecting one of the spatial coordinates, say $x^1 \longrightarrow -x^1$. If we take the parity transformation[†] of the spinor as $\psi(x^1, x^2, t) \rightarrow \psi'(-x^1, x^2, t) = \sigma^3\psi(x^1, x^2, t)$, then the transformed field obeys the equation where the mass has opposite sign, i.e.,

$$\left(\frac{1}{i}\gamma^\mu\partial_\mu - m\right)\psi'(x^1, x^2, t) = 0 \; . \tag{62}$$

Thus, only the massless two-component spinor can satisfy the Dirac equation which is parity-invariant.

On the other hand, unlike what happened in $1+1$ dimensions (and in all other even spacetime dimensions), the massless Dirac equation cannot be written as two independent equations and there is no analog of Weyl fermion in $2+1$ dimensions.

Also the only way to make a parity-invariant massive spinor field is to take *two* two-component fields $\psi_{(1)}$ and $\psi_{(2)}$, which satisfy Dirac equations

$$\left(\frac{1}{i}\gamma^\mu\partial_\mu + m\right)\psi_{(1)} = 0 \; , \qquad \left(\frac{1}{i}\gamma^\mu\partial_\mu - m\right)\psi_{(2)} = 0 \tag{63}$$

and have the parity transformations

$$\begin{aligned} \psi_{(1)}(x^1, x^2, t) &\rightarrow \psi'_{(1)}(-x^1, x^2, t) = \sigma^3\psi_{(2)}(x^1, x^2, t) \; , \\ \psi_{(2)}(x^1, x^2, t) &\rightarrow \psi'_{(2)}(-x^1, x^2, t) = \sigma^3\psi_{(1)}(x^1, x^2, t) \; . \end{aligned} \tag{64}$$

We then have a parity-invariant mass term, $-\int d^2x[m\bar{\psi}_{(1)}\psi_{(1)} - m\bar{\psi}_{(2)}\psi_{(2)}]$, in the action. Actually, both fields $\psi_{(1)}$ and $\psi_{(2)}$ could be taken to be Majorana spinors here. This would be fewest possible degrees of freedom of the fermi field which have parity symmetry. Then, can we not introduce a single complex spinor $\tilde{\psi} = \psi_{(1)} + i\psi_{(2)}$ with nonzero mass? Well, that is true, but this complex spinor does not satisfy the Dirac equation (36). This is related to the fact that, while (36) is invariant under the $U(1)$ phase change $\psi(x, t) \rightarrow e^{i\lambda}\psi(x, t)$, the parity-invariant mass term considered above is not invariant under this transformation (or, equivalently, under the $O(2)$ rotation involving $\psi_{(1)}$ and $\psi_{(2)}$). Given a system with a complex spinor (or two Majorana spinors), it is not possible to introduce a mass term which is invariant under both parity transformation and $U(1)$ phase change.

2. $3+1$ Dimensions

[†] Note that $\gamma^1 = i\sigma^3$ anticommutes with $\gamma^0 = \sigma^2$ and $\gamma^2 = i\sigma^1$; there exists no 2×2 matrix which anticommutes with γ^1 but commutes with γ^0 and γ^2.

In $3+1$ dimensions, the minimal representation of the Dirac-Clifford algebra is as 4×4 matrices. For example, one can take

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} , \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} , \gamma^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} . \quad (65)$$

With this choice, the Dirac equation is real, and the spinor can be chosen to be either *Majorana* or *Dirac*. When the mass is zero, with this representation of the γ-matrices, the Dirac equation cannot be presented as two separate equations. However, it can be block-diagonalized by a similarity transformation. The resulting two blocks lead to the *Weyl equations* for two-component spinors ψ_L and ψ_R:

$$\left(\frac{\partial}{\partial t} - \vec{\sigma} \cdot \vec{\nabla}\right) \psi_L = 0 \,, \quad \left(\frac{\partial}{\partial t} + \vec{\sigma} \cdot \vec{\nabla}\right) \psi_R = 0 \,. \quad (66)$$

Note that these equations are complex, so that fields must be taken to be complex; thus, in four spacetime dimensions, there is no Majorana-Weyl spinor.