엔트로피와 정보
Entropy and Information

최무영
서울대학교 물리천문학부

2010. 2. 고등과학원-서울대학교 이론물리 겨울 캠프

- 열기: 물리학이란 무엇인가
- 거시적 기술: 통계역학
- 엔트로피와 못되짚기
- 확률과 정보
- 맺기: 정보와 물리학

물리학이란 무엇인가

자연과학: 자연을 이루는 모든 것의 근본 이치를 과학적 방법으로 추구하며, 이를 통해서 다양한 자연현상에 대한 합리적 설명 및 예측을 가능하게 하는 학문 (자연 → 사회 ➡ 사회과학)

- 물리과학: 물리학, 화학, 천문학,
- 생명과학: 생물학, 의학

지구과학 (지질, 대기, 해양)

수학: 사고의 틀(논리) 자체 연구, 과학의 언어

물리학: 보편지식(이론) 체계 추구, 자연과학의 전형, ~ 물리

단편적 특정지식들을 하나의 합리적 체계 속에서 이해

← 상상력+논리

e.g. 낙하, 지구의 운동, 밀물과 썰물 ← 중력

자연현상의 실체: 물질 (대상계)

구성원
그들 사이의 상호작용 → 모든 현상

물리주의 physicalism

- 물리학의 방법

동역학: 고전역학과 양자역학, 비상대론과 상대론, 입자와 파동(마당)
통계역학: 뭇알갱이계, 거시적 기술 (엔트로피와 정보)

- 물리학의 분야: 물질의 구성 단계와 대상에 따른 분류

입자물리, 원자핵물리, 원자분자물리, 응집물질물리

광학, 플라스마 및 유체물리, 천체물리, 생물물리, 화학물리, …

통계물리 (방법: 통계역학)

물리법칙의 대칭성: e.g. 뉴턴의 운동 법칙 $a = F/m$

대칭성 변환에 대한 불변성

나란히 옮김, 돌림, 시간 진행

전하켤레, 홀짝성, 시간 되짚기

맞바꿈

게이지

대칭성 ↔ 보존

일상 경험 (거시세계): 시간 되짚기 비대칭

못되짚기irreversibility → 시간의 화살 (기억·예측)

엔트로피 및 열역학 둘째 법칙

물질 (뭇알갱이계): 대칭성이 절로 깨질 수 있음

spontaneous symmetry breaking

→ 정돈(질서) order

질서와 무질서

- 자유도가 적은 계: 동역학(*dynamics*)

  질서: 천체의 운동

  무질서 (혼돈chaos): 주사위, 빵 반죽, 플라스마, 날씨

  초기조건에 민감 → 예측 불가능, 결정론 붕괴

- 자유도가 많은 계 (뭇알갱이계): 통계역학(*statistical mechanics*)

  무질서: 마구잡이(random)   기체

  질서: 대칭성 깨짐   고체

  집단성질의 떠오름(emergence) (← 구성원들의 협동성)

  미시적 세부사항과 무관: 보편성(universality)

  상전이, 생명, 사회 현상 (촛불시위)

  부분의 합 ≠ 전체 → (인식론적) 환원주의 붕괴

  cf. 존재론적 환원주의

물과 얼음: H_2O 분자들의 집단

T

물 (대칭성 있음: 나란히움김, 돌림) 무질서 엔트로피

T_c ← 상전이 (대칭성 깨짐)

얼음 (대칭성이 깨져 있음) 질서 에너지

0

많은 구성원(원자, 분자)들 사이의 협동현상 → 떠오르는 성질

액체-고체, 자석, 초전도, 초기 우주, 기억 작용, DNA 풀어짐, 세포 분화, 피의 산소운반, 효소 작용, 여론 형성, 지각 작용, 도시 형성, 경기 변동과 공황, …

거시적 기술: 통계역학

Matter in our daily life (incl. biological systems)

macroscopic, many constituents *many-particle system*

e.g., air in this classroom ($N \sim 10^{25}$ molecules)

microscopic description: dynamics (classical or quantum)

(micro) state $\{q_i, p_i\}$ $6N$ (micro) variables

Can't specify
in practice!
in principle!

macroscopic description: statistical mechanics

(macro) state $\{p, T, \ldots\}$ a few macro variables

macro variables: collective degrees of freedom

external parameters + (internal) energy

Social system: individual states vs societal variables

(area, living level, technology, organization,…)

에너지, 일, 열 Energy, Work and Heat

What are these?

Energy levels E_n depends on external parameters $\{y_\alpha\}$

(mean) energy $E = \sum_n p_n E_n$ $\quad p_n$: prob. for (micro) state n

Change of energy E

via change of $\{y_\alpha\}$ (i.e., of E_n) : work done $\quad W \equiv -\Delta_y E$

via change of p_n: heat absorbed $\quad Q \equiv \Delta E$

Energy transfer bet. two (macro) systems: work + heat

$$\Delta E \equiv Q - W$$ (heat absorbed – work done) by the system

엔트로피 Entropy

To a given macro state $(E,\{y_\alpha\})$

⟵ many micro states correspond e.g. 윷놀이

accessible states

number of accessible states $\Omega(E,\{y_\alpha\})$

$\Omega > 1$ ⟶ missing information "entropy"

probability for the system in (macro) state $(E,\{y_\alpha\})$

$p(E,\{y_\alpha\}) \propto \Omega(E,\{y_\alpha\})$ postulate of equal a priori probability

macro state i (Ω_i small) $\underset{\nleftarrow}{\rightarrow}$ macro state f (Ω_f large)

irreversibility

e.g. 강의실 안의 공기: 앞에만 있는 상태 vs 고르게 퍼진 상태

$$\frac{p_i}{p_f} = \frac{\Omega_i}{\Omega_f} = \frac{(V/2)^N}{V^N} = 2^{-N} \sim 2^{-10^{25}} \approx 0$$

entropy $S \equiv k \log \Omega$ (Boltzmann)   function of (macro) state

irreversibility:

initial state $\rightarrow$ equilibrium state (S maximum)

i.e., $S \rightarrow \max$ or $\Delta S \geq 0$ isolated system

heat $đQ$ absorbed via a quasi-static process:

$$dS = đQ / T$$ (can be negative)

Clausius' definition, but S?, holonomy, T?, very limited

temperature $\frac{1}{T} \equiv \frac{\partial S}{\partial E}$ $\Longrightarrow$ energy $E = E(T, \{y_\alpha\})$

1st law of thermodynamics

$$Q = \Delta E + W$$ definition of heat

$\Rightarrow$ energy conservation

infinitesimal change: $\bar{d}Q = TdS = dE + \sum_{\alpha} X_{\alpha} dy_{\alpha}$

$$X_{\alpha} \equiv -\frac{\partial E}{\partial y_{\alpha}}$$ generalized force

2nd law of thermodynamics

$$\Delta S \geq 0$$ Spontaneous change in an isolated system is non-decreasing.

How can life survive the 2^{nd} law?

***Biological Question*:**

How can living organisms be so tightly ordered?

***Physical Idea*:**

The flow of energy/information can leave behind increased order.

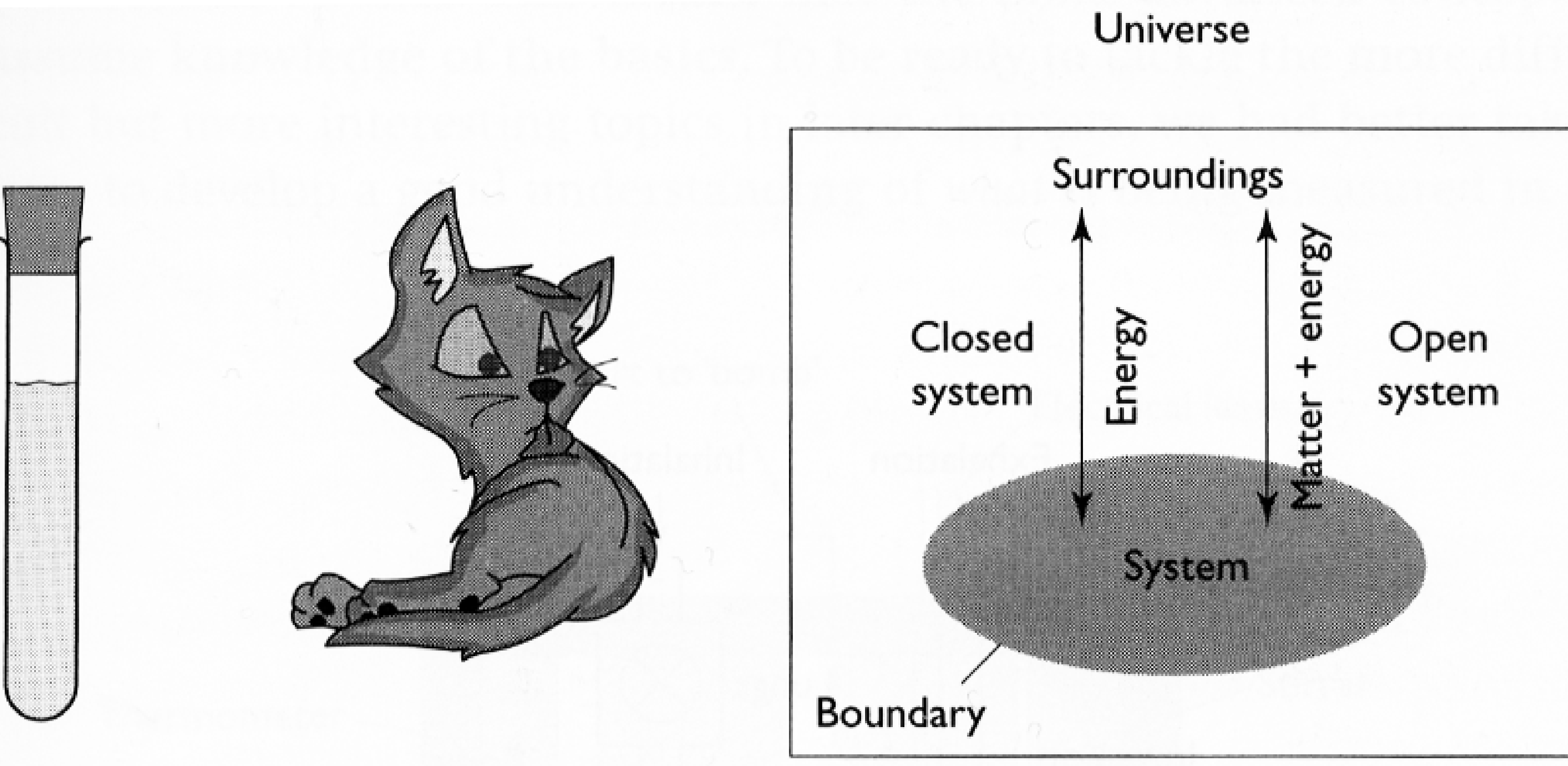
A biological system is a complex (many-particle) system, displaying life as cooperative phenomena.

⟹ breakdown of reductionism and determinism

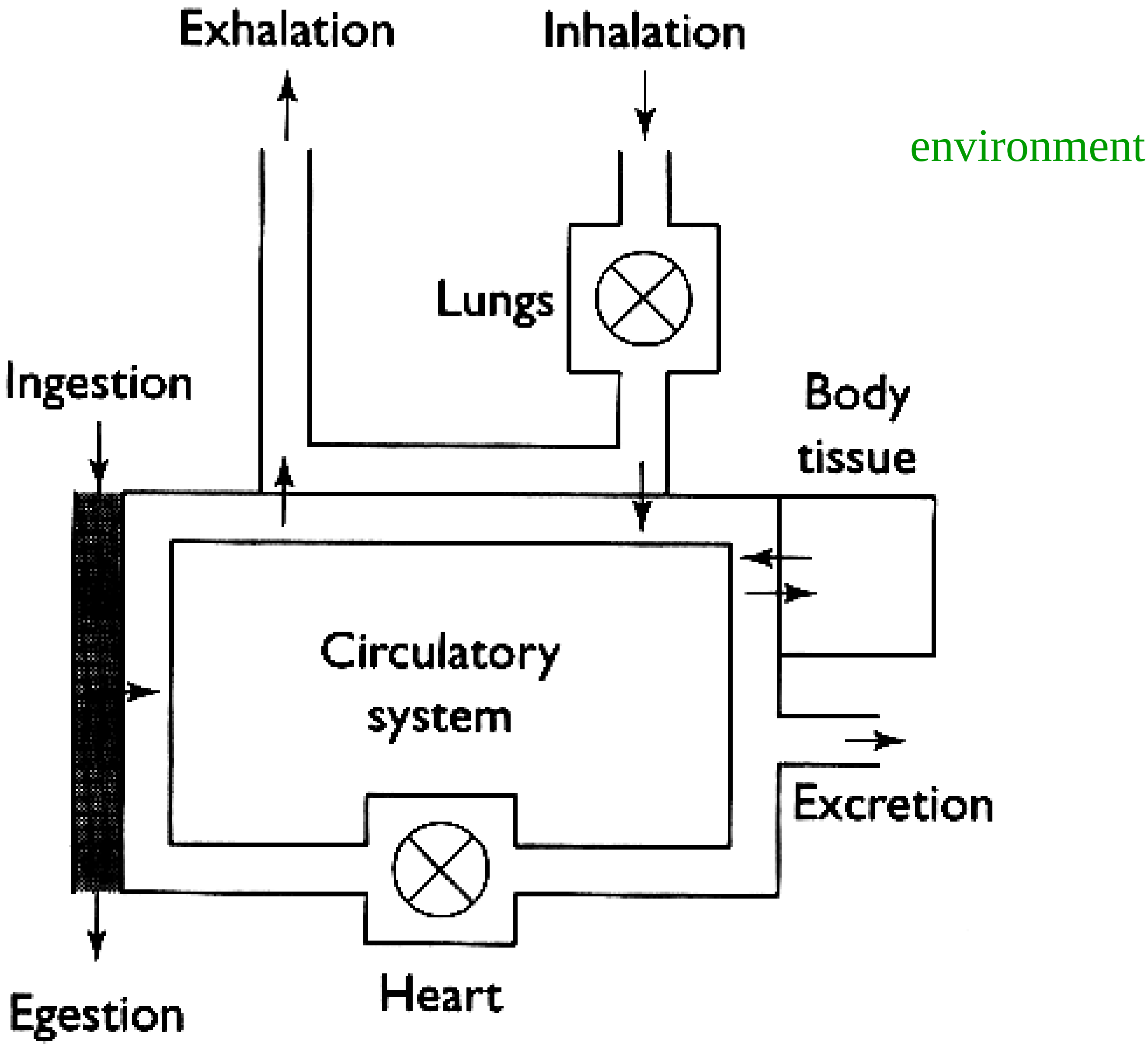
statistical mechanics
nonlinear dynamics; computational physics

System and Environment

not isolated → closed or open



Life as an open system



Thermodynamic potentials

single-component fluid: $\{y_\alpha\} = V; \quad X_\alpha = p$

$E(S, V) \quad (\Leftarrow dE = TdS - pdV)$

$F(T, V) = E - TS \quad (\Rightarrow dF = -SdT - pdV)$

$H(S, p) = E + pV \quad (\Rightarrow dH = TdS + Vdp)$

$G(T, p) = F + pV = E - TS + pV \quad (\Rightarrow dG = -SdT + Vdp)$

2nd law of thermodynamics

- isolated system: $S \rightarrow \max$
- system in contact with a heat reservoir: $F \rightarrow \min$
- system in contact with a heat reservoir at constant pressure:

  $G \rightarrow \min$

Energy flow, leaving behind increased order

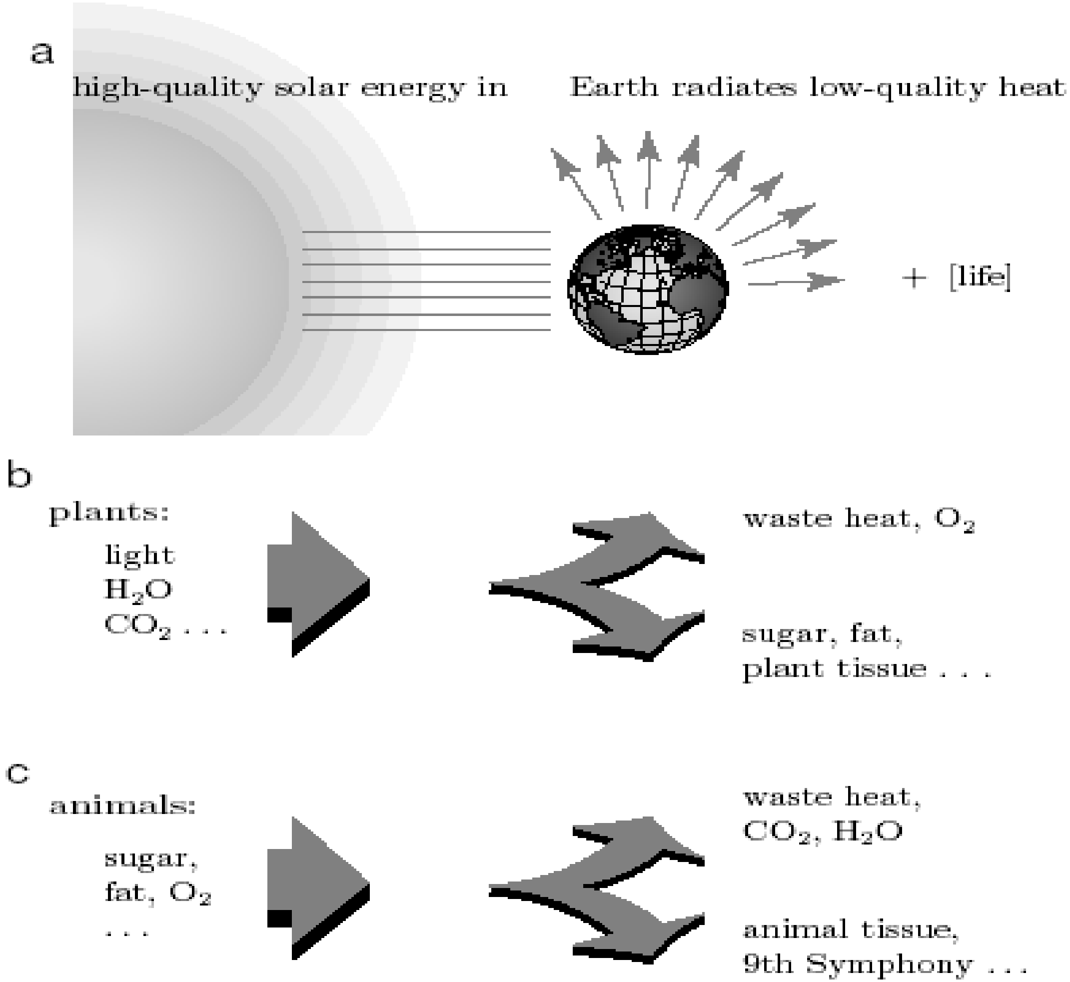


Figure 1.2: (Diagram.) (a) Energy budget of Earth's biosphere. Most of the incident high-quality energy is degraded to thermal energy and radiated into space, but some gets captured and used to create the order we see in life. (b) What plants do with energy: High-quality solar energy is partly used to upgrade low-energy molecules to high-energy molecules and the ordered structures they form; the rest is released in thermal form. (c) What animals do with energy: The high-quality energy in food molecules is partly used to do mechanical work and create ordered structures; the rest is released in thermal form.

Osmotic machine

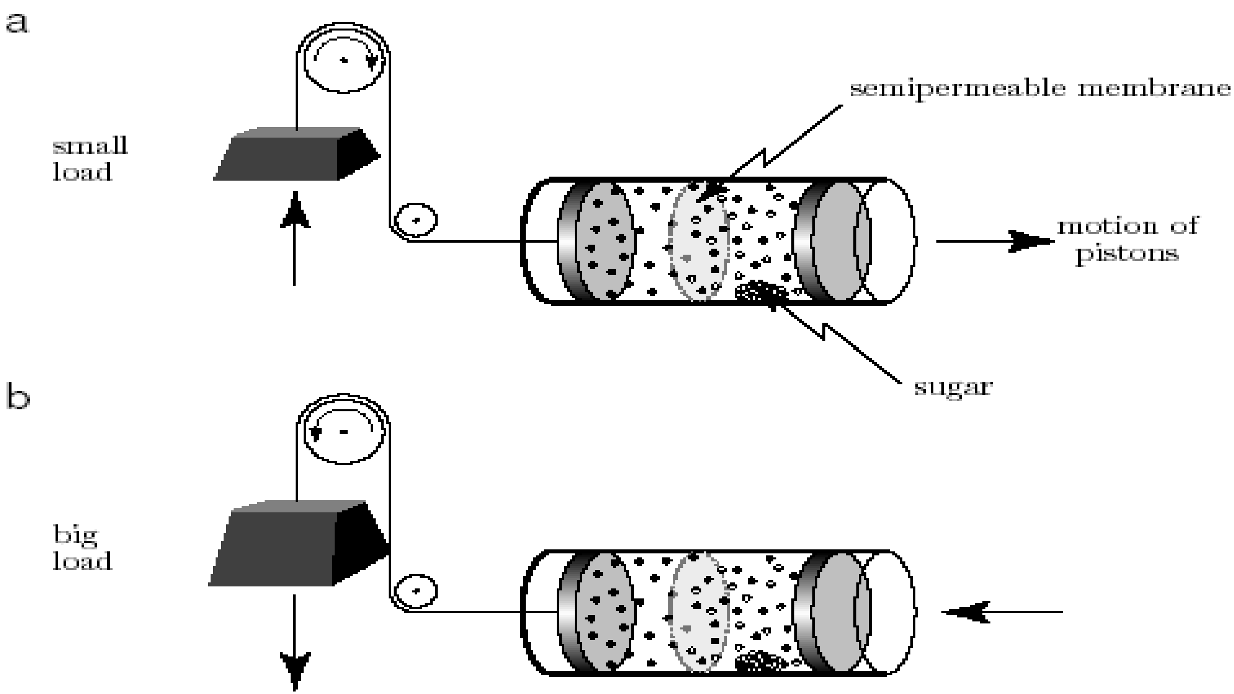


Figure 1.3: (Schematic.) A machine transducing free energy. A cylinder filled with water is separated into two chambers by a semipermeable membrane. The membrane is anchored to the cylinder. Two pistons slide freely, thus allowing the volumes of the two chambers to change as water molecules (*solid dots*) cross the membrane. The distance between the pistons stays fixed, however, because the water between them is incompressible. Sugar molecules (*open circles*) remain confined to the right-hand chamber. (a) Osmotic flow: As long as the weight is not too heavy, when we release the pistons, water crosses the membrane, thereby forcing both pistons to the right and lifting the weight. The sugar molecules then spread out into the increased volume of water on the right. (b) Reverse osmosis: If we pull hard enough, however, the pistons will move to the *left*, thereby increasing the concentration of the sugar solution in the right-hand chamber and generating heat.

osmotic flow

$Q>0 \rightarrow W>0$

heat →work   2nd law?

$\Delta V>0 \rightarrow \Delta S>0$

in price of molecular order

⟹ $\Delta F = \Delta E - T\Delta S < 0$

reverse osmosis

work →heat $(Q,W<0)$

$\Delta S<0$ increasing order!

e.g. water purification

엔트로피와 못되짚기

Entropy: measure of

- multiplicity (i.e., # of accessible states)
- # of arrangements
- Uncertainty
- randomness (disorder)
- homogeneity
- tendency toward spontaneous change
- energy dispersal
- degradation in usable energy (i,.e., unavailability of the energy to perform work)
- missing information
- lack of knowledge
- ability to store information

Missing information

Ω equally probable possibilities (a dice hidden in one of Ω boxes)

missing information (lack of info): $S(\Omega)$

1. $S(\Omega_1) < S(\Omega_2)$ if $\Omega_1 < \Omega_2$
2. $S(1) = 0$
3. $S(\Omega_1 \Omega_2) = S(\Omega_1) + S(\Omega_2)$

   e.g. Ω_1 groups, each consisting of Ω_2 boxes

   $\rightarrow$ First choose bet Ω_1 groups and choose bet Ω_2 boxes

   cf. If $S(\Omega_1 \Omega_2) = S(\Omega_1)S(\Omega_2)$, then $S(\Omega) = S(\Omega)S(1) = 0$!
4. Extend the def. for all real $\Omega \geq 1$, $S(\Omega)$: continuous

$\Rightarrow$ $S(\Omega)$ determined uniquely (up to constant)

$$S(\Omega) = k \log_a \Omega$$ entropy

$$k \equiv S(a)$$ units of info

$$\begin{cases} \text{initially, no info: } I = 0 \text{ and } S = k \log \Omega \\ \text{after observation, no missing info: } S = 0 \end{cases}$$

$\Longrightarrow$ info gain (amount of info obtained) $\Delta I = k \log \Omega$

Information and Entropy (missing information)

$$I = -S + I_0$$

probability assignment $\{p_i\}$

p_i: prob. for the sys. in state i $(1 \leq i \leq \Omega)$, $\Sigma_i p_i = 1$

ensemble of N (>> 1) such systems, broken into Ω groups (accrd. to states)

N_i systems in group i (i.e., in state i): $N_i = Np_i$

total # of distinct ordering: $\Omega_N = \dfrac{N!}{N_1! N_2! \cdots} = \dfrac{N!}{\prod_{i=1}^{\Omega} N_i!}$

missing info per system $(N \to \infty)$

$$S = \frac{1}{N} k \log \Omega_N = \frac{k}{N}\left(\log N! - \sum_i \log N_i! \right) = -k \sum_i p_i \log p_i$$

Information (theoretic) entropy

Given prob. measure $p(x)$ for $x \in X = \{x_1, x_2, ..., x_\Omega\}$

$$S(X) \equiv -k \sum_x p(x) \log p(x) \quad \text{(Shannon)} \quad \textit{missing information}$$

complete info: $p(x) = \delta_{x,x_1} \Longrightarrow S = 0$ (min)

least info: $p(x) = 1/\Omega$ (equally likely) $\Longrightarrow S = k \ln \Omega$ (max)

$k \equiv 1$

Various forms

$$\begin{cases} x \in X, & \text{prob. measure } p(x) \text{ and } q(x) \\ y \in Y, & \text{prob. measure } p(y) \end{cases}$$

joint prob. and conditional prob.

$$p(x, y) = p(x|y)\, p(y)$$

joint entropy

$$S(X,Y) \equiv -\sum_{x,y} p(x,y) \log p(x,y)$$

X, Y: indep., i.e., $p(x, y) = p(x)\, p(y)$ $\rightarrow$ $S(X,Y) = S(X) + S(Y)$

conditional entropy

$$S(X|Y) \equiv -\sum_{y} p(y) \sum_{x} p(x|y) \log p(x|y) = -\sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(y)}$$

$$= S(X,Y) - S(Y)$$

relative entropy (Kullback-Leibler divergence)

$$S(p\|q) \equiv -\sum_{x} p(x) \log q(x) - S(p) = \sum_{x} p(x) \log \frac{p(x)}{q(x)}$$

cross entropy

$$S(p,q) \equiv S(p) + S(p\|q) = -\sum_{x} p(x) \log q(x)$$

Continuous variable

discrete probability $p(x)$ vs continuos probability density $f(x)$

$$p(x) = f(x)dx \quad \text{with some partitioning } dx \quad \sum_x p(x) = \int dx\, f(x) = 1$$

information entropy

$$S(p) \equiv -\sum_x p(x) \log p(x) = -\sum_x f(x)dx \log f(x)dx$$

$$= -\sum_x dx\, f(x) \log f(x) - \log dx$$

$$\to S[f] + C_I \geq 0$$

differential entropy $S[f] \equiv -\int dx\, f(x) \log f(x) \gtrless 0$

information capacity $C_I \equiv -\log dx \xrightarrow[dx \to 0]{} \infty$

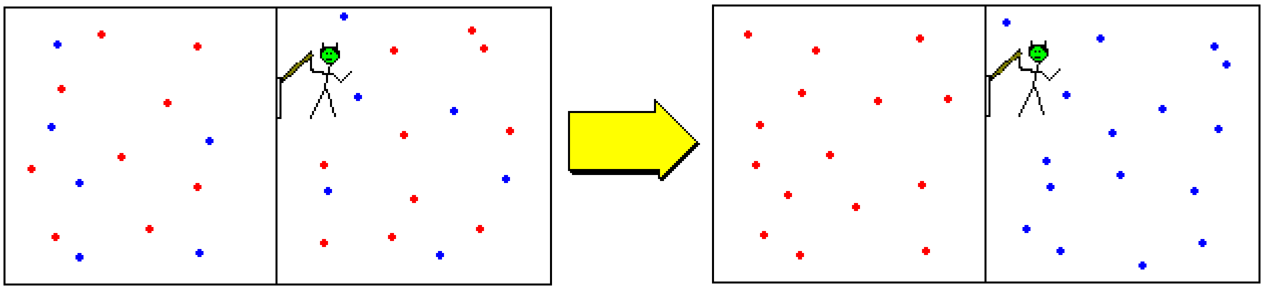
Almost all real numbers has infinite digits, thus we need infinite amount of information: incomputable!

Relative entropy, free from divergence, corresponds to info.

$$S[f \| g] = \int dx\, f(x) \log \frac{f(x)}{g(x)}$$

Maxwell's demon

1. Birth: Maxwell (1867)



빠른 (뜨거운) 분자 느린 (차가운) 분자

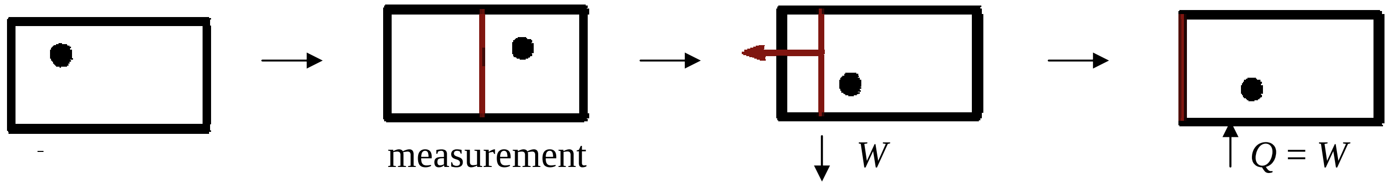
violation of the 2nd law!

intelligent being necessary to work against thermal fluctuations

Smoluchowski

2. Exorcism: Szilard (1929) and Brillouin (1949)

Szilard's model: intelligent being operating a heat engine



System returns to the initial state: $\Delta S_{sys} = 0$

Heat Q flows from the environment: $\Delta S_{env} < 0$

$\Rightarrow$ $\Delta S_{tot} < 0$??

However, measurement is required to determine in which part the molecule is.

$\rightarrow$ entropy production $k \log 2$ ($\leftarrow$ two states of memory)

information (observer) and entropy (system) tied!

Measurement: information acquisition

$\rightarrow$ dissipative, increasing entropy

2^{nd} law OK!

3. Resurrection: Landauer (1961) and Bennett (1973, 1982)

Landauer: in computing process

most operations possible with arbitrarily little dissipation

erasure of info: irreversible $\Longrightarrow$ 2^{nd} law OK

Bennett: reversible computing automaton

arbitrarily little dissipation possible

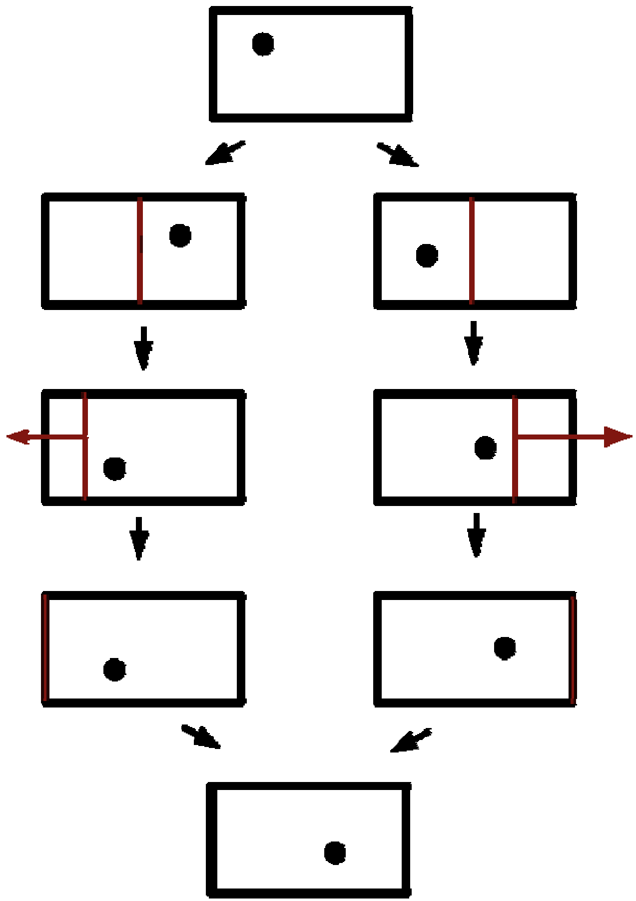
erasure of demon's memory $\Longrightarrow$ $\Delta S_{tot} = 0$, 2^{nd} law OK

It is not measurement but erasure that saves the 2^{nd} law!

Is a reversible computer realizable?

Is memory erasure always essential?

Szilard's engine



	system	demon	environment
	$k \log 2$	0	S_0
measur.	dec.	inc.	
	0	$k \log 2$	S_0
expans.	inc.		$\leftarrow Q$ dec.
	$k \log 2$	$k \log 2$	S_0 - $k \log 2$
erasure		dec. $\rightarrow Q$	inc.
	$k \log 2$	0	S_0

Measurement establishes correlations bet. states of particles and demon's mind. Reversible measurement possible.

Nature of entropy

lack of knowledge, measure of our ignorance:

→ purely subjective? "epistemic view"

An ice cube melts when I became ignorant of it?

This is absurd!

macroscopic variables

epistemic?: which we happen to be willing and able to observe

physical?: own dynamics, physical consequences

e.g. amount of work performed by a heat engine

both objective and subjective

If purely objective, only one level of description

(cf. Boltzmann, hydrodyn., thermodyn., etc.)

a set of macro states ↔ a partitioning of the phase space

"(coarse grained) prob. space"

확률과 정보

probability assignment ⟺ amount of information

확률 Probability

1. Classical definition

principle of equal a priori probabilities, $p(x) = 1/\Omega$ "belief"

e.g., coin flip, dice, microcanonical ensemble

2. Frequency interpretation

$$p(x) = \lim_{N\to\infty} \frac{n(x)}{N}$$

But $N < \infty$ in reality! cf. ergodicity

3. Bayesian probability: measure of the state of knowledge

$\left\{ \begin{array}{l} \text{objective: rationality and consistency, extension of logic} \\ \text{subjective: personal belief} \end{array} \right.$ non-informative prior → classical def.

e.g. maximum entropy method (Jaynes), Bayesian inference

정보 Information

Essential constituents of the world (natural and artificial)

Why not seriously considered so far?

1. unnecessary for description of natural phenomena

   IT: communication (transmission), computer (process), …

   BT: genetic info (DNA)

   QT: quantum info

2. vague and obscure, lacking precise definition

   energy: abstract and no intrinsic def., but physical “reality”

3. regarded as obvious (like space and time)

   cf. kinematic quantities (e.g., position and velocity) = info

   interpretation in terms of energy (e.g., heat and potential)

   human: inherently info

정보의 속성

구분가능성 distinguishability

- 최소단위 bit ← 예/아니오

물질 vs 정신 matter-mind or mind-body

- 물질? 정보기술(IT)
- What is matter? Never mind!
- 정신? 같은 정보가 사람마다 다른 의미 (주관성)
- What is mind? No matter!
- 물질과 정신의 연결점

나르개(carrier; 물질)에 담겨 있으나 그를 제외한 무엇 (추상적)

- 나르개 바꾸기: coding

기호와 의미

- 의미와 관계없이 기호 자체: 정보기술(IT)
- 메시지의 의미: 정보환경(prob. space)이 전제되어야 의미 결정

Example

정보의 의미

한글을 아는 사람: 의미 이해
서체도 분별 가능할 수도

한글을 모르는 사람: 알 수 없는 기호일 뿐
2차원의 함수로 해석할 수도

Difference in interpretation ← difference in the prob. space in the brain

You see as much as you know!

자연현상과 해석

자연현상
- 실체: 물질 (시공간, 질량, 전기량, 운동량, 에너지, …)
- 해석: 정보 (~엔트로피)

- 대상 자체의 성격
- 대상에 대한 정보가 전해 질 수 있는 정도

정보의 획득(측정) 및 제거 과정 cf. Maxwell's demon

Everything is particles. 동역학 (입자모형)

Everything is fields. 마당이론 (파동모형 ← 양자역학)

energy

Everything is information. 통계역학

entropy

J.A. Wheeler

과학의 목표는 사물 자체가 아니라 그들 사이의 관계 (실체) H. Poincaré

물리학은 존재론이 아니라 인식론을 연구 N. Bohr

정보: 관계의 소통, 세계에 대한 앎은 물질이 아니라 정보

Information

$X = \{x\}$ with prob. measure $p(x)$

Information entropy

$$S(X) \equiv -\sum_x p(x) \log p(x) = -\sum_x p(x)[I(x) - I_0] = -\bar{I}(X) + I_0$$

self-information $I(x) \equiv \log p(x) + I_0$

(average) self-information or information content of X

$$\bar{I}(X) \equiv \sum_x p(x) I(x) \equiv I$$

Simplest case: $p(x) = 1/\Omega$ with $\Omega = 2$

e.g. prob. for an unknown digit X to be 0 and to be 1 are the same, ½.

$$S = -\sum_x p(x) \log p(x) = \log \Omega = \log 2 \; (= 1 \text{ for base } 2)$$

When the digit is determined, say 1, the entropy becomes zero. The entropy is reduced by 1 bit and we gain 1 bit of information.

Classical information

- Reducible to bits   e.g., 스무고개
- Can be copied without disturbing it
- Can be erased
- Cannot travel faster than c or backward in time (causality)

Quantum information

- Reducible to qubits
- Cannot be read or copied without disturbance
- Entanglement (non-separable)

정보와 물리학

- 물리학: 보편지식 (이론) 체계 추구

  "이론과학"

- 20c 물리학: 기본 원리
  - 환원주의와 결정론
  - 간단한 현상 (제한적, 예외적)

- 21c 물리학: 자연의 해석
  - 전체론과 예측불가능 *new paradigm*
  - 복잡한 현상, 복잡계 (다양, 근원적)

➔ 복잡계의 물리 통합과학?

e.g., biological physics, econophysics, sociophysics, …

핵심역할: 엔트로피와 정보

어록

Everything is information.

J.A. Wheeler

The law that entropy always increases - the second law of thermodynamics - holds the supreme position among the laws of Nature.

A. Eddington

Nobody knows what entropy really is.

J. von Neumann

The future belongs to those who can manipulate entropy; those who understand but energy will be only accountants.

F. Keffer