Bose-Einstein condensation

Lecturer: Yong-il Shin (SNU)

2010 KIAS-SNU Physics camp

Outline

1. What is Bose-Einstein condensation?
2. BEC in ultracold atomic gases
3. Phase coherence of BEC
4. Superfluidity and BEC
5. BEC in an optical lattice

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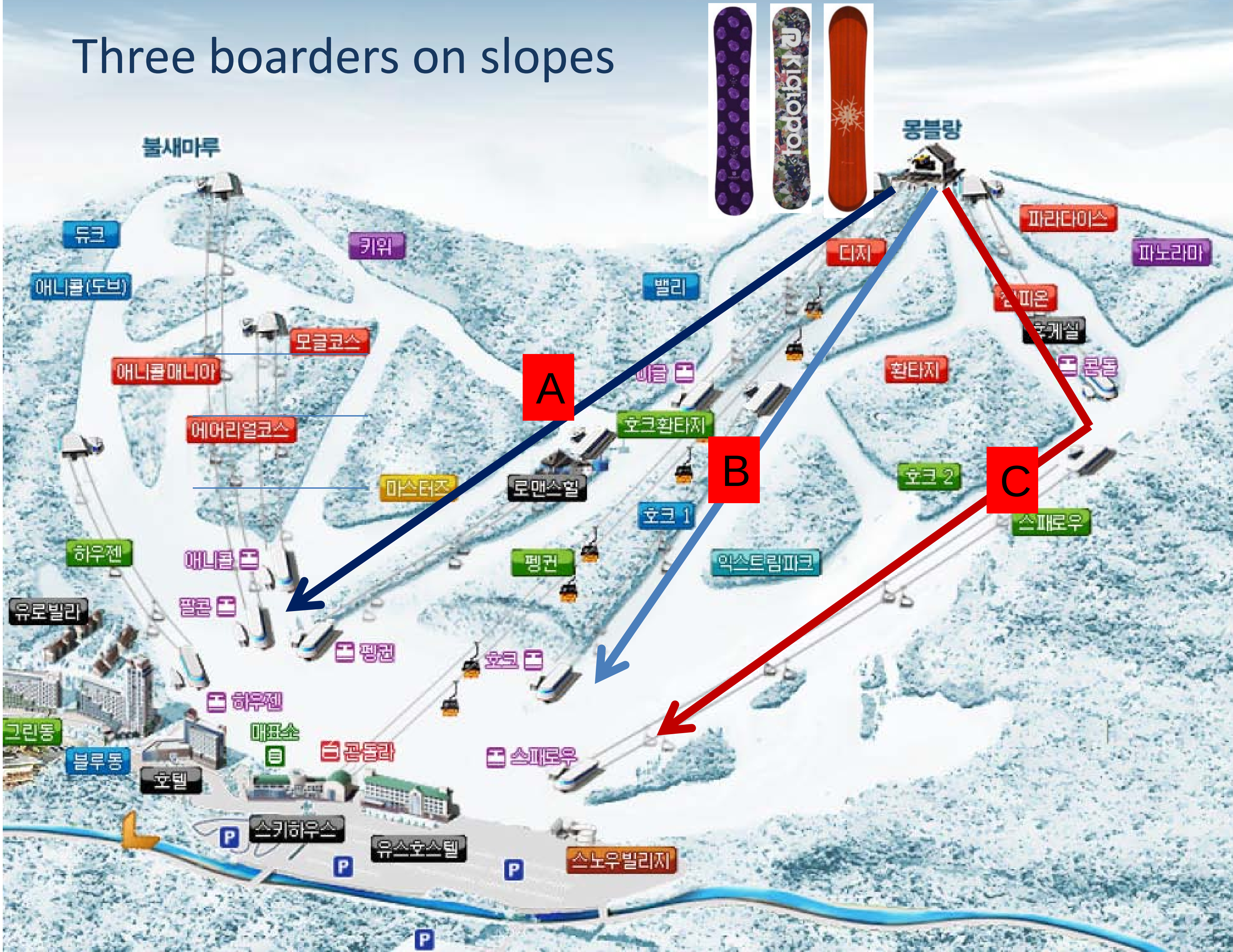
Three boarders on slopes



Classical counting

Total number of cases: 27

Probability of having all of them in the same slope: 1/9

Quantum counting

Here, we can't distinguish them.
They all look same.

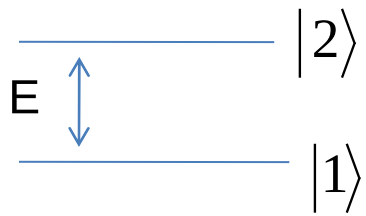
Total number of cases: 10

Probability of having all of them in the same slope: **3/10**

Indistinguishability makes them more likely to be together.

Saturation of occupation

Example)   N-particles in a two-level system



$$p = e^{-E/k_B T}$$

Classical counting

2^N

$$N_2 = N\frac{p}{1+p}$$

Quantum counting

due to the indistinguishability of particles

N+1

$$N_2 = \frac{p}{1-p} - (N+1)\frac{p^{N+1}}{1-p^{N+1}}$$

In a thermodynamic limit $N \rightarrow \infty$,
the occupation number of the excited state is saturated

Bose statistics

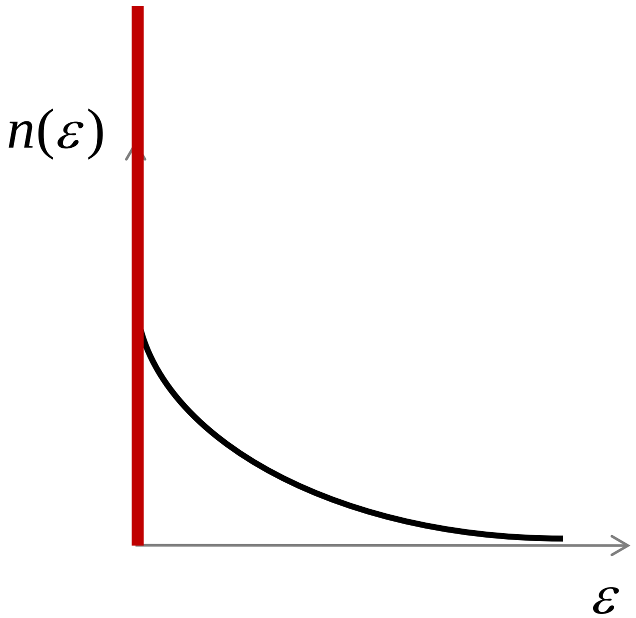
Occupation number of a state with energy ε_i

$$n_i = \frac{1}{e^{(\epsilon_i - \mu)/kT} - 1}$$

Total number of particles

Density of states

$$N = \int d\epsilon \frac{g(\epsilon)}{e^{(\epsilon - \mu)/kT} - 1} \leq N_C(T)$$



When $N > N_C$ *, the remaining particles are put into the ground state with* $\mu = \varepsilon_0$ *.*

Bose-Einstein condensate: Macroscopic occupation of a single quantum state

Criterion of Bose-Einstein condensation

σ : density matrix of a given many-body state of N-bosons

$\sigma_1 \equiv N \ \mathrm{tr}_{2 \dots N}(\sigma),$: single-particle density matrix

➡ $\{\psi_i\}, \{n_i\}$: corresponding eigenfunctions and values

Macroscopic occupation of a single quantum state

$$\exists n_0 \sim O(N) \Longleftrightarrow \mathrm{BEC}$$

ψ_0 : wavefunction of a condensate

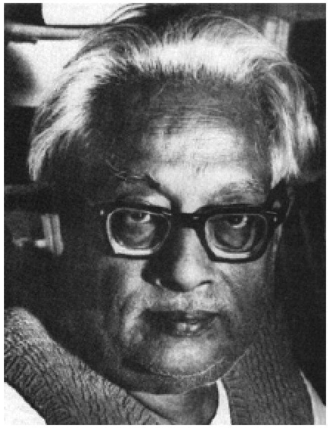
Penrose & Onsager (1956)

$$\lim_{|x-y|\to\infty} \langle \hat{\psi}^\dagger(x) \hat{\psi}(y) \rangle = f^*(x) f(y) \neq 0,$$

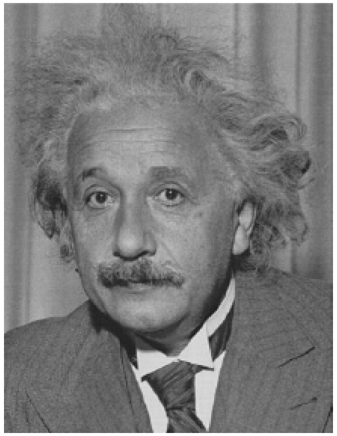
ψ_0, is one of possible solution for $f(x)$

Yang (1962): *off-diagonal long-range order* (ODLRO)

Birth of the BEC idea (1920's)



S. N. Bose



A. Einstein

- Bose derived Plank distribution of Black-body radiation with a new photon counting way, but failed to publish his results.

- Einstein immediately agreed with Bose, and they described the indistinguishability of photons and Bose-Einstein statistics.

- Einstein extended this idea to include systems with a conserved particle number, adopting de Broglie's new idea of matter waves.

- Einstein pointed a peculiar feature of the distribution: at low temperature it saturates.

- Schrodinger first heard about de Broglie's idea from reading Einstein's paper and later he developed his wave equation.

de Broglie

E. Schrodinger

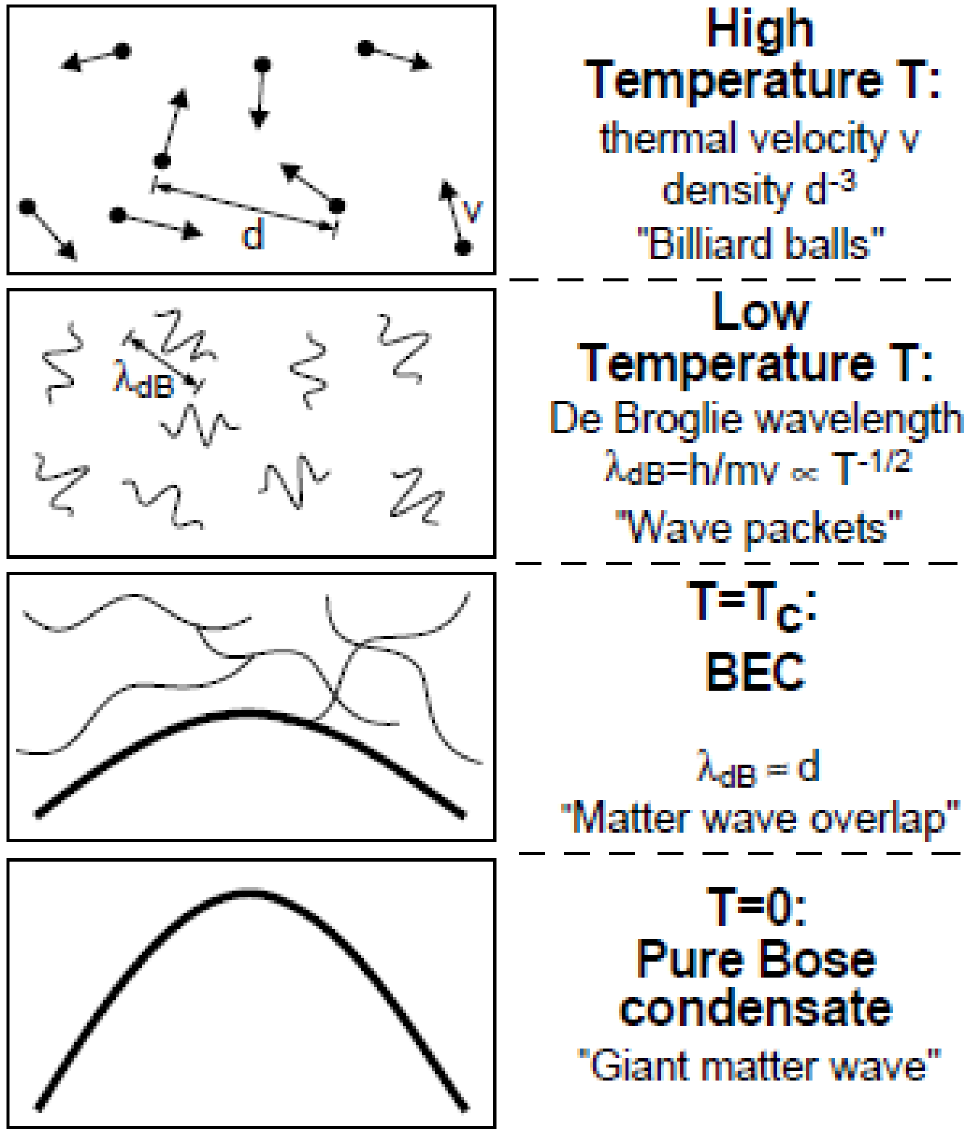
Matter wave picture of BEC

de Broglie's wavelength

$$\lambda_{dB} = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}$$

Critical condition of BEC

$$n\lambda_{dB}^{\ 3} \cong 2.612$$



BEC systems

- Superfluid Helium
- Lasers and Masers (macroscopic occupation in the same state)
- Superconductors
- Ultracold atomic gases

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Bose-Einstein Condensation of Paraexcitons in Stressed Cu_2O

Jia Ling Lin and J. P. Wolfe

Physics Department and Materials Research Laboratory, University of Illino

(Received 19 March 1993)

oexciton ground state in C
nic gas produced
m develops an ex
excitons.

NATURE | VOL 418 | 15 AUGUST 2002 | www.nature.com/nature

Macroscopically ordered state in an exciton system

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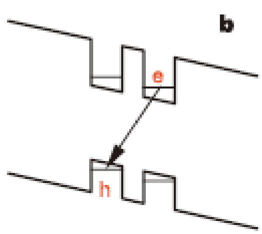

Vol 443|28 September 2006|doi:10.1038/nature05131 nature

ARTICLES

Bose–Einstein condensation of exciton polaritons

J. Kasprzak[1], M. Richard[2], S. Kundermann[2], A. Baas[2], P. Jeambrun[2], J. M. J. Keeling[3], F. M. Marchetti[4], M. H. Szymańska[5], R. André[1], J. L. Staehli[2], V. Savona[2], P. B. Littlewood[4], B. Deveaud[2] & Le Si Dang[1]

ndensation (BEC), superfluidity, and
um effects to a macroscopic scale. BEC has,
oms at temperatures below 200 nanokelvin.
can take place. Promising candidate systems
y coupled to electronic excitations, leading to
es lighter than rubidium atoms, thus
ere we detail a comprehensive set of
al density, we observe massive occupation
t 19 K, an increase of temporal coherence,
of which indicate the spontaneous onset of a

Bose–Einstein condensation in magnetic insulators

nature physics | VOL 4 | MARCH 2008 | www.nature.com/naturephysics

The Bose–Einstein condensate (BEC) is a fascinating state of matter predicted to occur for particles obeying Bose statistics. Although the BEC has been observed with bosonic atoms in liquid helium and cold gases, the concept is much more general. We here review analogous states, where excitations in magnetic insulators form the BEC. In antiferromagnets, elementary excitations are magnons, quasiparticles with integer spin and Bose statistics. In certain experiments their density can be controlled by an applied magnetic field leading to the formation of a BEC. Furthermore, interactions between the excitations and the interplay with the crystalline lattice produce very rich physics compared with the canonical BEC. Studies of magnon condensation in a growing number of magnetic materials thus provide a unique window into an exciting world of quantum phase transitions and exotic quantum states, with striking parallels to phenomena studied in ultracold atomic gases in optical lattices.

Outline

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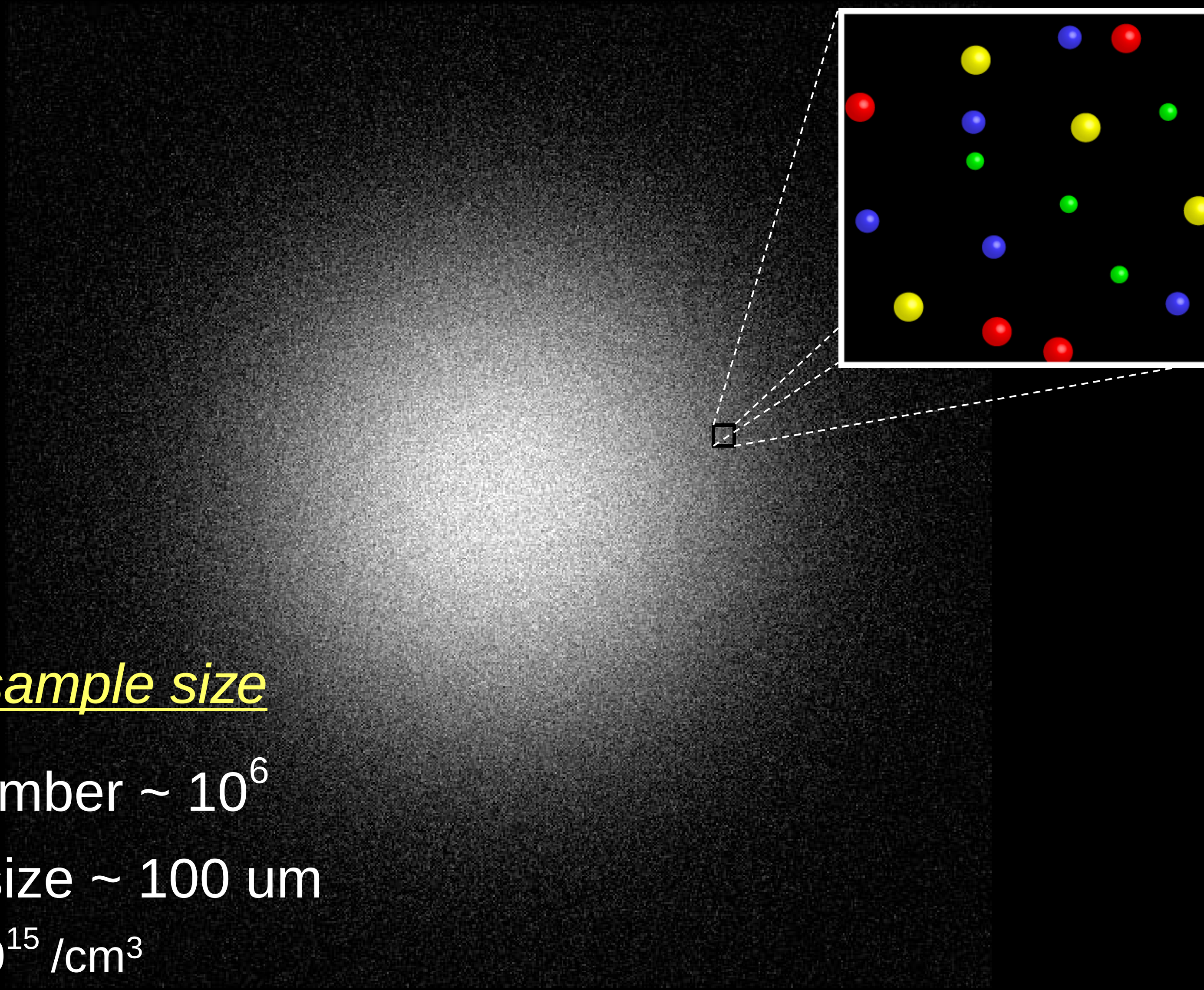
Ultracold Atom Cloud



Typical sample size

Atom number ~ 10^6

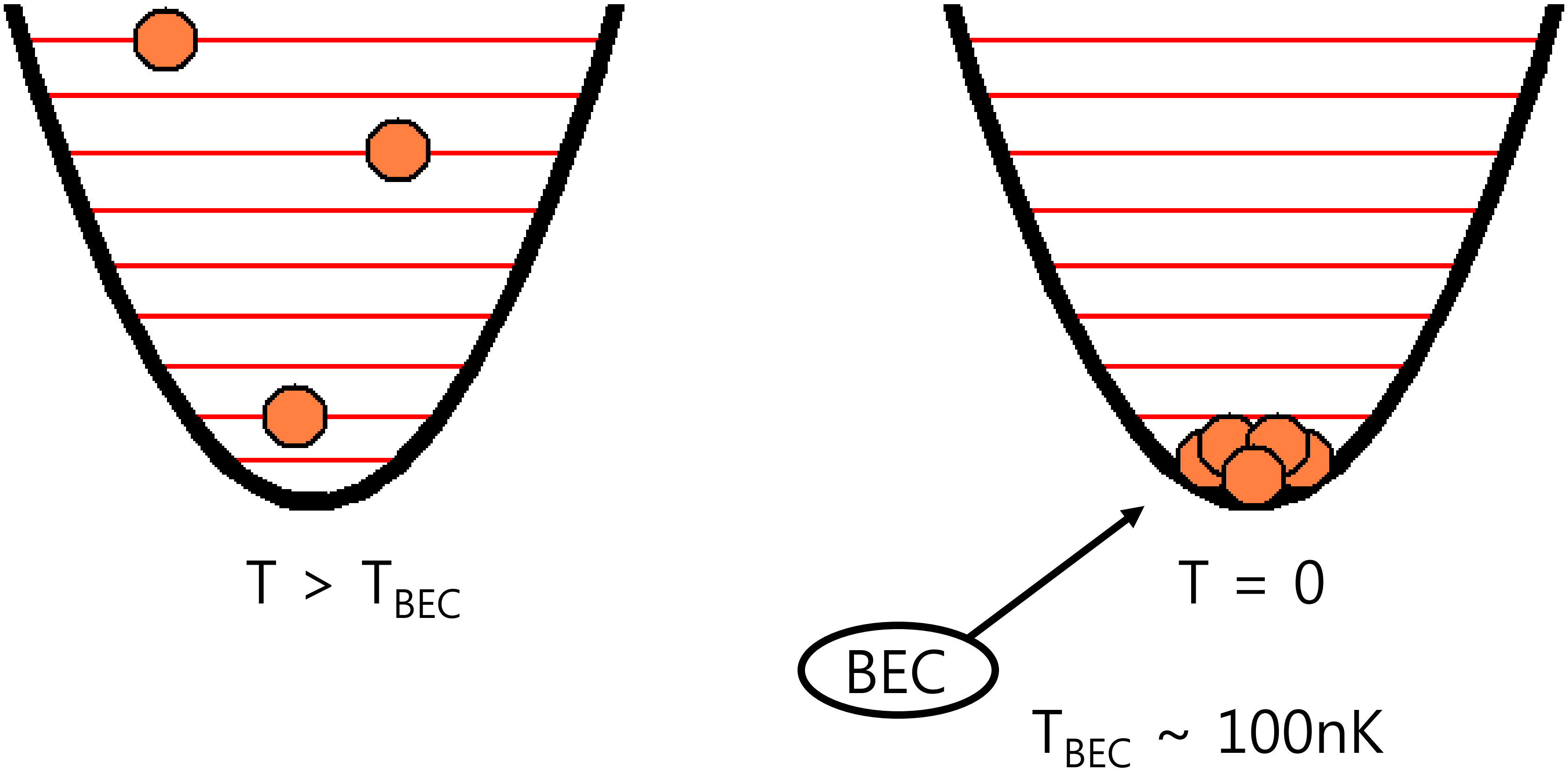
Spatial size ~ 100 um

$n=10^{11} \sim 10^{15}$ /cm^3

T= 100 nK (~ 1 Hz)

Bose-Einstein condensate (BEC)

A Bose-Einstein condensate is the macroscopic occupation of the ground state of a system.

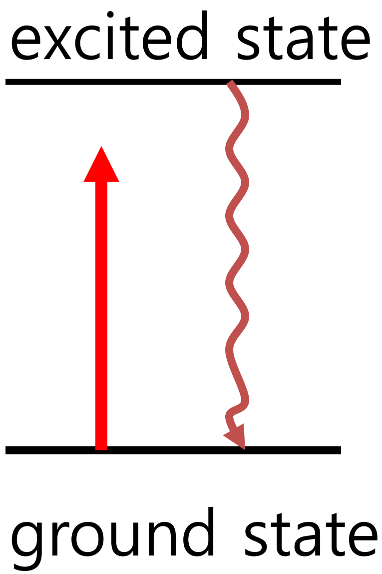


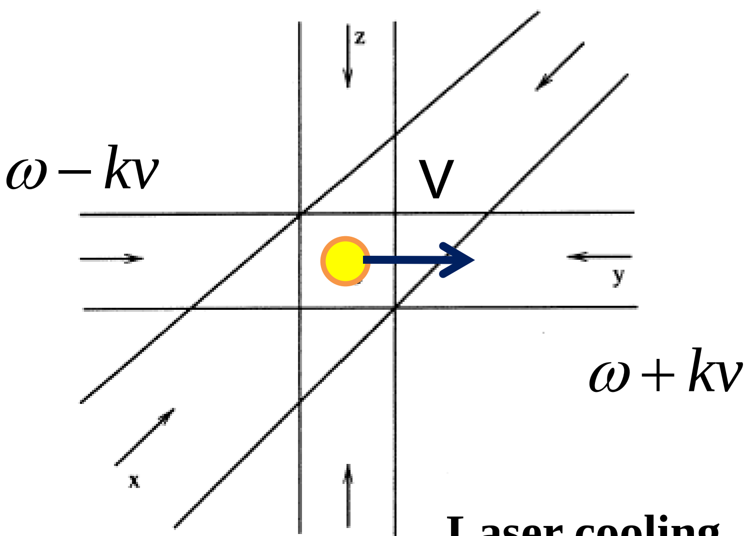
How to cool down atoms ?

Laser cooling

- photon has momentum
- atom absorbs and emits photons

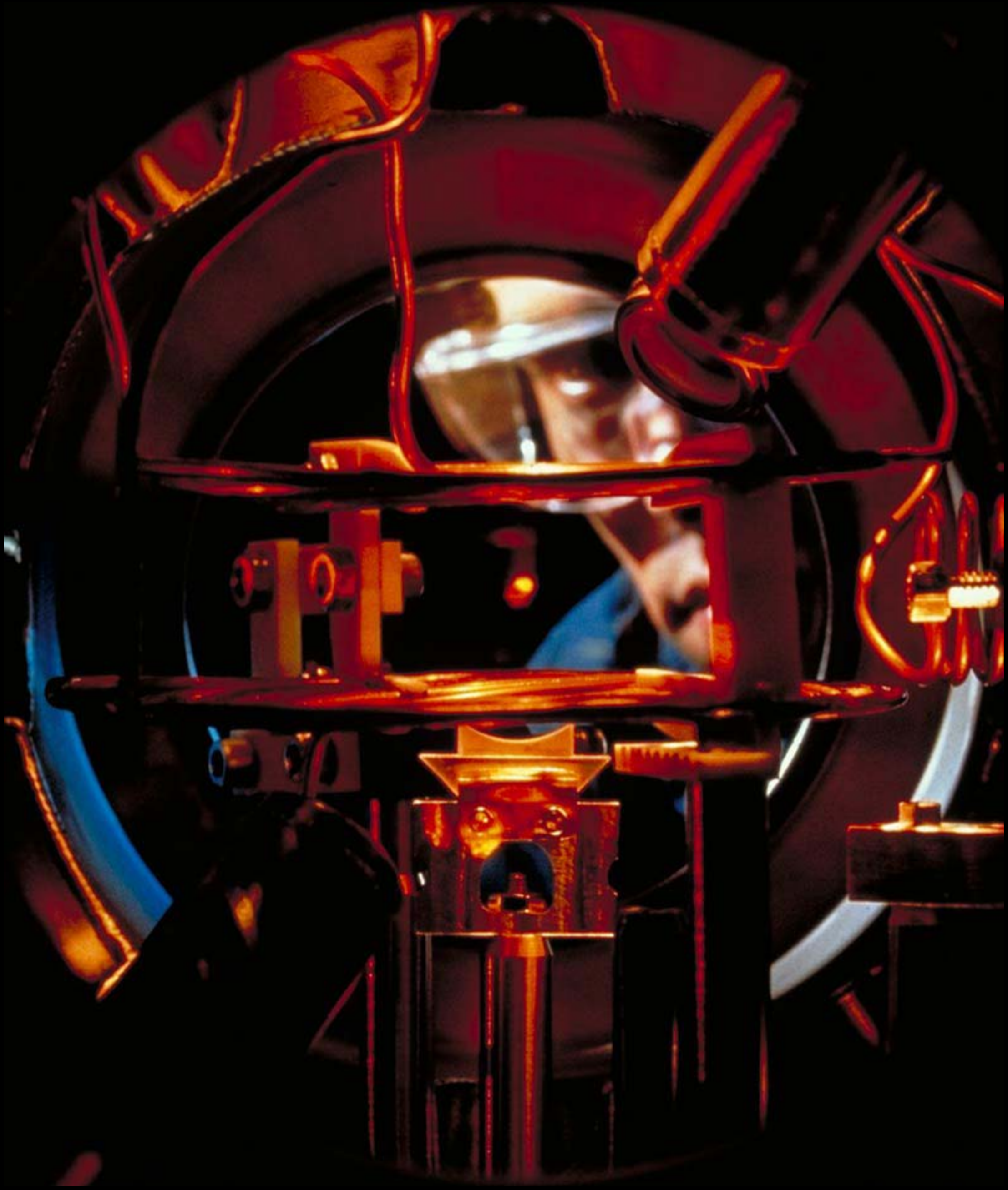
- Doppler effect : Optical molasses





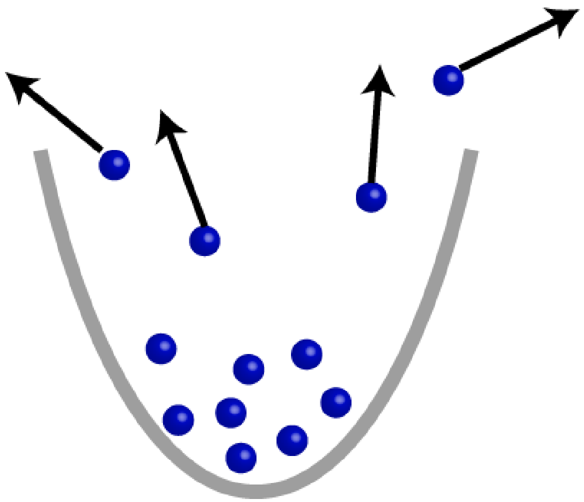
Laser cooling
(1997 Nobel prize)

T ~ 100 μK

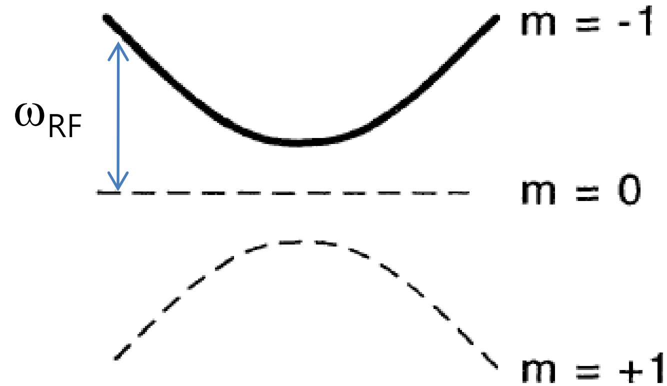


How to cool down atoms ?

Evaporative cooling



Method 1: transitions to untrapable states

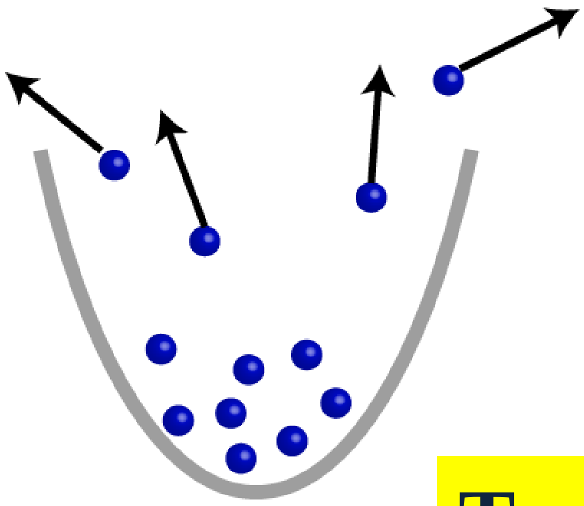


Method 2: Reducing the trap depth

Removing the tail of thermal distribution leads to lower average energy, i.e., cooling the sample.

How to cool down atoms ?

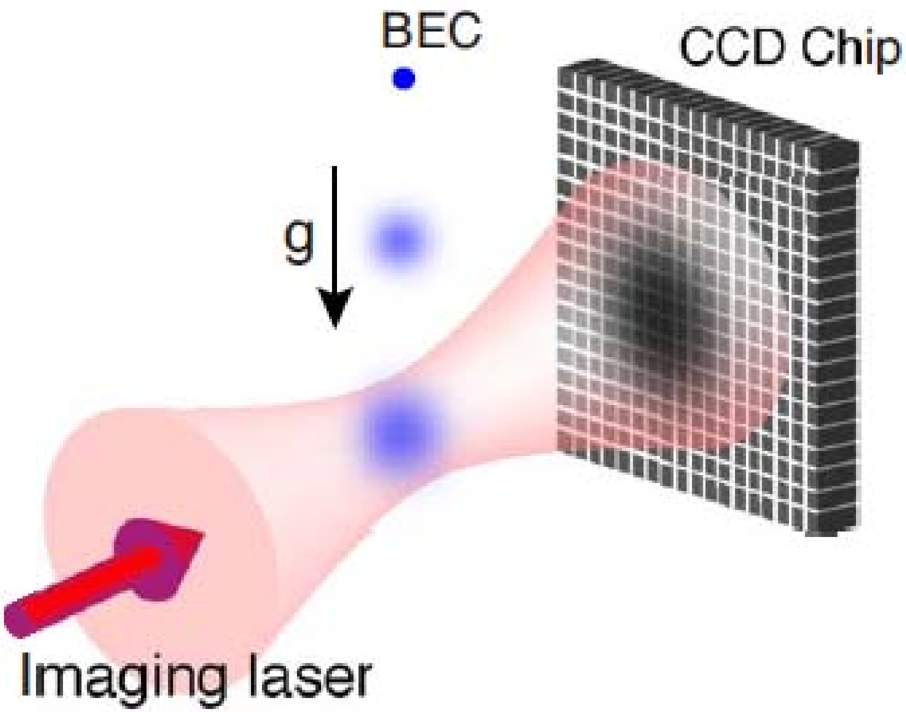
Evaporative cooling



T ~ 100 nK

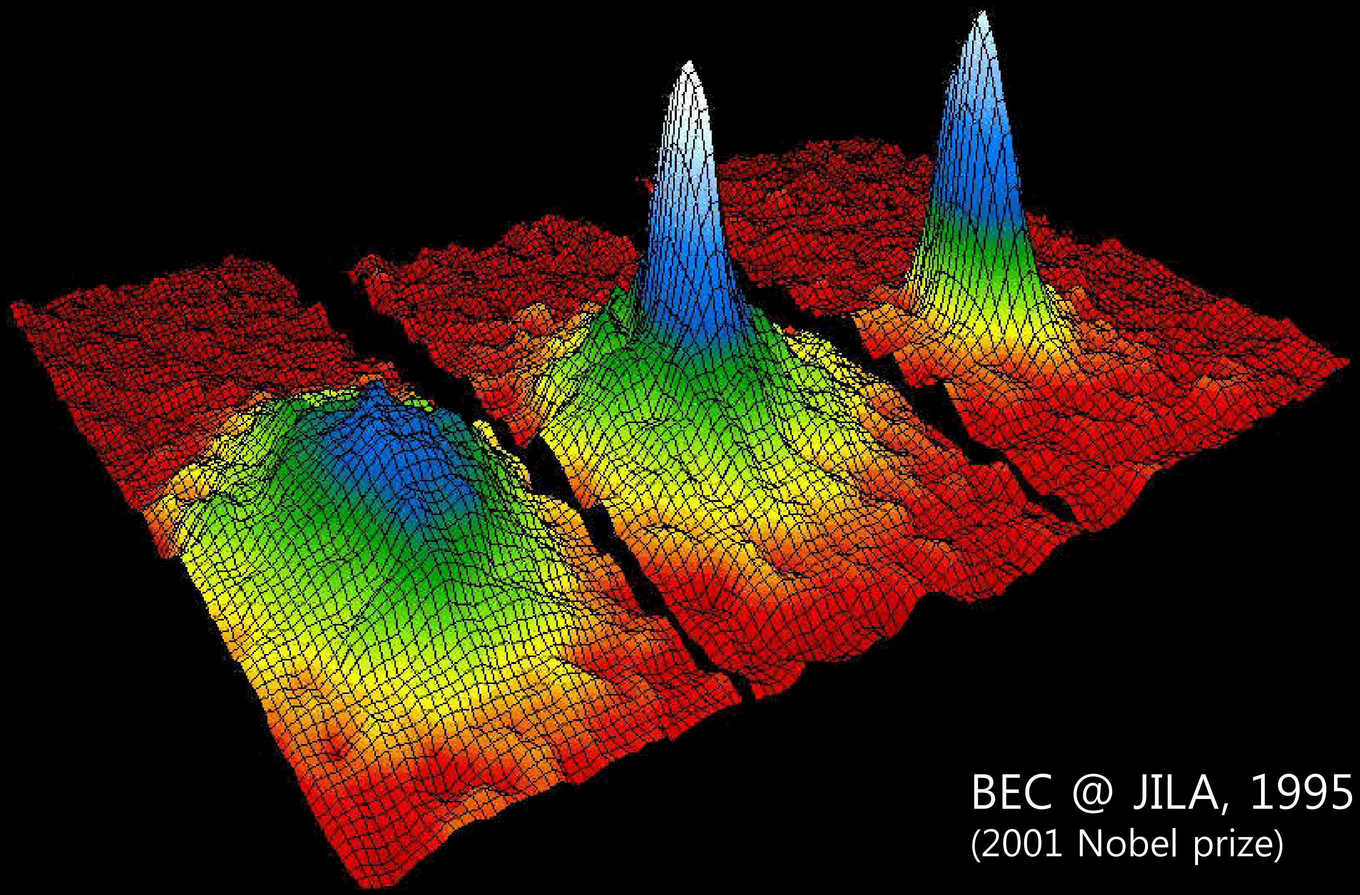
Removing the tail of thermal distribution leads to lower average energy, i.e., cooling the sample.

Time-of-flight Imaging

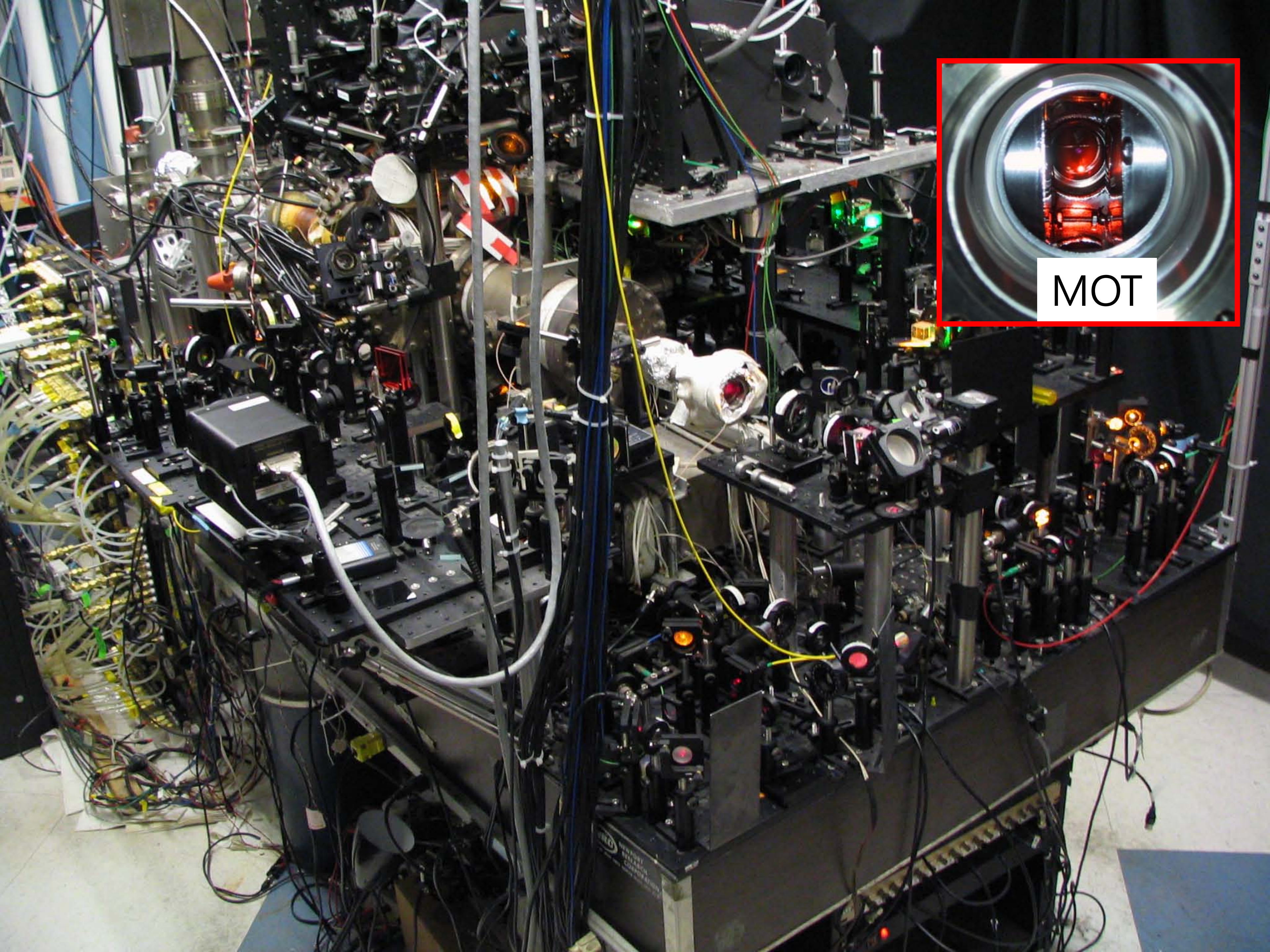


The expanded cloud reveals the momentum distribution of the sample.

Bose-Einstein condensation in a dilute gas



BEC @ JILA, 1995
(2001 Nobel prize)


MOT

Many-body Hamiltonian in cold atom gases

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + V_{ext}(r_i)\right] + \sum_{i<j} V(r_i - r_j)$$

Model system

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + V_{ext}(r_i)\right] + U_0 \sum_{i<j} \delta(r_i - r_j)$$

Scattering problem

For a given Interparticle potential $V(r)$

$$\psi = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

$f(\theta)$: Scattering amplitude

Partial wave description

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\, f_l(k) P_l(\cos\theta)$$

$$f_l(k) = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{1}{k\cot\delta_l - ik}$$

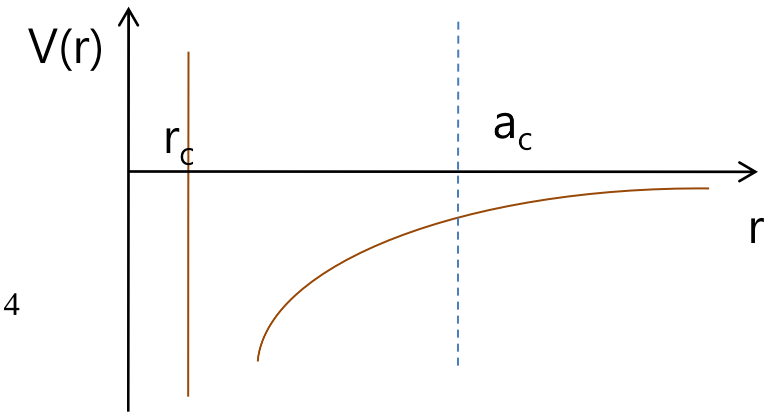
Phase shift

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Cold atom collisions

van der Waals attraction

$$V(r) = \begin{cases} -C_6/r^6 & \text{if } r > r_c \\ \infty & \text{if } r \leqslant r_c, \end{cases}$$



Characteristic length $a_c = \left(\frac{2m_r C_6}{\hbar^2} \right)^{1/4}$

For non-zero *l*, $V_{eff}(r) = V(r) + \frac{h^2 l(l+1)}{2mr^2}$

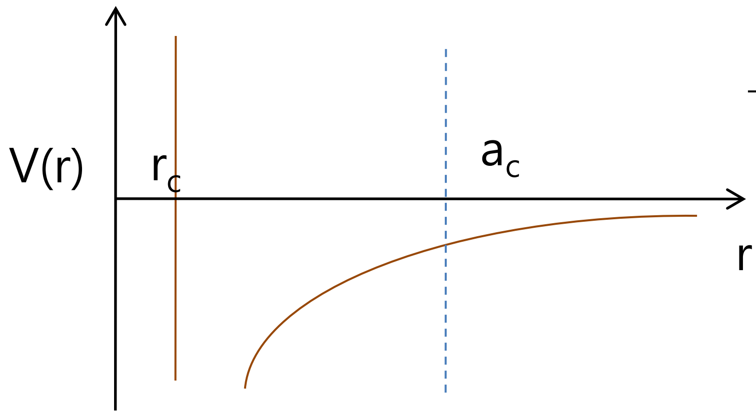
Centrifugal barrier $E_c \approx \frac{\hbar^2 l^2}{m_r a_c^2} \approx 1\,\mathrm{mK}$

For gases in the sub-milikelvin regime, only s-wave collisions are relevant.

$$f(k) = \frac{1}{k \cot \delta_0(k) - ik} \Rightarrow f(k) = -a/(1 + ika)$$

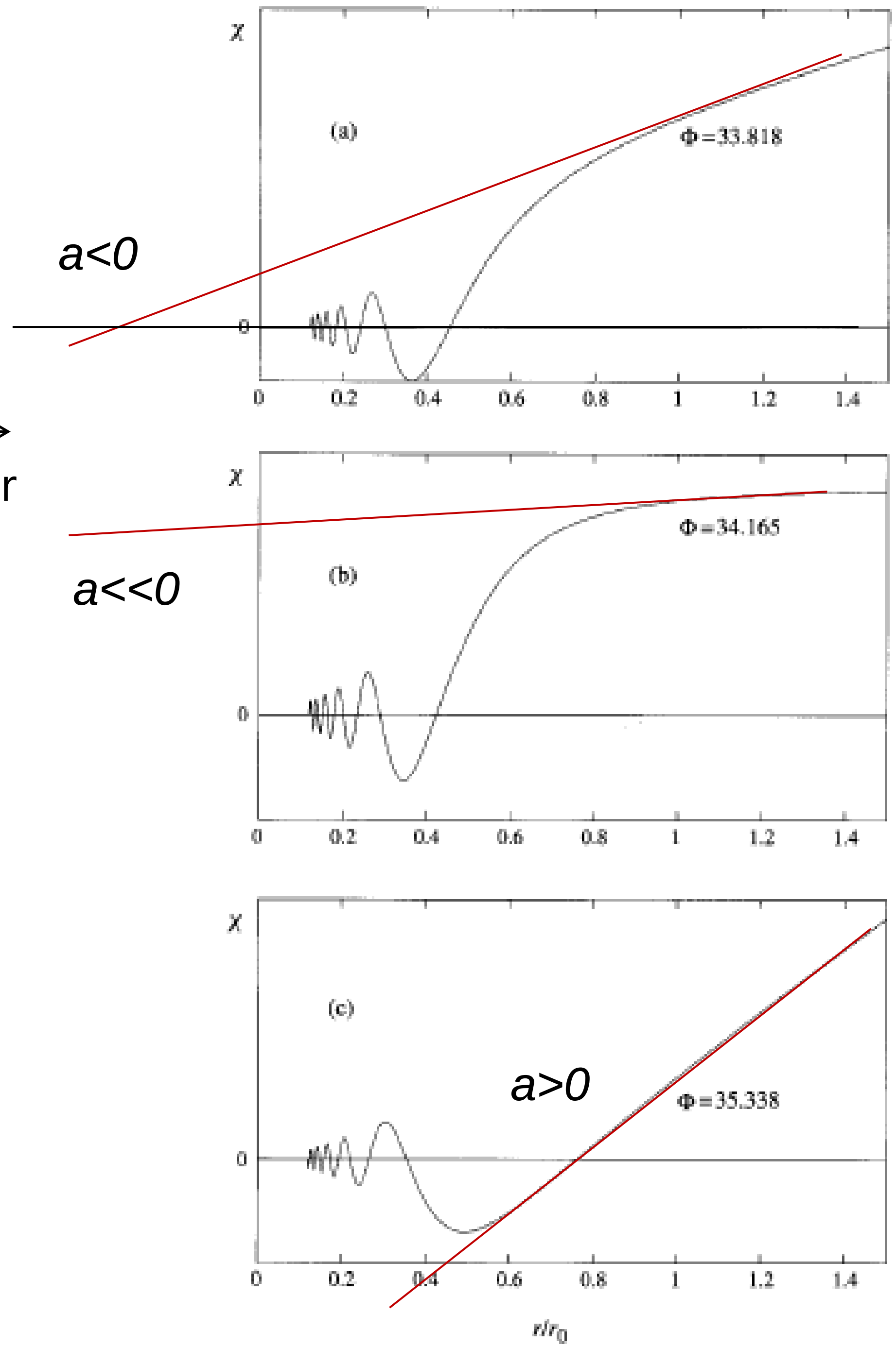
s-wave scattering length

Physical meaning of scattering length

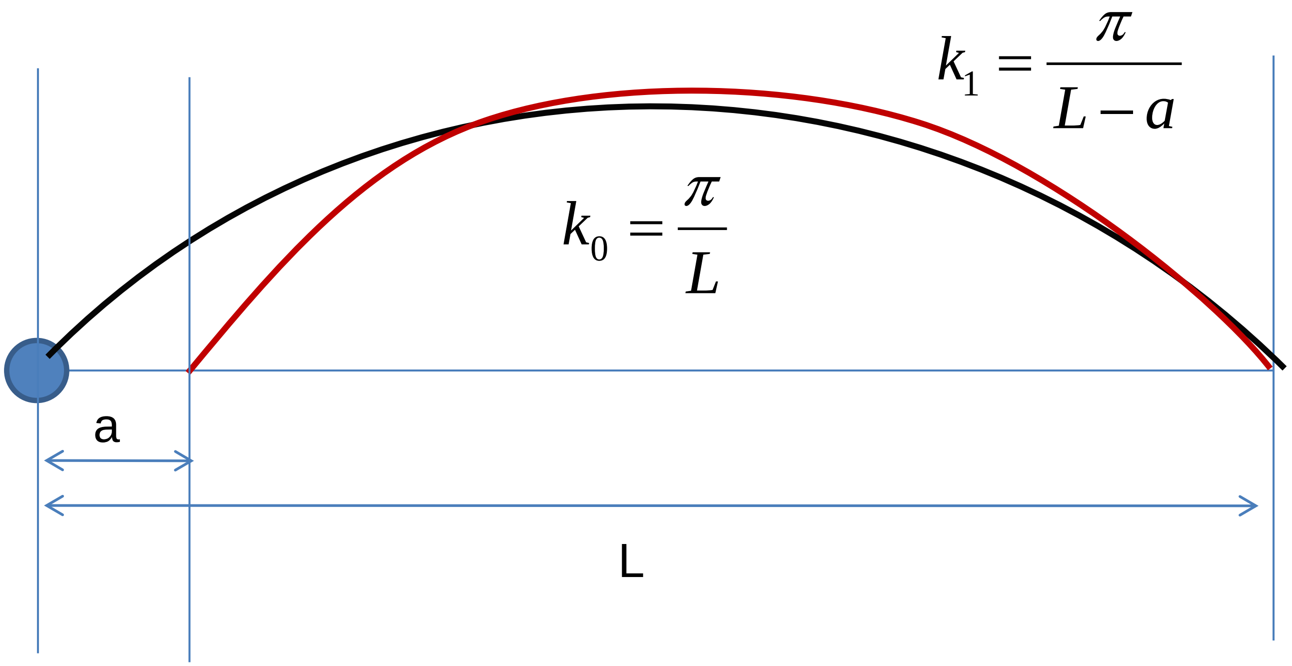


At $r >> a_c$

$$\psi_{out} \cong \frac{e^{i\delta_0}\sin(kr+\delta_0)}{kr}$$



Sign of scattering length and energy shift



Corresponding energy shift?

$$\delta E = \frac{\hbar^2 k_1{}^2}{2m} - \frac{\hbar^2 k_0{}^2}{2m} \simeq \frac{\hbar^2}{2m}\left(\frac{2\pi a}{L^3}\right) \sim \frac{\hbar^2 n a}{m}$$

Positive *a* : repulsive
Negative *a* : attractive

Effective potential

In the regime of ultracold collision, $ka_c \ll 1$
The two-body collision is completely specified by a single parameter, *a*

$$f(k) = -a/(1+ika) \Rightarrow V(\mathbf{x})(\cdots) = \frac{4\pi\hbar^2 a}{2M_r}\delta(\mathbf{x})\frac{\partial}{\partial r}(r\cdots)$$

Effective pseudopotential

Realizing the toy-model Hamiltonian,

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + V_{ext}(r_i)\right] + \frac{4\pi\hbar^2 a}{m}\sum_{i<j}\delta(r_i - r_j)$$

Mean-field description of a dilute Bose gas

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + V_{ext}(r_i) \right] + U_0 \sum_{i<j} \delta(r_i - r_j)$$

A simplest approximation for many-body states
➔ a product of a single-particle wavefunction:

$$\Psi_{\mathrm{GP}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \prod_{i=1}^{N} \phi_1(\mathbf{x}_i)$$

Gross-Pitaevskii (GP) equation

$$i\hbar \frac{\partial}{\partial t} \Phi(r,t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{ext}(r) + U_0 |\Phi(r,t)|^2 \right] \Phi(r,t)$$

Wave function of condensate

Outline

1. What is Bose-Einstein condensation?
2. Ultracold atomic gases
3. Phase coherence of BEC
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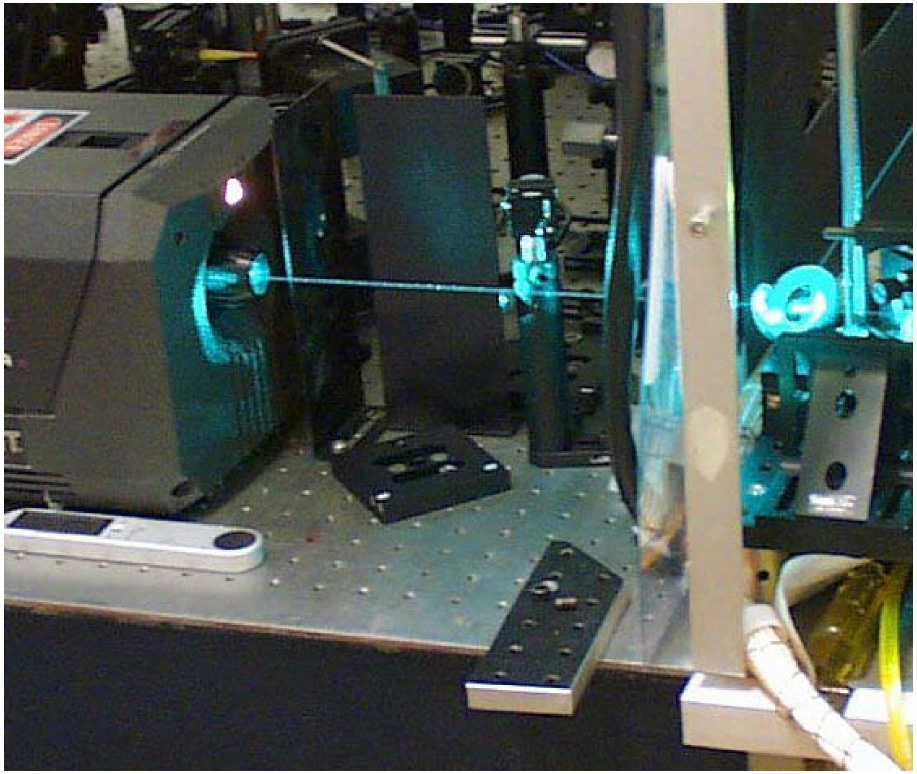
Ordinary light

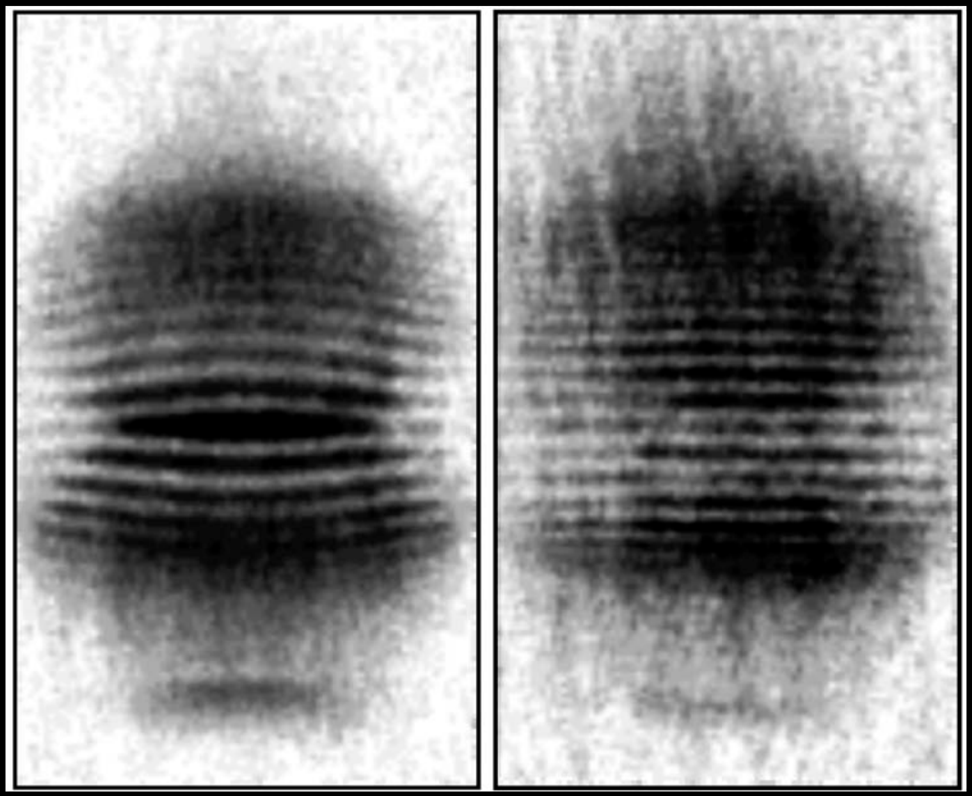
divergent
incoherent
many small waves
many modes

Laser light



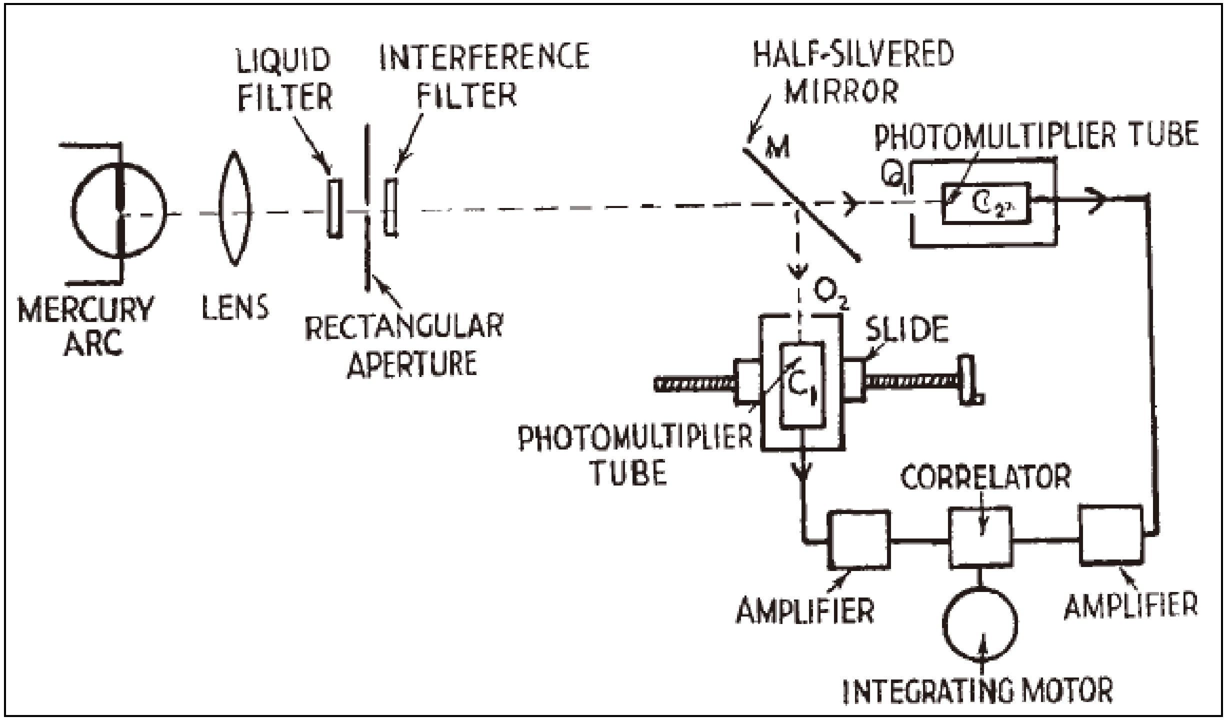
diffraction limited (directional)
coherent
one big wave
single mode (monochromatic)

Interference of two BECs



Interference @ MIT, 1997
(2001 Nobel Prize)

Hanbury Brown – Twiss Effect



Hanbury Brown & Twiss, Nature 177 (1956)

Photon bunching in light emitted by a chaotic source
➔ Highlight the importance of two-photon correlations
➔ Modern quantum optics

Quantum theory of optical coherence

How to describe the state of light *Glauber, PRL 10 (1963)*

$$E = E^{(+)} + E^{(-)}$$

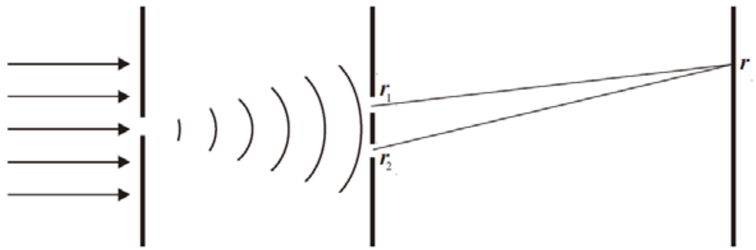
$E^{(+)} \to a$, $E^{(-)} \to a^{+}$

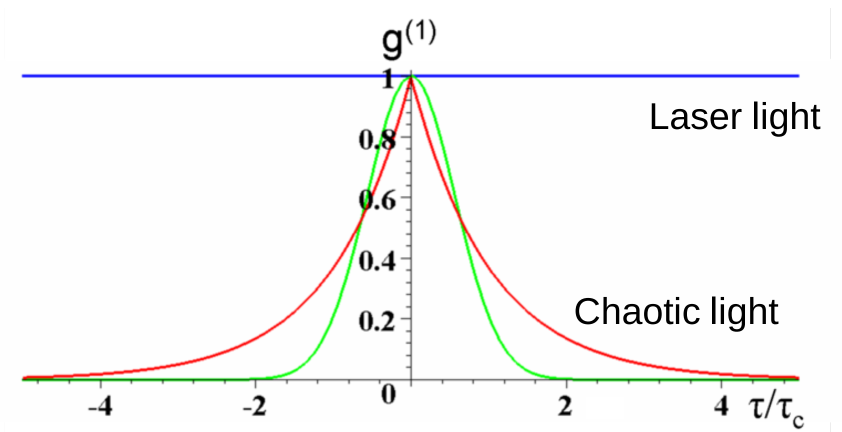
$$a\,a^{\dagger} - a^{\dagger}a = 1$$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

First-order coherence function

$$G^{(1)}(r_1,t_1;r_2,t_2) = \langle E^{(-)}(r_1,t_1)E^{(+)}(r_2,t_2)\rangle$$





Correlations in many-body systems

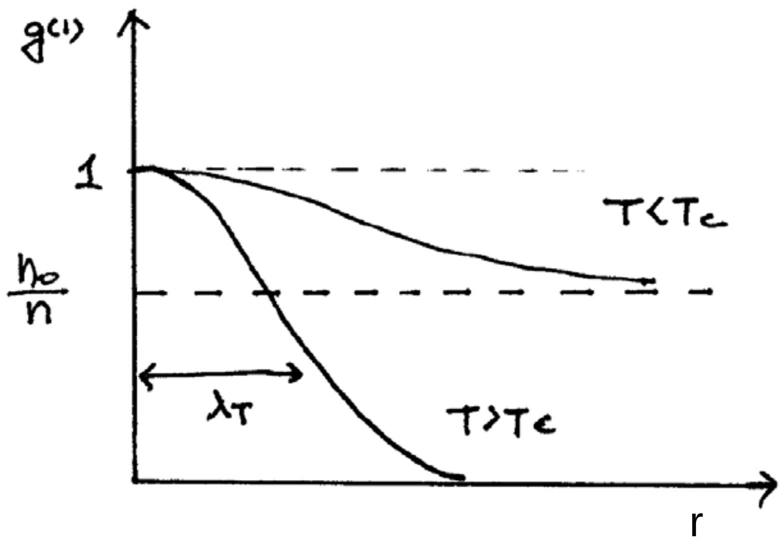
First-order coherence function

$$G^{(1)}(\vec{x}_1,\vec{x}_2)=\left\langle \hat{\psi}^+(\vec{x}_1)\hat{\psi}(\vec{x}_2)\right\rangle$$ one-particle density matrix

$$\lim_{|x_1-x_2|\to\infty} G^{(1)}(\vec{x}_1,\vec{x}_2)=n_0\neq 0$$: condensate fraction

$$g^{(1)}(x_1,x_2)=\frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}}\leq 1$$ Normalized first-order coherence function

For a translational invariant system



For a classical gas $n_k\sim e^{-\frac{p^2}{2mk_BT}}$

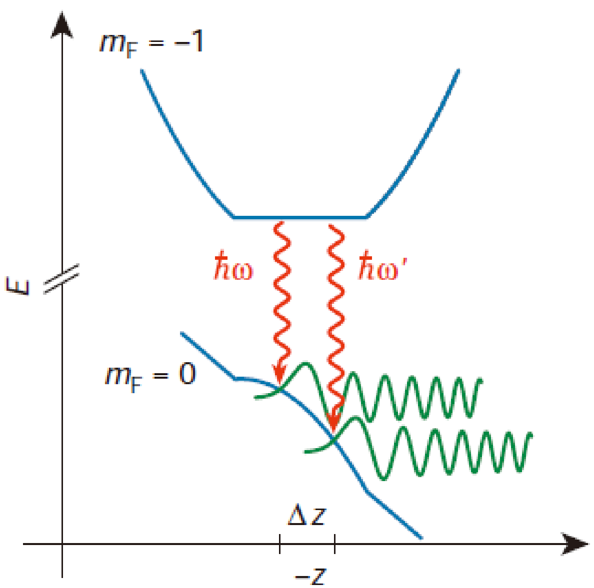
$$g^{(1)}_{cl}(r)=\exp\left(-\frac{\pi r^2}{\lambda_T}\right)$$

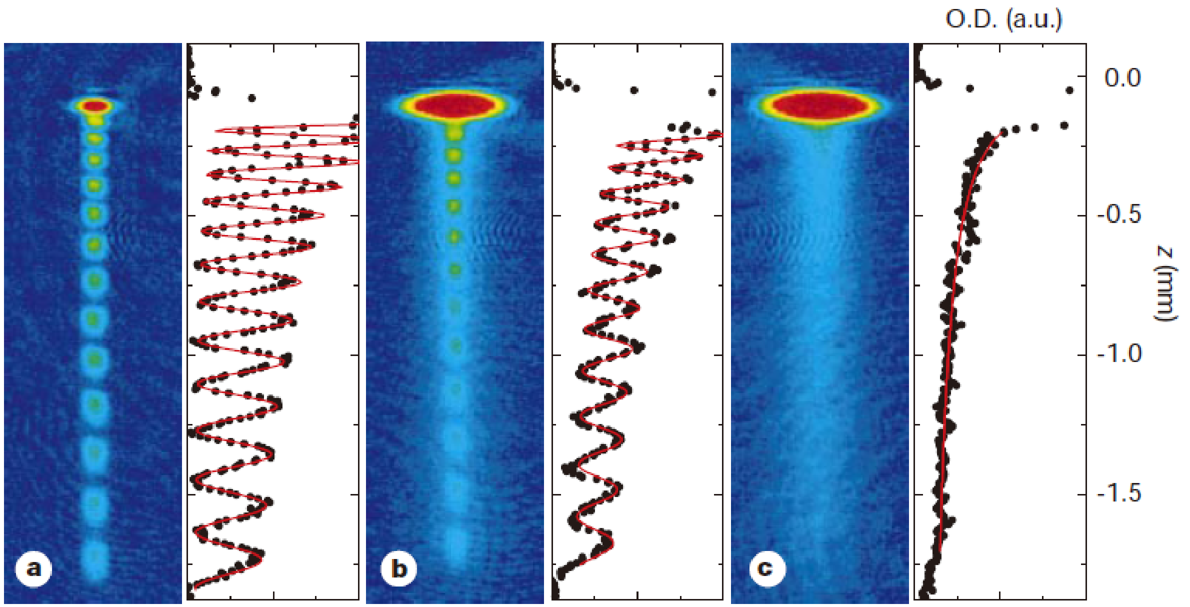
At T~T$_c$ $$g^{(1)}(r)\sim\exp\left(-\frac{r}{\xi(T)}\right)$$

With a BEC $$\lim_{r\to\infty} g^{(1)}(r)=n_0/n$$

Spatial coherence of a trapped Bose gas

Bloch et al., Nature 403 (2000)



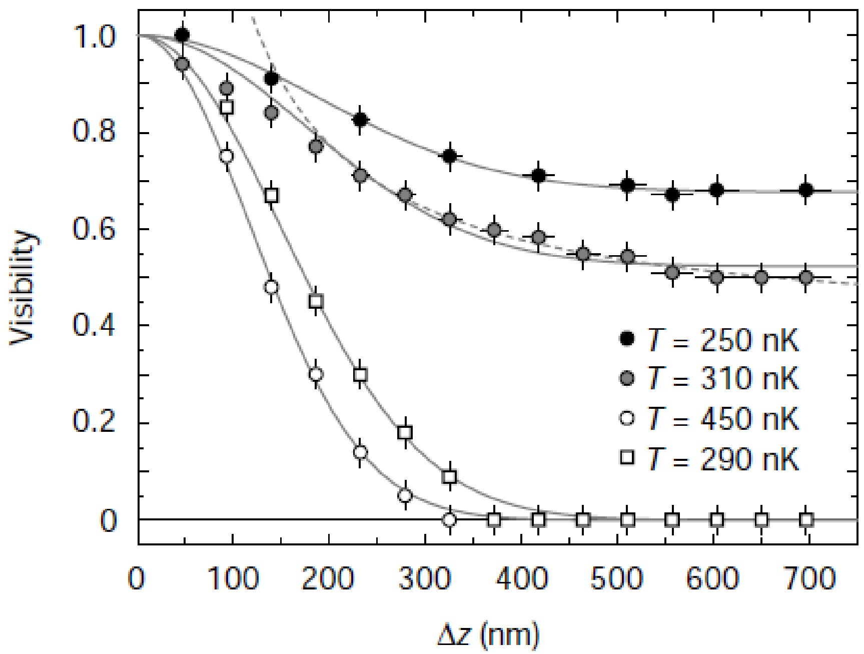


$T \ll T_c$ $T \sim T_c$ $T \gg T_c$

- Two-slit experiment to measure spatial coherence
- Using two rf waves, outcouple two atomic beams in different positions
- Visibility of the interference pattern indicates spatial coherence

Spatial coherence of a trapped Bose gas

Bloch et al., Nature 403 (2000)



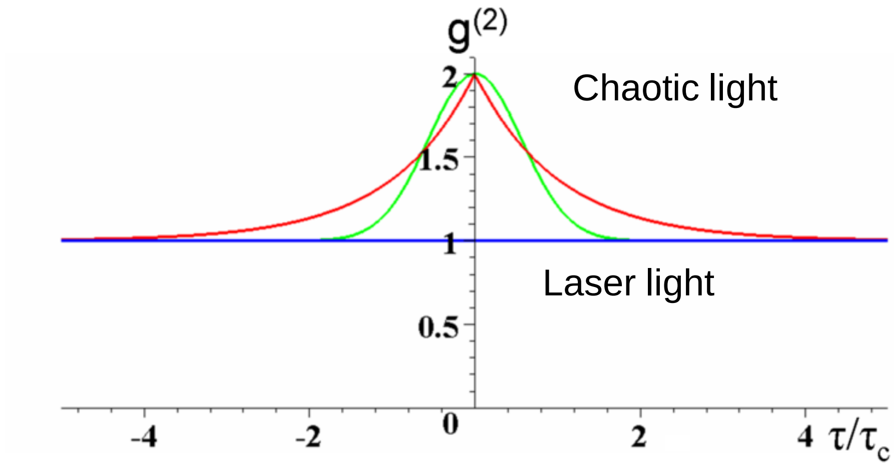
$$V = \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}} = g^{(1)}(x_1, x_2) \leq 1$$

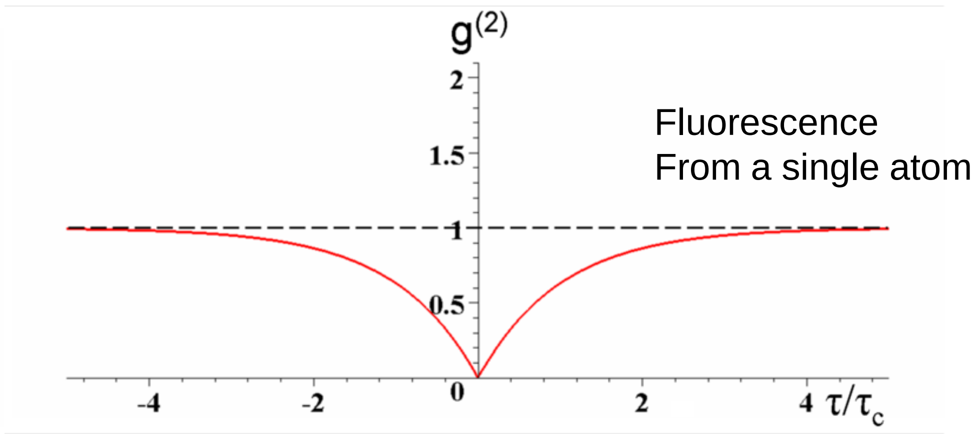
Quantum theory of optical coherence (2)

How to describe the state of light *Glauber, PRL 10 (1963)*

Second-order coherence function

$$G^{(2)}(r_1,t_1;r_2,t_2)=\left\langle E^{(-)}(r_1,t_1)E^{(-)}(r_2,t_2)E^{(+)}(r_2,t_2)E^{(+)}(r_1,t_1)\right\rangle$$





Correlations in many-body systems (2)

Second-order coherence function

$$G^{(2)}(\vec{x}_1,\vec{x}_2)=\left\langle \hat{\psi}^+(\vec{x}_1)\hat{\psi}^+(\vec{x}_2)\hat{\psi}(\vec{x}_2)\hat{\psi}(\vec{x}_1)\right\rangle$$

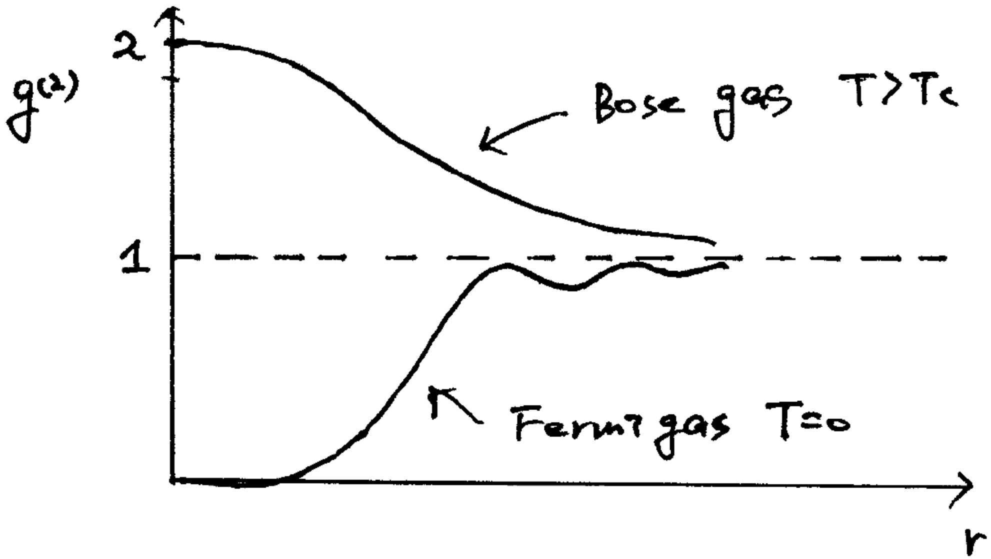
Density-density correlation function

$$\left\langle \hat{n}(\vec{x}_1)\hat{n}(\vec{x}_2)\right\rangle = n(\vec{x}_1)\delta(\vec{x}_1-\vec{x}_2)+n(\vec{x}_1)n(\vec{x}_2)g^{(2)}(\vec{x}_1,\vec{x}_2)$$

$$g^{(2)}(x_1,x_2)=\frac{G^{(2)}(x_1,x_2)}{n(x_1)n(x_2)}$$

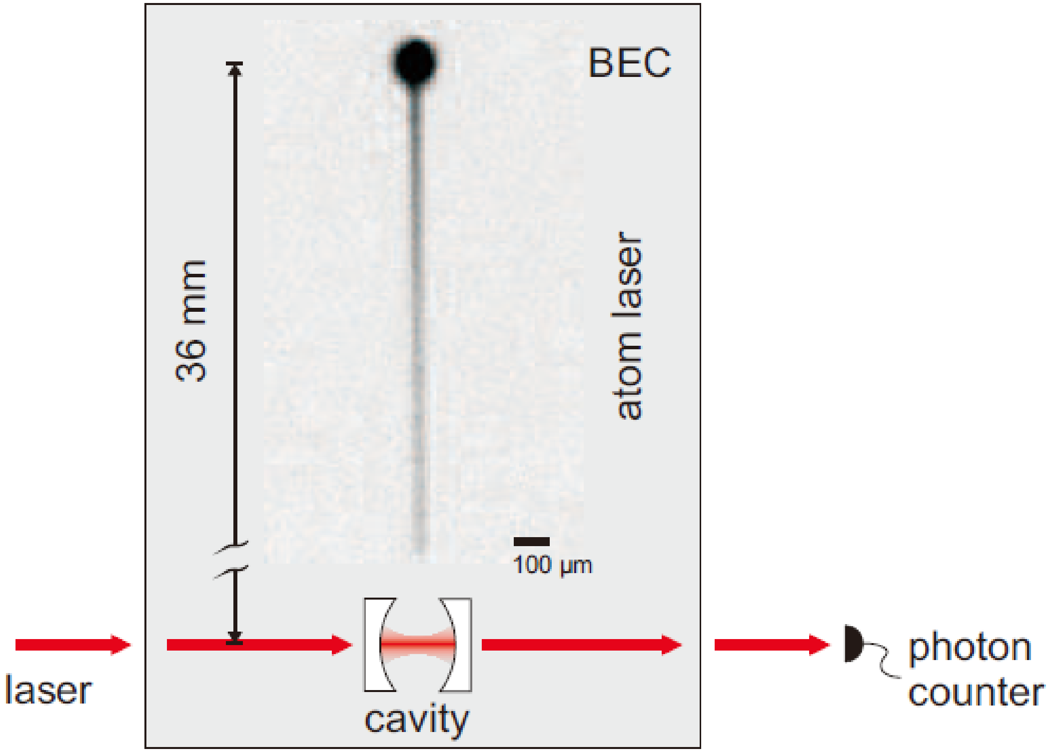
Normalized second-order coherence function

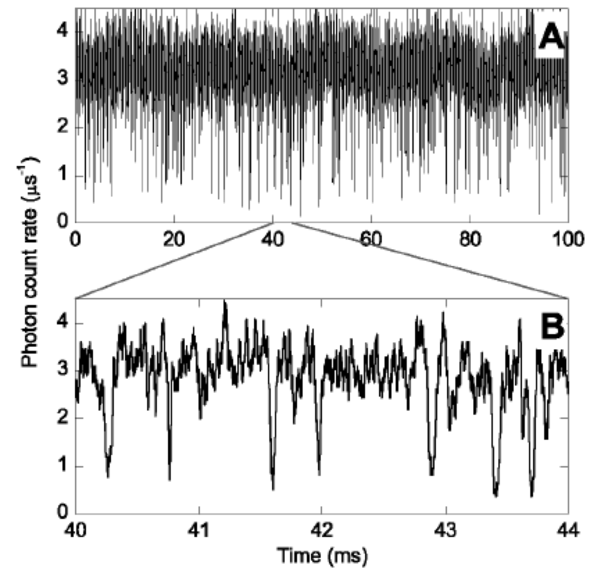
$4\pi r^2 n^2 g^{(2)}(r)dr$ Prob. To have another particle in a shell [r, r+dr]



Higher order phase coherence

Ottl et al., PRL 95 (2005)

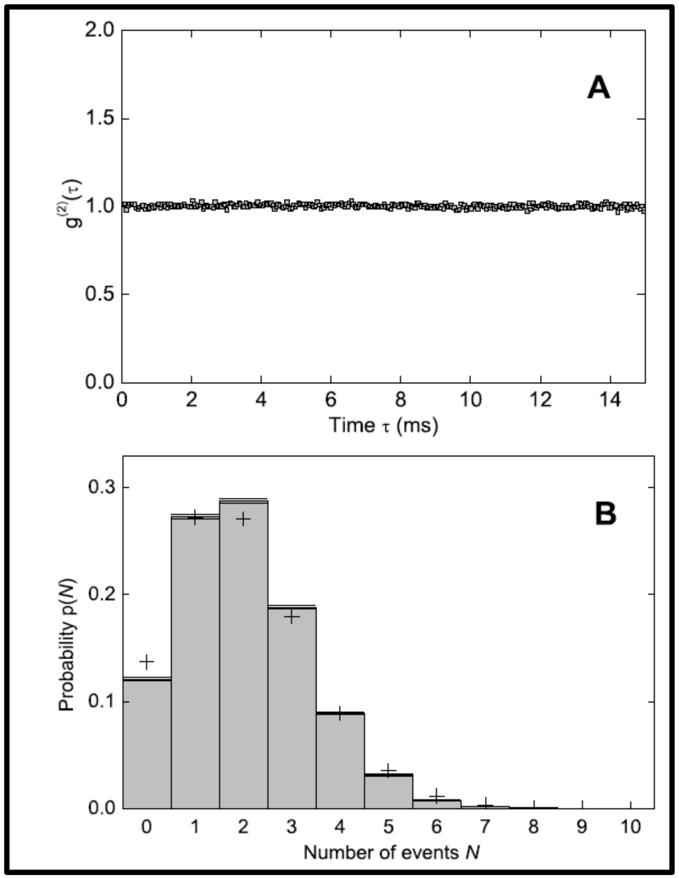




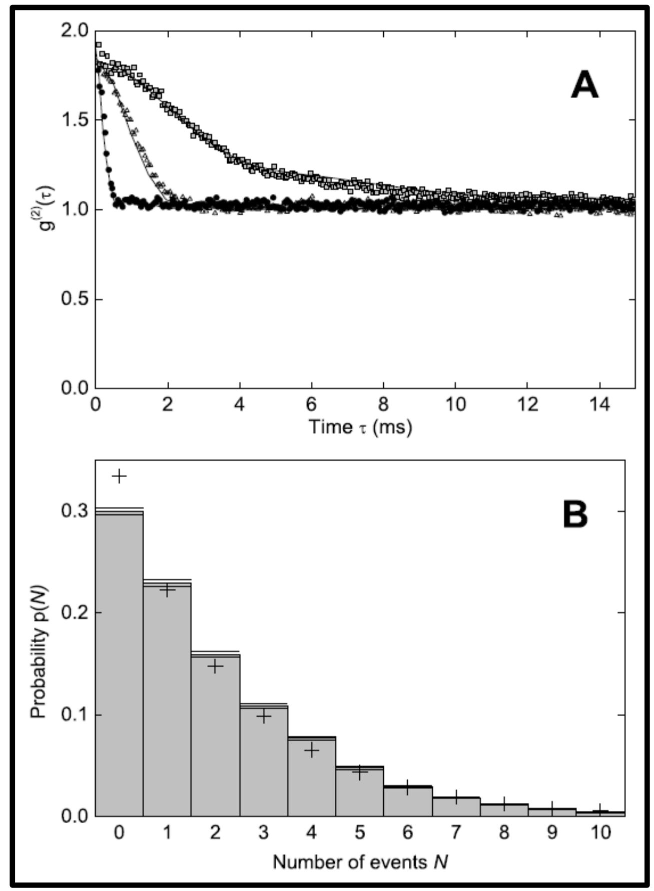
Higher order phase coherence

Ottl et al., PRL 95 (2005)

Coherent outcoupling

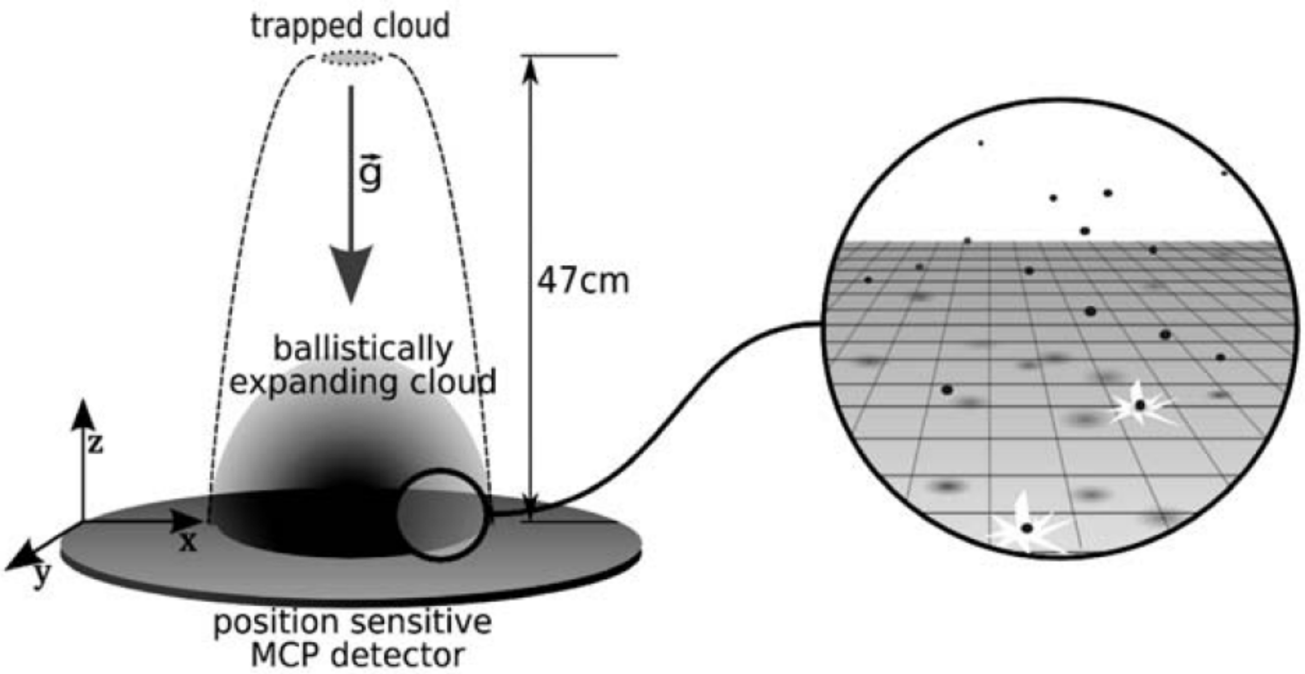


Incoherent outcoupling



Bunching and anti-bunching

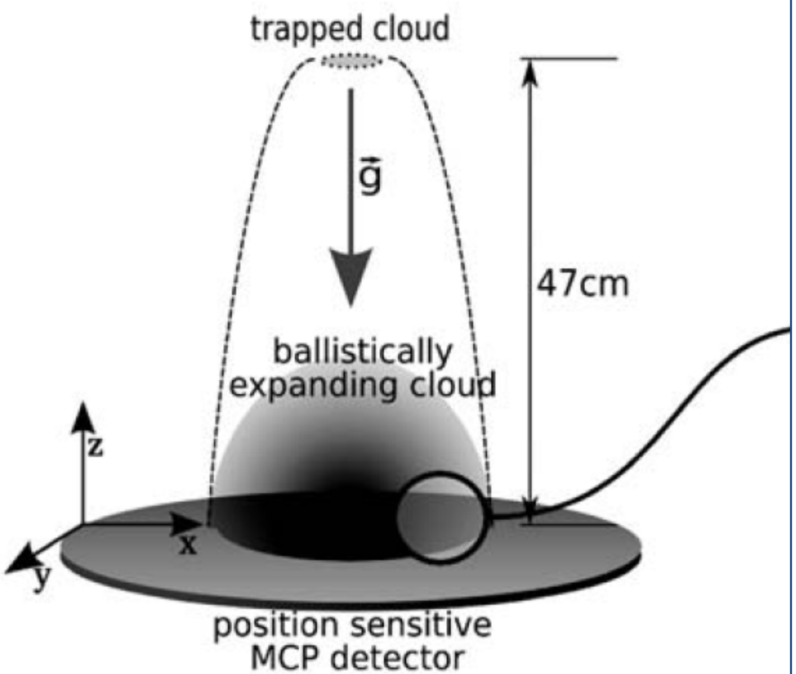
Using $^3He^*$ (fermion) and $^4He^*$ (boson)

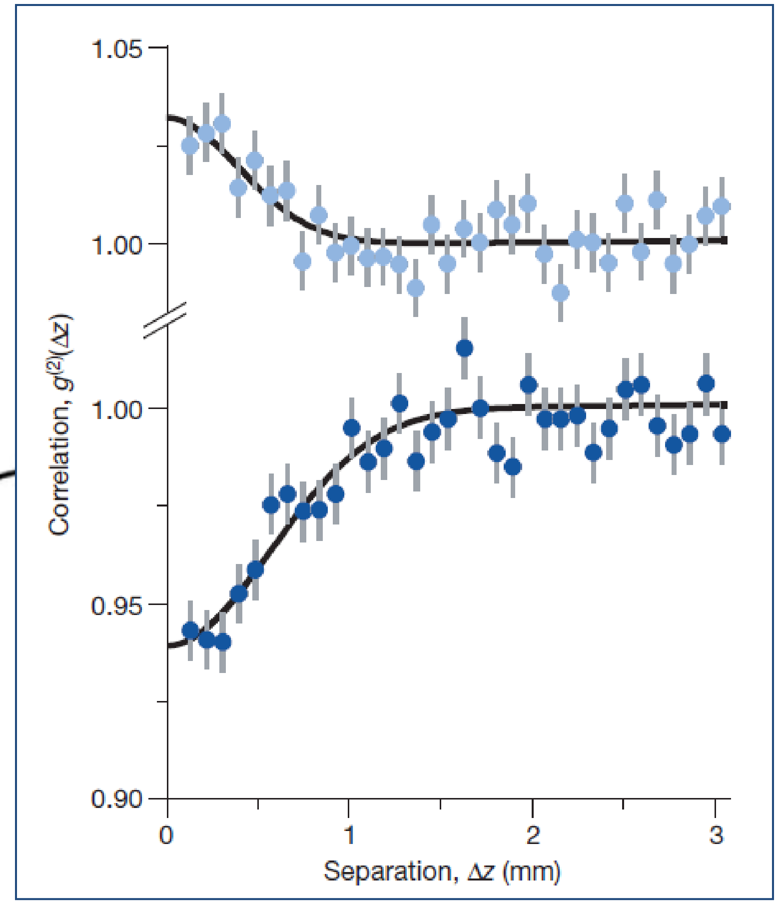


Schellekens et al., Science 310 (2005) / Jeltes et al., Nature 445 (2007)

Bunching and anti-bunching

Using ^{3}He* (fermion) and ^{4}He* (boson)



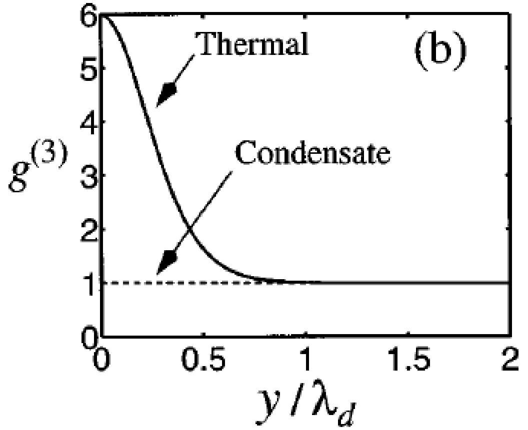


Schellekens et al., Science 310 (2005) / Jeltes et al., Nature 445 (2007)

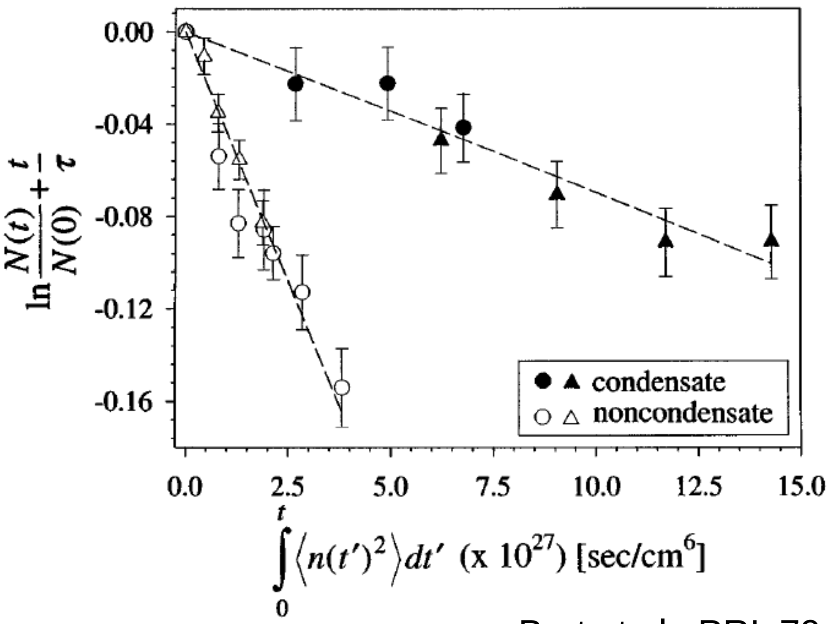
Higher order phase coherence

$$g_{th}^{(3)}(0) = 3! = 6$$

$$g_{BEC}^{(3)}(0) = 1$$



Three-body decay rate is six-times smaller for condensates.



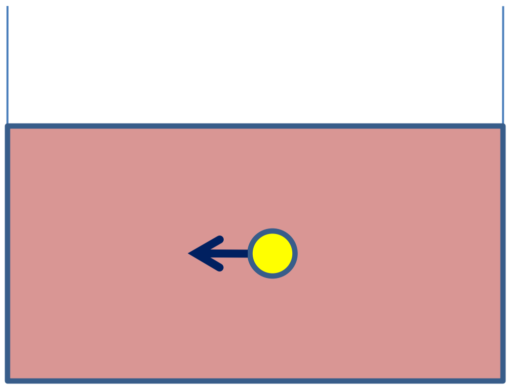
Burt et al., PRL 79 (1997)

Outline

1. What is Bose-Einstein condensation?
2. Ultracold atomic gases
3. Phase coherence of BEC
4. Superfluidity and BEC
5. BEC in an optical lattice

Superfluid

Superfluid, having a phenomenological definition, can flow without dissipation.

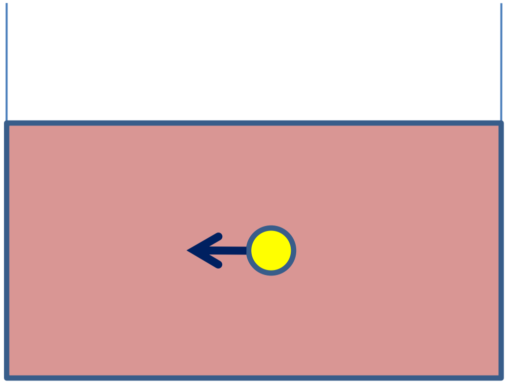


Q) Can this particle excite this fluid,
or give its kinetic energy to this fluid?

Landau Criterion of Superfluidity

$$\varepsilon = \frac{1}{2}mv^2 - \frac{1}{2}m(v-\delta v)^2 \approx mv\delta v$$

$$p = mv - m(v-\delta v) = m\delta v$$



If the particle excite the fluid, $$v = \frac{\varepsilon(p)}{p}$$ ← Excitation energy for momentum *p*

Critical velocity $$\upsilon_c = \left[\frac{\varepsilon(p)}{p}\right]_{\min}$$

L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Excitation spectrum of Superfluid Helium

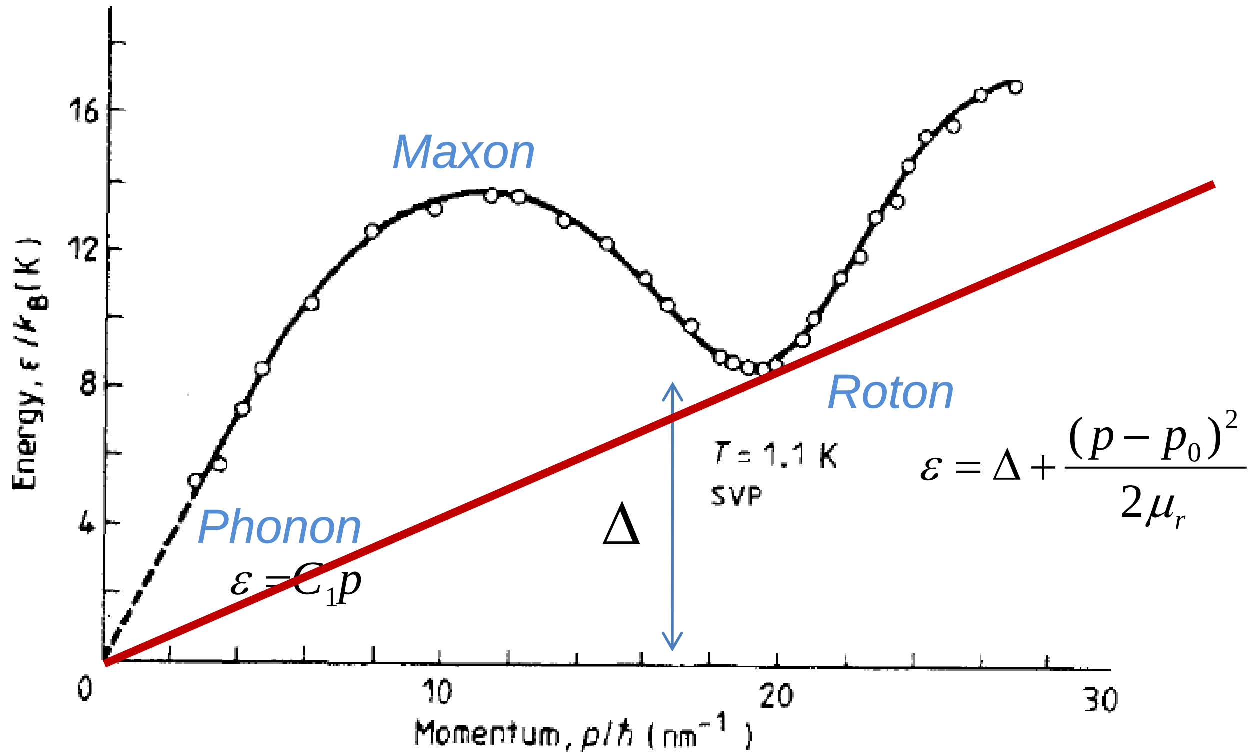
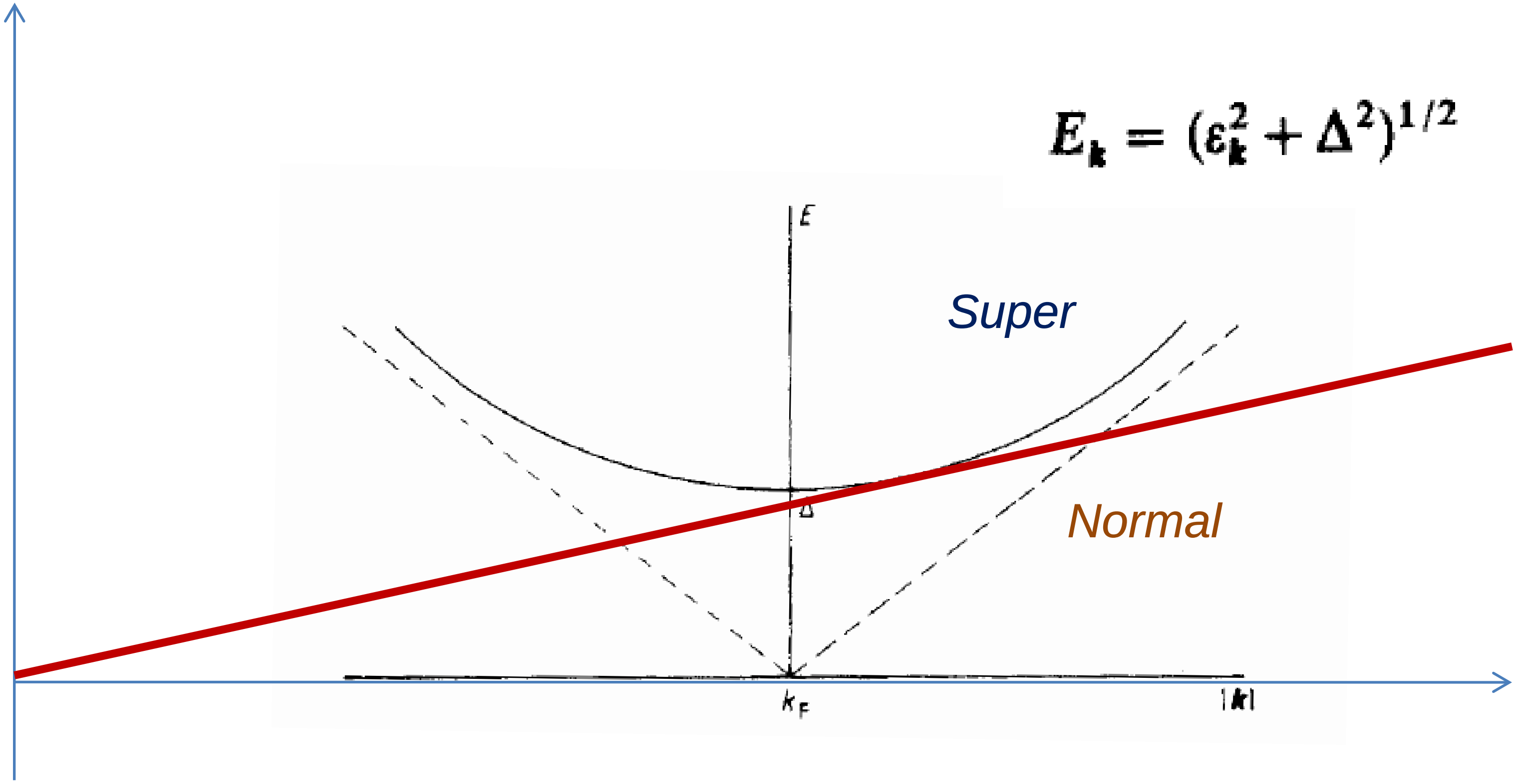


Figure 2.6 He II excitation spectrum obtained from neutron-scattering experiments (Henshaw and Woods 1961).

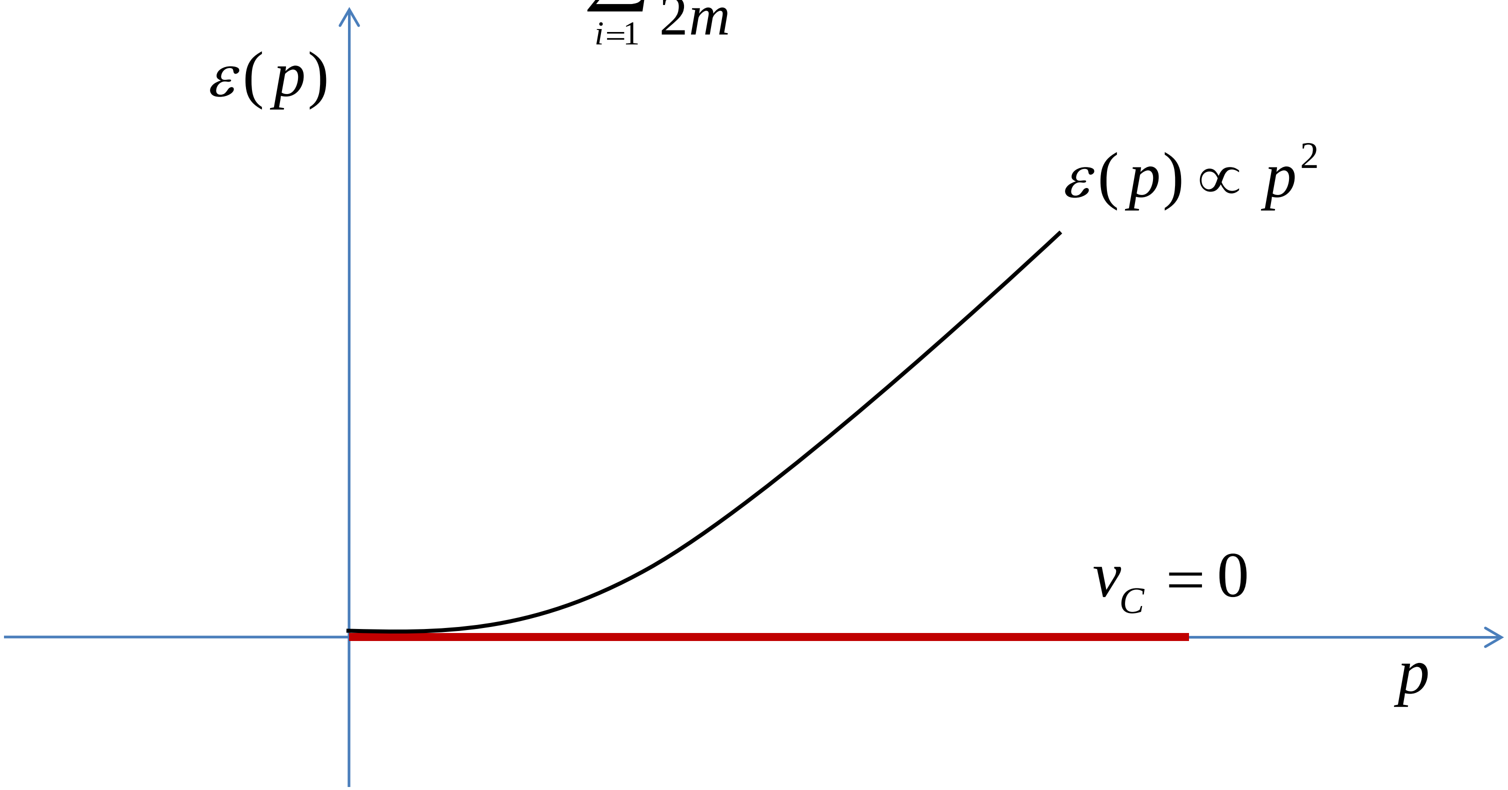
Excitation spectrum of Superconductor

$$E_k = (\varepsilon_k^2 + \Delta^2)^{1/2}$$



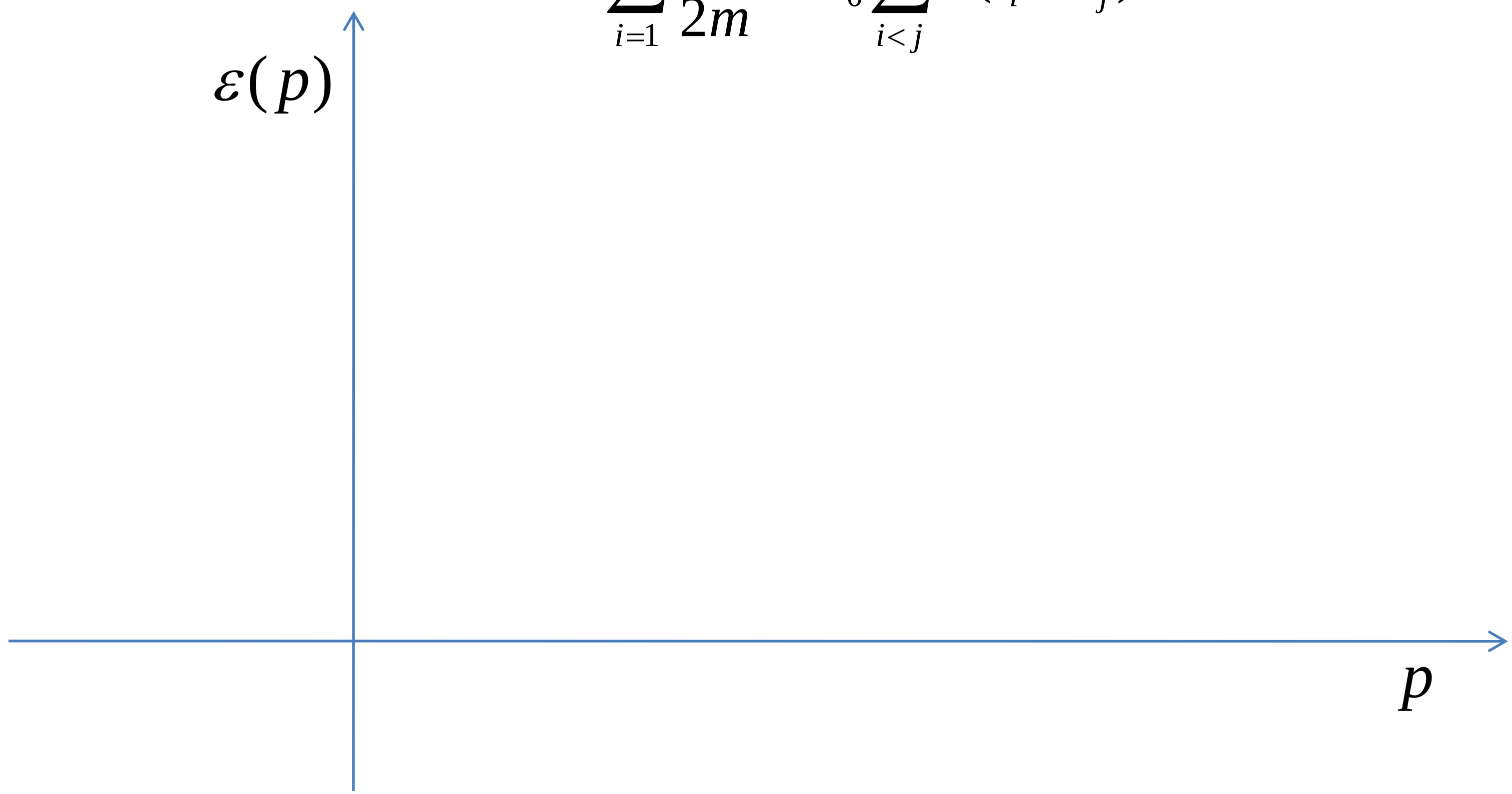
Excitation spectrum of a non-interacting Bose gas

$$H = \sum_{i=1}^{N} \frac{p^2}{2m}$$



Excitation spectrum of a ~~non~~-interacting Bose gas

$$H = \sum_{i=1}^{N} \frac{p^2}{2m} + U_0 \sum_{i<j} \delta(r_i - r_j)$$



Microscopic theory of a Bose gas at T=0

$$\hat{H} = \sum_{\mathbf{p}} \epsilon_p^0 \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{U_0}{2V} \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} \hat{a}_{\mathbf{p}+\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}'-\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}'} \hat{a}_{\mathbf{p}},$$

Bogoliubov approximation: replacing a_0 with c-number $N_0^{1/2}$

$$\Rightarrow \quad \hat{H} = \frac{N^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p}\neq 0)} \left[(\epsilon_p^0 + n_0 U_0) \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{n_0 U_0}{2} (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}}) \right],$$

Diagonalize with canonical transformation:

$$\hat{a}_{\mathbf{p}} = u_p \hat{\alpha}_{\mathbf{p}} - v_p \hat{\alpha}_{-\mathbf{p}}^\dagger, \quad \hat{a}_{-\mathbf{p}} = u_p \hat{\alpha}_{-\mathbf{p}} - v_p \hat{\alpha}_{\mathbf{p}}^\dagger,$$

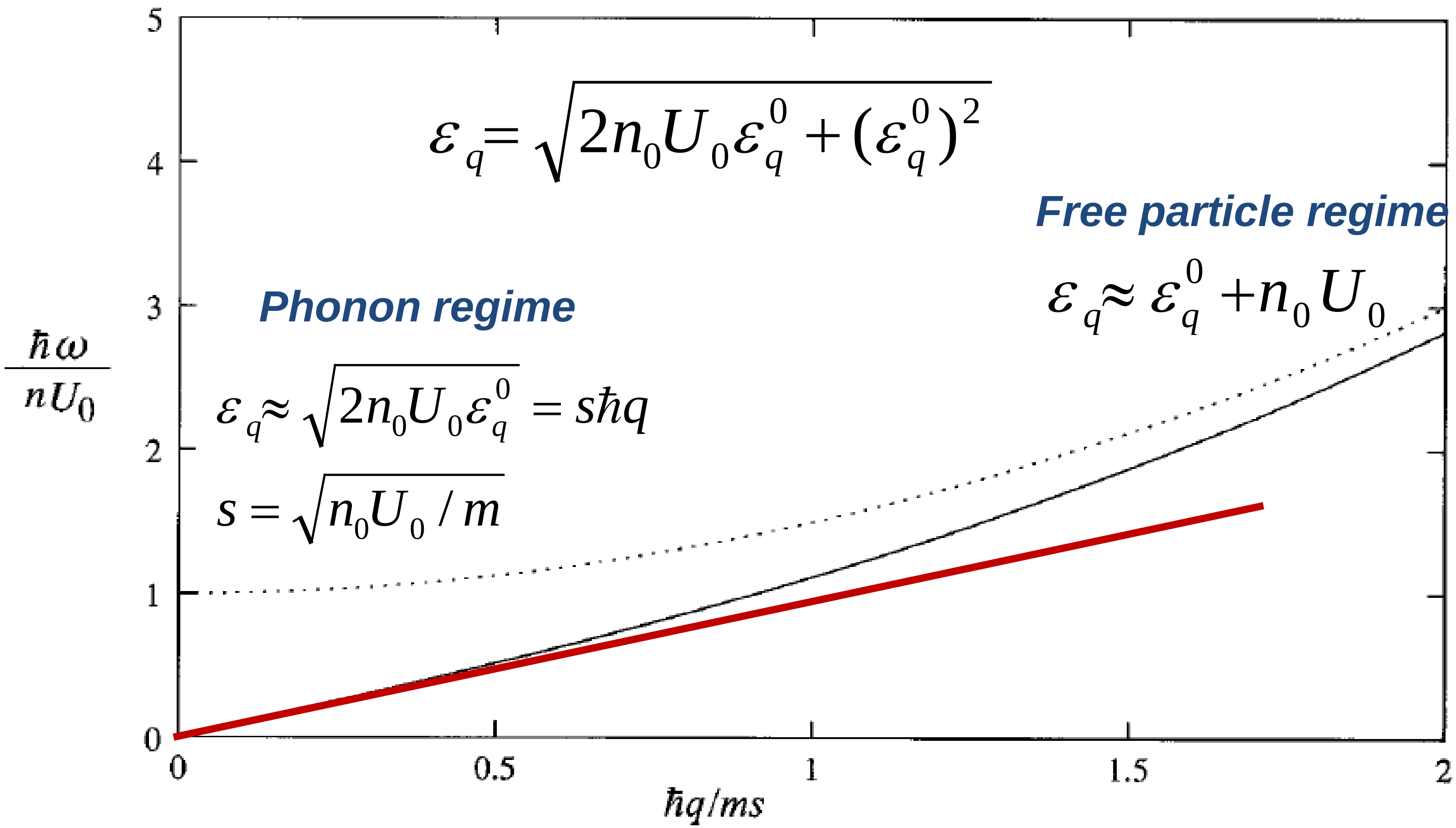
$$\Rightarrow \quad H = \frac{N^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p}\neq 0)} \epsilon_p \hat{\alpha}_{\mathbf{p}}^\dagger \hat{\alpha}_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}(\mathbf{p}\neq 0)} (\epsilon_p^0 + n_0 U_0 - \epsilon_p)$$

$$\varepsilon_q = \sqrt{2 n_0 U_0 \varepsilon_q^0 + (\varepsilon_q^0)^2}$$

$$u_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} + 1\right) \quad v_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} - 1\right) \quad u_p^2 - v_p^2 = 1$$

$$\xi_p = \epsilon_p^0 + n_0 U_0$$

Elementary excitation of an interacting Bose gas



$$H = \sum_{i=1}^{N} \frac{p^2}{2m} + U_0 \sum_{i<j} \delta(r_i - r_j)$$

Many-body ground state

Dominant scattering processes at T~0

$$(0)+(0)\Leftrightarrow(+p)+(-p)$$

Two atoms in condensate collide into +p and –p atoms.

$$n_p=\left\langle a_p^+ a_p\right\rangle = v_p^2 \neq 0$$

Non-condensed atom number

$$n_{\mathrm{ex}}=\frac{1}{V}\sum_{\mathbf{p}(\mathbf{p}\neq 0)} v_p^2=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}v_p^2=\frac{1}{3\pi^2}\left(\frac{ms}{\hbar}\right)^3$$

$$\frac{n_{\mathrm{ex}}}{n}=\frac{8}{3\sqrt{\pi}}(na^3)^{1/2}$$

Quantum depletion

Outline

1. What is Bose-Einstein condensation?
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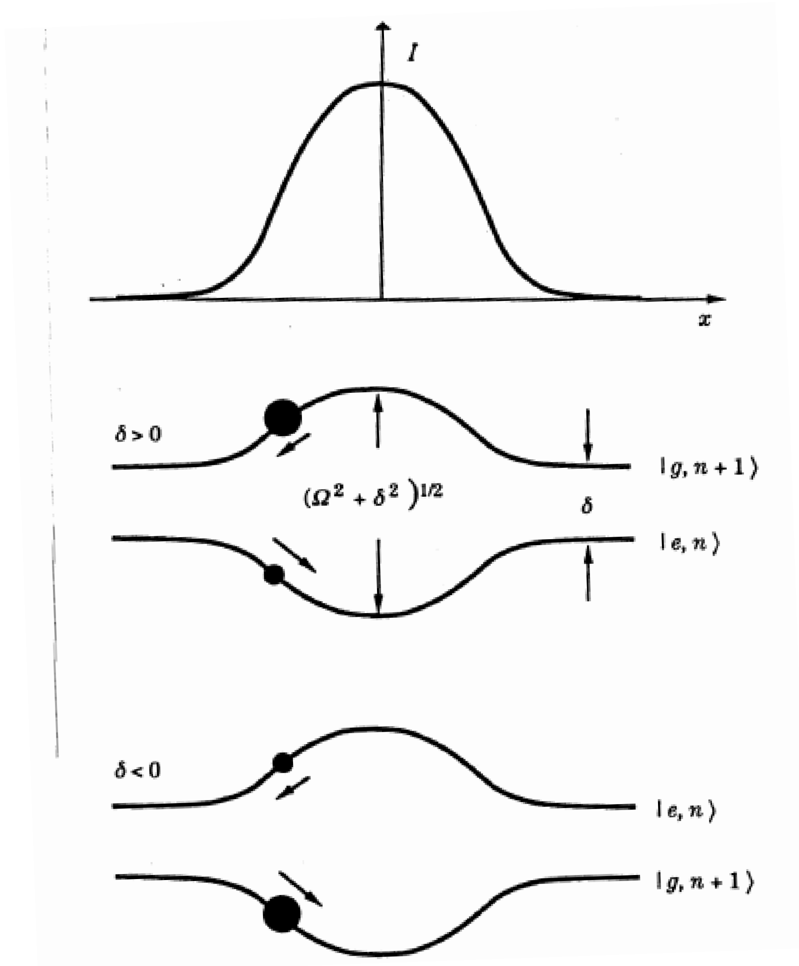
Optical dipole trap



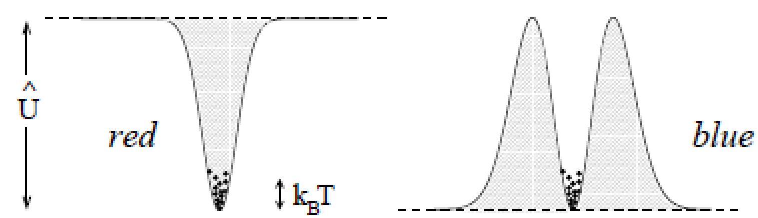
$$U_{\rm dip} = -\frac{1}{2}\langle \mathbf{pE}\rangle = -\frac{1}{2\epsilon_0 c}\,\mathrm{Re}(\alpha)\,I$$

Complex polarizability

Far detuning limit $(\Delta << \Gamma)$

$$U_{\rm dip}(\mathbf{r}) = \frac{3\pi c^2}{2\omega_0^3}\,\frac{\Gamma}{\Delta}\,I(\mathbf{r})\,,$$

$$\Gamma_{\rm sc}(\mathbf{r}) = \frac{3\pi c^2}{2\hbar\omega_0^3}\left(\frac{\Gamma}{\Delta}\right)^2 I(\mathbf{r})\,.$$



R. Grimm, *et al,* Adv. At., Mol., Opt. Phys. 42, 95 (2000)

Optical lattice

When two laser beams overlap, they interfere, leading to a periodic pattern of the intensity, i.e. a periodic potential for atoms.

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_1(\mathbf{r},t) + \mathbf{E}_2(\mathbf{r},t) \\ &= E_1 \mathbf{e}_1 \exp[-i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)] + E_2 \mathbf{e}_2 \exp[-i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)]\end{aligned}$$

$$\begin{aligned}I(\mathbf{r},t) &\propto \mathbf{E} \cdot \mathbf{E}^*(\mathbf{r},t) \\ &= E_1^2 + E_2^2 + 2(\mathbf{e}_1 \cdot \mathbf{e}_2)\mathrm{Re}\left[E_1 E_2^* e^{-i((\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r} - (\omega_1 - \omega_2)t)}\right]\end{aligned}$$

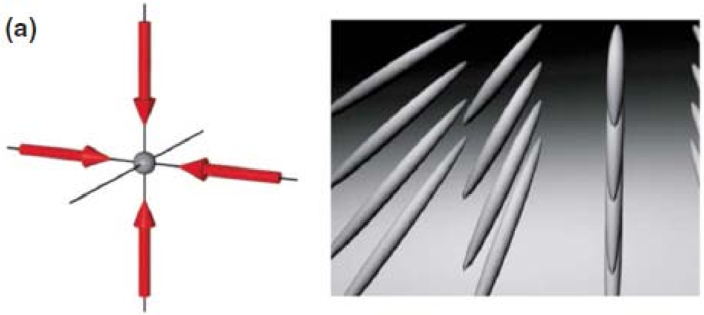
$\omega_1 = \omega_2$ Standing potential

$\omega_1 \neq \omega_2$ Moving lattice potential

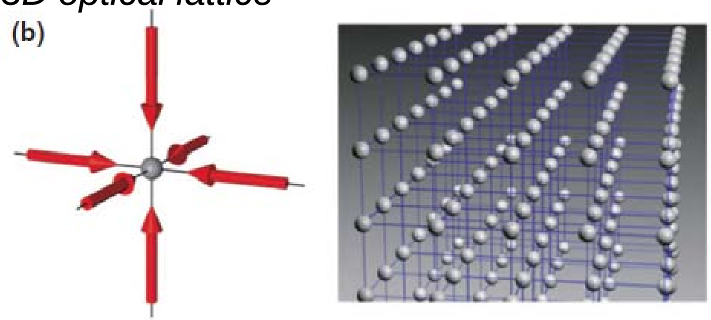
$V \sim \sin\left(\left[2k \sin(\frac{\theta}{2})\right] z\right)$ Lattice period is controlled by the angle between the two beams.

Optical lattice

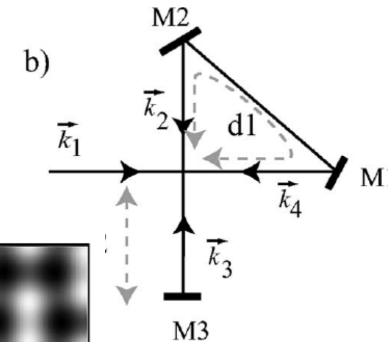
2D optical lattice / quantum wire, 1D physics

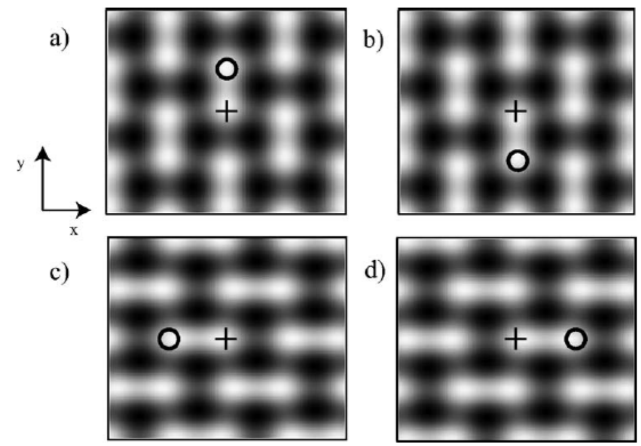


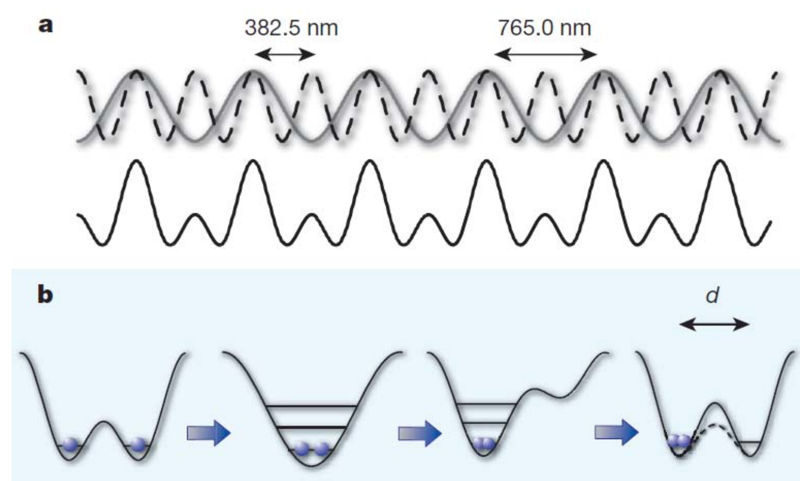
3D optical lattice



Superlattice potentials







Optical lattice

Atoms moving in an optical lattice have the same basic physics as electrons in a crystal lattice in solids.

Lattice constant

Solid crystal	$\sim 10^{-10}$m
Optical lattice	$\sim 10^{-7}$m

Lattice barrier height

Solid crystal	$\sim 10^{5}$ K
Optical lattice	$\sim 10^{-5}$ K

Optical lattice: Magnifying laboratory for condensed matter physics.

Band structure

The presence of an optical lattice modifies the single-particle energy spectrum to a band structure.

$$E\psi(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x)$$

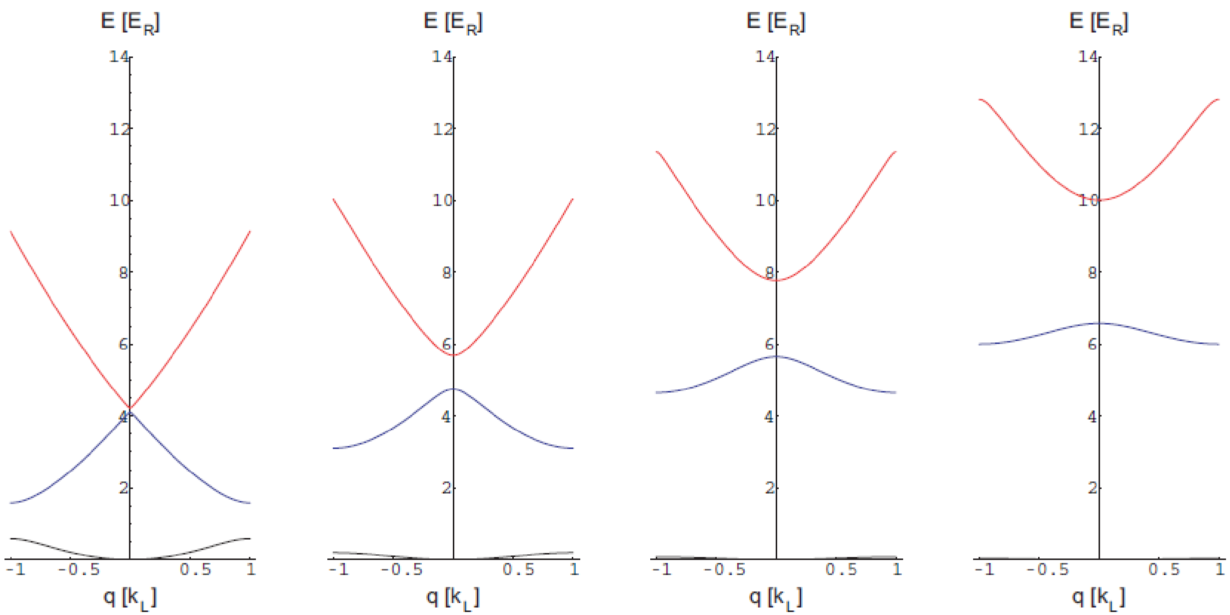
$$\begin{aligned} V_{latt}(x) &= V_{latt}\sin^2(k_L x) = V_{latt}\left(\frac{e^{ik_L x} - e^{-ik_L x}}{2i}\right)^2 \\ &= \frac{V_{latt}}{4}\left(2 - e^{i2k_L x} - e^{i2k_L x}\right) \end{aligned}$$

Bloch wavefunction $\psi(x) = e^{ikx}u(x)$

$$u(x+d) = u(x)$$

$$V_K = \begin{cases} V_{latt}/2 & K = 0 \\ -V_{latt}/4 & K = \pm K_x \\ 0 & \text{otherwise} \end{cases} \qquad K_x = 2k_L$$

Energy band structure



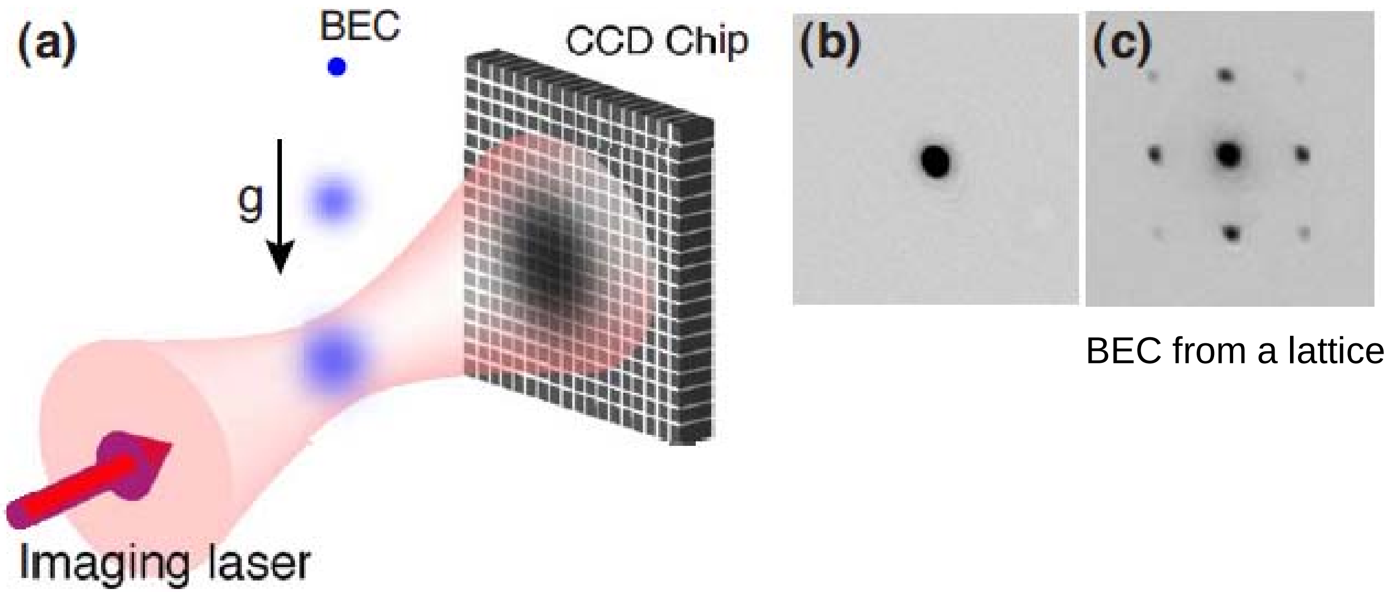
V = 2 E_R V = 6 E_R V = 10 E_R V = 14 E_R

$$\psi_{n,\mathbf{q}}(\mathbf{r})$$

Bloch wave function
for n[th] band with momentum *q*
-distributed over all lattice sites

Time-of-flight image of a BEC in an optical lattice

BEC from a harmonic trap



BEC from a lattice

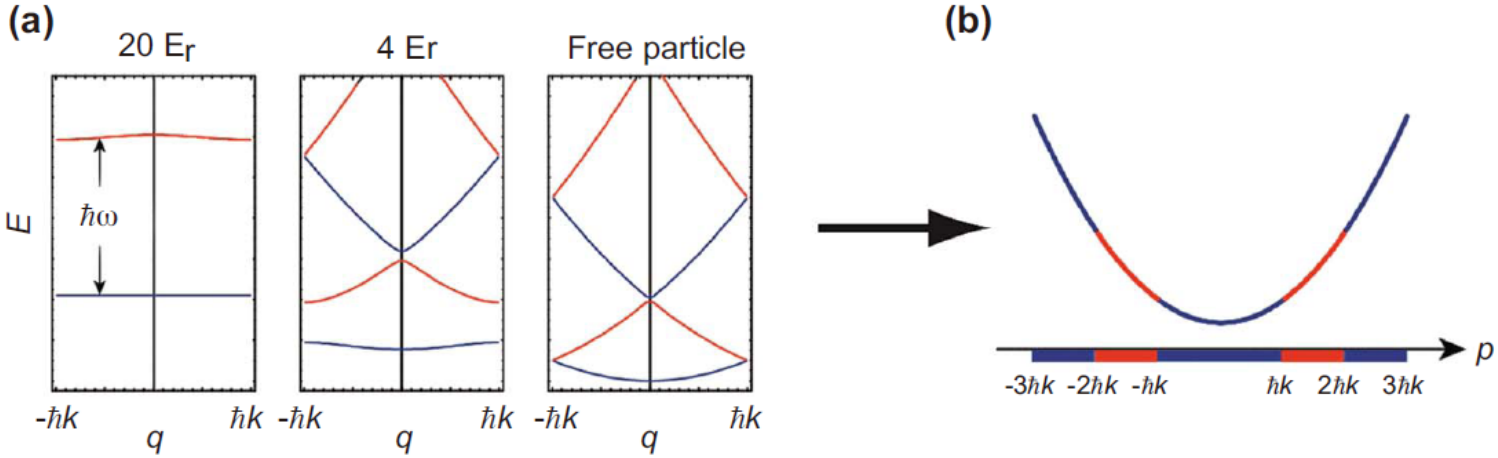
- Sudden release from a trap
- Revealing the in-trap momentum distribution.
- Diffraction from an optical grating

the one-particle density matrix $G^{(1)}(\mathbf{R},\mathbf{R}')=\langle \hat{a}^{\dagger}_{\mathbf{R}}\hat{a}_{\mathbf{R}'}\rangle$

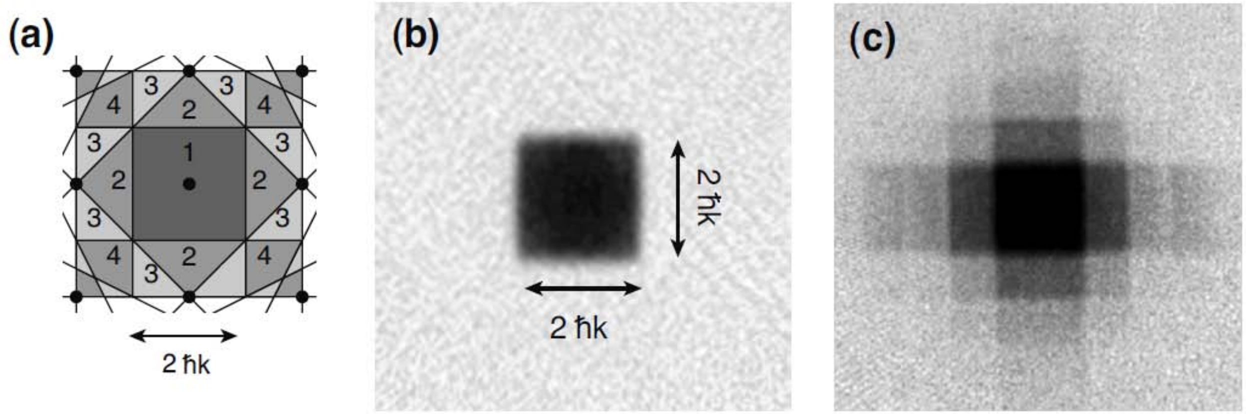
$$\mathcal{G}(\mathbf{k}) = \sum_{\mathbf{R},\mathbf{R}'} e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} G^{(1)}(\mathbf{R},\mathbf{R}')$$

Adiabatic mapping of quasimomentum

Under adiabatic transformation of the lattice depth the quasimomentum q is preserved during slow turn-off process

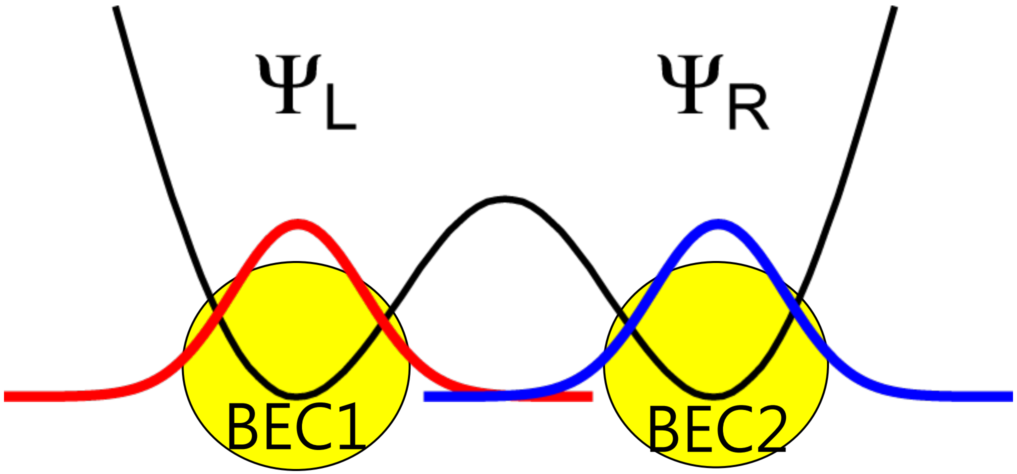


Brillouin zones



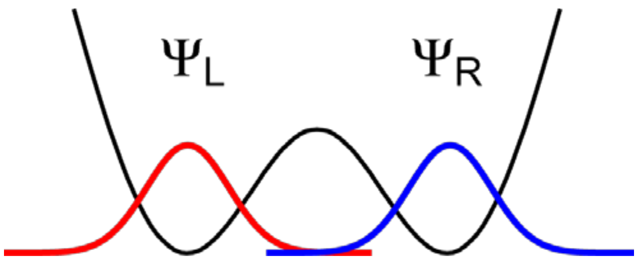
PRL **87**, 160405 (2001).

BEC in a double-well potential



- Relative phase of two condensates
- Tunneling of particles between the wells
- Time evolution of the phase and the atom number: Josephson dynamics

Simple description



Two-mode approximation

$$H = \frac{1}{2} U \sum_{i=1,2} a_i^+ a_i^+ a_i a_i - J(a_2^+ a_1 + a_1^+ a_2) + \cdots$$

Interaction term | Tunneling term

Ground state for the non-interacting case U=0

$$H = -J a_+^+ a_+ + J a_-^+ a_-$$

$a_+ = (a_1 + a_2)/\sqrt{2}$ Symmetric state

$a_- = (a_1 - a_2)/\sqrt{2}$ Anti-symmetric state

- Symmetric ground state
- If we start with a BEC in one well, it will oscillate at $2J/h$

Coherent state and number state

$$H = \frac{1}{2} U \sum_{i=1,2} a_i^+ a_i^+ a_i a_i - J(a_2^+ a_1 + a_1^+ a_2) + \cdots$$

Non-interacting case U=0

$$\left|\psi_{coh}\right\rangle \propto (a_1^{\ +} + e^{i\phi} a_2^{\ +})^N \left|0\right\rangle$$

: Coherent state
with a well-defined relative phase

Particle number uncertainty ~ $\sqrt{N}$

Strongly-interacting case U>>J

$$\left|\psi_{num}\right\rangle \propto (a_1^{\ +})^{\frac{N}{2}} (a_2^{\ +})^{\frac{N}{2}} \left|0\right\rangle$$

: Number (Fock) state
with well-defined atom numbers

Bose-Hubbard model

- ❖ Both the thermal and mean interaction energies at a single site are much smaller than the separation to the first excited band.
  ➔ *Only the lowest band is involved*.

- ❖ Wannier functions decay essentially within a singel lattice constant.
  ➔ *Only the hopping to nearest neighbors are counted*.

- ❖ In the limit of a sufficient deep optical lattice.

$$\hat{H} = -J \sum_{\langle \mathbf{R},\mathbf{R}' \rangle} \hat{a}^{\dagger}_{\mathbf{R}} \hat{a}_{\mathbf{R}'} + \frac{U}{2} \sum_{\mathbf{R}} \hat{n}_{\mathbf{R}} (\hat{n}_{\mathbf{R}} - 1)$$

Kinetic energy
Hopping to nearest neighbors

On-site interactions

Superfluid-Mott-insulator transition

$$\hat{H} = -J \sum_{\langle \mathbf{R},\mathbf{R}' \rangle} \hat{a}^\dagger_{\mathbf{R}} \hat{a}_{\mathbf{R}'} + \frac{U}{2} \sum_{\mathbf{R}} \hat{n}_{\mathbf{R}} (\hat{n}_{\mathbf{R}} - 1)$$

The many-body ground state is determined via the competition between the kinetic energy and the interaction energy.

Superfluid phase

$$|\Psi_N\rangle (U=0) = \frac{1}{\sqrt{N!}} \left(\frac{1}{\sqrt{N_L}} \sum_{\mathbf{R}} \hat{a}^\dagger_{\mathbf{R}} \right)^N |0\rangle$$

$$\bar{n} = N / N_L$$

$$\Delta n = \sqrt{\bar{n}}$$

Mott-insulating phase

$$|\Psi_{N=N_L}\rangle (J=0) = \left(\prod_{\mathbf{R}} \hat{a}^\dagger_{\mathbf{R}} \right) |0\rangle$$

$$\bar{n} = 1$$

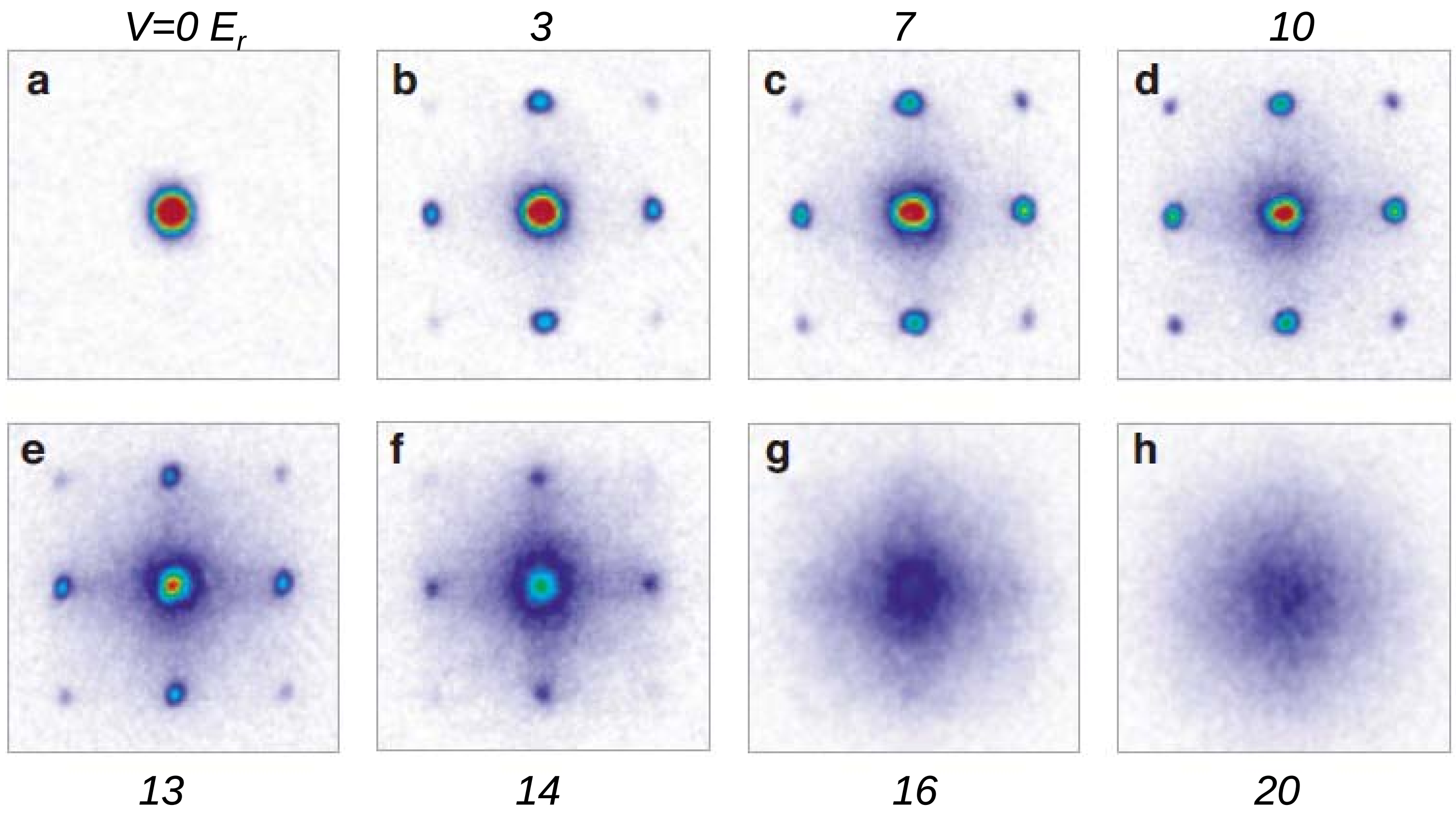
$$\Delta n = 0$$

$U/J \sim (a/d) \exp(2\sqrt{V_0/E_r})$ can be controlled by the lattice intensity.

Quantum phase transition from SF to Mott-Insulating phase

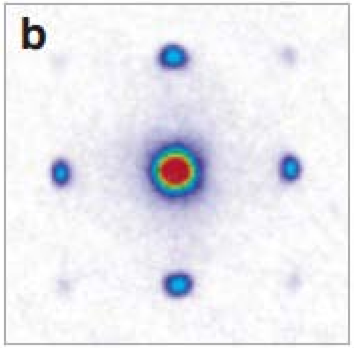
Superfluid-Mott-insulator transition

Nature **415**, 39 (2001).



Interference peaks disappear ➔ Loosing superfluidity

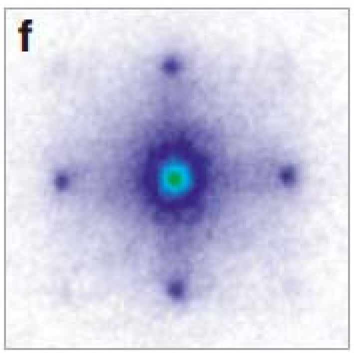
Phase coherence in SF-to-MI transition



$$n(\mathbf{k}) \sim |\tilde{w}(\mathbf{k})|^2 \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} G^{(1)}(\mathbf{R})$$

$$\mathbf{k} = M\mathbf{x}/\hbar t$$

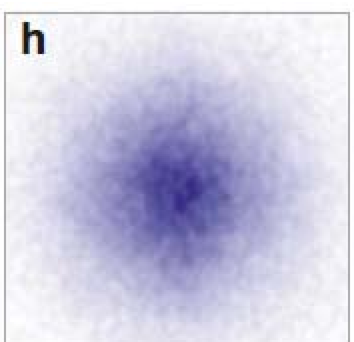
Existence of BEC
➔Superfluid phase



With non-zero *J*, a coherent admixture of particle-hole pairs

$$|\Psi\rangle_{U/J} \approx |\Psi\rangle_{U/J\to\infty} + \frac{J}{U} \sum_{\langle \mathbf{R},\mathbf{R}'\rangle} \hat{a}^\dagger_{\mathbf{R}} \hat{a}_{\mathbf{R}'} |\Psi\rangle_{U/J\to\infty}$$

Short range coherence



Perfect Mott regime, J=0

$G^{(1)}(\mathbf{R})$ vanishes exponentially beyond R=0.

The momentum distribution is a structureless Guassian.

Summary

- Ultracold atom gases

  : Model system for many-body physics

- BEC has laser-like properties
- BEC with interactions is a superfluid
- Optical lattice systems simulate solid state physics.