Classical and Semiclassical Analyses of Normal Mode Splitting

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I. SEMICLASSICAL ANALYSIS OF NORMAL MODE SPLITTING

In the semiclassical approach, the field is treated classically (Maxwell equation) and the atom is treated quantum mechanically (Schrödinger equtaion). Let us first derive the Maxwell-Schrödinger Equations to be applied to the normal mode splitting in an atom-cavity system.

A. Derivation of the Maxwell-Schrödinger Equations

1. Differential Equation for the Cavity Field

Let us assume that a single two-level atom is located in the cavity. The atom is near resonant with the TEM_{00} mode of the cavity. Starting from one of the Maxwell equations in Gaussian units,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

taking a curl on both sides and substituting

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

and using the relation $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, we obtain

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{3}$$

The electric field can be decomposed into the normal modes of the cavity:

$$\mathbf{E}(\mathbf{x},t) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(\mathbf{x},t) = \sum_{\mathbf{k}} \mathbf{E}_{0\mathbf{k}} f_{\mathbf{k}}(\mathbf{x}) \exp(-i\omega_{\mathbf{k}} t), \tag{4}$$

where $\mathbf{k}$ and $\omega_{\mathbf{k}}$ are the wave vector and the normal mode frequency associated with the normal mode $\mathbf{E}_{\mathbf{k}}$, respectively, and $f_{\mathbf{k}}(\mathbf{x})$ is the mode function summarizing the spatial dependence of the normal mode $\mathbf{E}_{\mathbf{k}}$. For instance, the mode function for the TEM_{00} mode of a Fabry-Perot cavity with its axis along the z axis is given by

$$f(\mathbf{x}) = \cos kz \exp\left[-(x^2+y^2)/w^2\right] \tag{5}$$

with w being the mode waist. The mode volume of a particular normal mode is given by

$$V_{\mathbf{k}} = \int |f_{\mathbf{k}}(\mathbf{x})|^2\, d^3\mathbf{x}. \tag{6}$$

We assume that only a particular normal mode $\mathbf{E}_{\mathbf{k}}$ with a normal mode frequency of $\omega_{\mathbf{k}}$ is resonant with the atom. This particular normal mode is usually the TEM_{00} mode of the cavity with its mode function given by Eq. (5). Then, Eq. (3) is replaced by

$$\nabla(\nabla\cdot\mathbf{E}_{\mathbf{k}}) - \nabla^2\mathbf{E}_{\mathbf{k}} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_{\mathbf{k}}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{7}$$

We assume $\mathbf{E}_{\mathbf{k}}$ is polarized along the y direction, as is $\mathbf{P}$. By noting that $\mathbf{E}_{\mathbf{k}} \equiv E_{\mathbf{k}}\hat{\mathbf{y}} \propto \exp(\pm ikz - i\omega t)$ with $\hat{\mathbf{y}}$ being the unit vector in the y direction, we find in the slowly-varying-envelope approximation that

$$\nabla\cdot\mathbf{E}_{\mathbf{k}} \simeq \pm ik\hat{\mathbf{z}}\cdot E\hat{\mathbf{y}} = 0,$$
$$\nabla^2\mathbf{E}_{\mathbf{k}} \simeq -k^2\mathbf{E}_{\mathbf{k}};$$

thus, the above wave equation can be reduced to

$$k^2\mathbf{E}_{\mathbf{k}} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}_{\mathbf{k}}}{\partial t^2} \simeq \frac{4\pi\omega_0^2}{c^2}\mathbf{P}, \tag{8}$$

where we assumed that the polarization has a slow time variation on top of a fast oscillation at a frequency of ω_0. The polarization induced by the single atom located at $\mathbf{x}_0$ can be written as

$$\mathbf{P}(\mathbf{x},t) = \hat{\mathbf{y}}p(t)\delta(\mathbf{x}-\mathbf{x}_0), \tag{9}$$

We also define $E_{0\mathbf{k}}$, the amplitude of the normal mode, as $E_{0\mathbf{k}} \equiv \hat{\mathbf{y}}\cdot\mathbf{E}_{0\mathbf{k}}\exp(-i\omega t)$. Then, by multiplying $f_{\mathbf{k}}(\mathbf{x})$ to both sides of Eq. (8) and integrating the resulting equation over space, we obtain

$$k^2E_{0\mathbf{k}}(t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_{0\mathbf{k}}(t) \simeq \frac{4\pi\omega_0^2}{c^2}\frac{p(t)}{V_{\mathbf{k}}}f_{\mathbf{k}}(\mathbf{x}_0). \tag{10}$$

Defining the frequency of the field as $\omega \equiv kc$ and dropping off the subscript $\mathbf{k}$ from now on, we get

$$\ddot{E}_0 + \omega^2E_0 = \frac{4\pi\omega_0^2 p}{V}f(\mathbf{x}_0). \tag{11}$$

The cavity damping is introduced phenomenologically. Let us define $2\gamma_c$ as the energy decay rate of the cavity. For $p=0$ (*i.e.*, the homogeneous equation), the cavity field then goes like $\exp(i\omega-\gamma_c)t$, so

$$\begin{aligned}\dot{E}_0 &= (i\omega-\gamma_c)E_0,\\ \ddot{E}_0 &= (i\omega-\gamma_c)^2E_0 \simeq -\omega^2E_0 - 2i\omega\gamma_cE_0\\ &\simeq -\omega^2E_0 - 2\gamma_c\dot{E}_0,\end{aligned}$$

where we assumed $\omega \gg \gamma_c$. This relation yields an homogeneous differential equation

$$\ddot{E}_0 + 2\gamma_c\dot{E}_0 + \omega^2E_0 = 0, \tag{12}$$

which in turn instruct us to write down a differential equation for the cavity field including the cavity damping as

$$\ddot{E}_0 + 2\gamma_c \dot{E}_0 + \omega^2 E_0 = \frac{4\pi\omega_0^2 p}{V} f(\mathbf{x}_0). \tag{13}$$

2. Differential Equation for the Atomic Dipole Moment and Population Inversion

We start from a density matrix equation,

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho], \tag{14}$$

which is equivalent to the Schrödinger equation for a pure state as considered here. The atom has two levels with the upper level denoted by a and the lower level by b. The matrix elements for the Hamiltonian operator H are

$$H_{aa} = \hbar\omega_a, \;\; H_{bb} = \hbar\omega_b, \;\; H_{ab} = -\mu E(t), \tag{15}$$

where $E(t) \equiv \mathbf{E}(\mathbf{x}_0, t) \cdot \hat{\mathbf{y}} = \varepsilon_0 \cos\omega t$ with $\mathbf{x}_0$ being the atomic position, and μ (chosen to be real) being the matrix element for the dipole moment operator $\hat{\mu}$ along the y direction. After the rotating wave approximation, we obtain a set of differential equations for the density matrix elements:

$$\dot{\rho}_{aa} = -\frac{i}{2}\Omega e^{i\omega t}\rho_{ab} + c.c. - 2\gamma_p \rho_{aa}, \tag{16a}$$

$$\dot{\rho}_{ab} = -\frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) - i\omega_0 \rho_{ab} - \gamma_p \rho_{ab}, \tag{16b}$$

$$\dot{\rho}_{bb} = -\dot{\rho}_{aa}, \tag{16c}$$

where atomic radiative damping is included phenomenologically with $2\gamma_p$, the spontaneous emission rate of level a. Equation (16c) means that the lower level b does not decay. We define $\Omega \equiv \mu\varepsilon_0/\hbar$, the Rabi frequency associated with the cavity field, and "*c.c.*" indicates the complex conjugate of the preceding expression.

The induced dipole moment p of the atom is given by

$$p = \mathrm{Tr}(\hat{\mu}\rho) = \mu(\rho_{ab} + \rho_{ba}). \tag{17}$$

We are interested in the differential equation for p. Straightforward differentiation of p gives

$$\begin{aligned} \frac{\dot{p}}{\mu} &= \dot{\rho}_{ab} + \dot{\rho}_{ba} \\ &= -(i\omega_0 + \gamma_p)\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) + c.c., \end{aligned} \tag{18a}$$

$$\begin{aligned} \frac{\ddot{p}}{\mu} &\simeq -(i\omega_0 + \gamma_p)\dot{\rho}_{ab} - \frac{1}{2}\Omega\omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) + c.c. \\ &\simeq (i\omega_0 + \gamma_p)^2 \rho_{ab} - (i\omega_0 + \gamma)\frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) \\ &\quad -\frac{1}{2}\Omega\omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) + c.c. \\ &\simeq -\omega_0^2 \rho_{ab} + 2i\omega_0\gamma_p\rho_{ab} - \Omega\omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) + c.c. \\ &\simeq -\omega_0^2 \frac{p}{\mu} - 2\gamma_p \frac{\dot{p}}{\mu} - 2\Omega\omega_0 \cos\omega t(\rho_{aa} - \rho_{bb}), \end{aligned} \tag{18b}$$

where we used $\omega \simeq \omega_0 \gg \gamma_p, \Omega$. Therefore, the differential equation for p becomes

$$\ddot{p} + 2\gamma_p \dot{p} + \omega_0^2 p = -\frac{2\mu^2}{\hbar}\omega_0 E r, \tag{19}$$

where $r \equiv \rho_{aa} - \rho_{bb}$ is the population inversion. Since the above equation contains r, we also need a differential equation for r. By differentiating the definition of r, we find

$$\dot{r} = \dot{\rho}_{aa} - \dot{\rho}_{bb} = (-i\Omega e^{i\omega t}\rho_{ab} + c.c.) - 4\gamma_p \rho_{aa}. \tag{20}$$

Using the relation $\rho_{aa} + \rho_{bb} = 1$ for a single atom, we obtain

$$\begin{aligned} &\dot{r} + 2\gamma_p(r+1) \\ &= -i\Omega\cos\omega t(\rho_{ab} - \rho_{ba}) + \Omega\sin\omega t(\rho_{ab} + \rho_{ba}) \\ &\simeq \frac{\Omega}{\omega_0}\cos\omega t\frac{\dot{p}}{\mu} + \Omega\sin\omega t\frac{p}{\mu} \\ &= \frac{\varepsilon}{\hbar\omega_0}\cos\omega t\dot{p} + \frac{\varepsilon}{\hbar}\sin\omega t p \\ &\simeq \frac{E\dot{p} - \dot{E}p}{\hbar\omega_0} \simeq \frac{2E\dot{p}}{\hbar\omega_0}. \end{aligned} \tag{21}$$

Equations (19) and (21) are the Maxwell-Schrödinger equations for an arbitrary driving field $E(t)$. For an atom in the cavity considered in the preceding section, we identify $E(t) = E_0(t)f(\mathbf{x}_0)$. Therefore, the proper forms of the atomic part of the Maxwell-Schrödinger equations are

$$\ddot{p} + 2\gamma_p \dot{p} + \omega_0^2 p = -\frac{2\mu^2}{\hbar}\omega_0 E_0 r f(\mathbf{x}_0), \tag{22}$$

$$\dot{r} + 2\gamma_p(r+1) = \frac{2E_0\dot{p}}{\hbar\omega_0} f(\mathbf{x}_0). \tag{23}$$

B. Application to Normal Mode Splitting

Consider a two level atom interacting with a single-mode of a cavity. Assume that the atom with a resonance frequency ω_0 is placed at an anti-node of the cavity mode (so $f(\mathbf{x}_0) = 1$). The cavity with a resonance frequency ω_c, which is near resonant with the atom, is externally driven by a probe laser beam with a frequency of ω, which is near resonant with the cavity. Assume that the probe laser is so weak that the atom is almost in its lower state in steady state. In this situation, we do not need the equation for the population inversion. We can simply set $r = -N$ for N atoms. Our coupled equations are then

$$\ddot{E} + 2\gamma_c \dot{E} + \omega_c^2 E = \frac{4\pi\omega_0^2 p}{V} + \xi E_L, \tag{24}$$

$$\ddot{p} + 2\gamma_p \dot{p} + \omega_0^2 p = \frac{2\mu^2}{\hbar}\omega_0 E N, \tag{25}$$

where ξ is the probe laser coupling coefficient, E_L is the probe laser amplitude and we dropped the subscript 0 in the electric field amplitude. In a steady state

$$\left(-\omega^2 - 2i\omega\gamma_c + \omega_c^2\right) E = \frac{4\pi\omega_0^2 p}{V} + \xi E_L, \tag{26}$$

$$\left(-\omega^2 - 2i\omega\gamma_p + \omega_0^2\right) p = \frac{2\mu^2}{\hbar}\omega_0 E N, \tag{27}$$

By the near resonance condition

$$\omega^2 - \omega_{0,c}^2 \approx 2\omega_0(\omega - \omega_{0,c}), \tag{28}$$

and thus

$$(\Delta_c + i\gamma_c)E = -\frac{2\pi\omega_0 p}{V} - \frac{\xi E_L}{2\omega_0}, \tag{29}$$

$$(\Delta_p + i\gamma_p)p = -\frac{\mu^2}{\hbar} E N. \tag{30}$$

Solving for E gives

$$E = \frac{i\left(\frac{\xi E_L}{2\omega_0}\right)(\gamma_p - i\Delta_p)}{N g_0^2 + (\gamma_p - i\Delta_p)(\gamma_c - i\Delta_c)}, \tag{31}$$

where g_0 is the atom-cavity coupling constant given by

$$g_0 = \frac{\mu}{\hbar}\sqrt{\frac{2\pi\hbar\omega_0}{V}}. \tag{32}$$

Probe transmittance of the atom-cavity system is proportional to $|E|^2$.

$$\mathcal{T} \propto \left|\frac{\gamma_p - i\Delta_p}{N g_0^2 + (\gamma_p - i\Delta_p)(\gamma_c - i\Delta_c)}\right|^2 \tag{33}$$

The denominator can be expanded as

$$\begin{aligned}
&\left[N g_0^2 + (\gamma_p - i\Delta_p)(\gamma_c - i\Delta_c)\right] \\
&\qquad\times \left[N g_0^2 + (\gamma_p + i\Delta_p)(\gamma_c + i\Delta_c)\right] \\
&= (N g_0^2)^2 + 2(\gamma_c\gamma_p - \Delta_c\Delta_p)(N g_0^2) \\
&\qquad + (\gamma_p^2 + \Delta_p^2)(\gamma_c^2 + \Delta_c^2).
\end{aligned} \tag{34}$$

When $\omega_c = \omega_0$ (*i.e.*, $\Delta_c = \Delta_p \equiv \Delta$), it is simplified to

$$\begin{aligned}
&(N g_0^2)^2 + 2(\gamma_c\gamma_p - \Delta^2)(N g_0^2) + (\gamma_p^2 + \Delta^2)(\gamma_c^2 + \Delta^2) \\
&= \Delta^4 - 2\left[N g_0^2 - (\gamma_c^2 + \gamma_p^2)/2\right]\Delta^2 + \left(N g_0^2 + \gamma_c\gamma_p\right)^2 \\
&= \Delta^4 - 2(N g_0^2 - \gamma_-^2 - \gamma_+^2)\Delta^2 + (N g_0^2 - \gamma_-^2 + \gamma_+^2)^2 \\
&= \Delta^4 - 2(\Omega^2 - \gamma_+^2)\Delta^2 + (\Omega^2 + \gamma_+^2)^2 \\
&= \Delta^4 + 2(\Omega^2 + \gamma_+^2)\Delta^2 + (\Omega^2 + \gamma_+^2)^2 - 4\Omega^2\Delta^2 \\
&= (\Delta^2 + \Omega^2 + \gamma_+^2)^2 - (2\Omega\Delta)^2 \\
&= (\Delta^2 + \Omega^2 + \gamma_+^2 - 2\Omega\Delta)(\Delta^2 + \Omega^2 + \gamma_+^2 + 2\Omega\Delta) \\
&= \left[(\Delta - \Omega)^2 + \gamma_+^2\right]\left[(\Delta + \Omega)^2 + \gamma_+^2\right]
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\gamma_+ &\equiv \frac{1}{2}(\gamma_c + \gamma_p), \\
\gamma_- &\equiv \frac{1}{2}(\gamma_c - \gamma_p), \\
\Omega &\equiv \sqrt{N g_0^2 - \gamma_-^2}.
\end{aligned} \tag{36}$$

The transmittance then becomes

$$\mathcal{T} \propto \frac{\Delta^2 + \gamma_p^2}{\left[(\Delta - \Omega)^2 + \gamma_+^2\right]\left[(\Delta + \Omega)^2 + \gamma_+^2\right]}, \tag{37}$$

which is a product of two Lorentzian curves centered around $\omega_0 \pm \Omega$. In the limit of strong coupling, *i.e.*, $g_0 \gg \gamma_c, \gamma_p$, the separation between the two peaks is

$$2\Omega = 2\sqrt{N g_0^2 - \gamma_-^2} \simeq 2\sqrt{N} g_0. \tag{38}$$

II. CLASSICAL ANALYSIS OF NORMAL MODE SPLITTING

In this approach, both atom and field are treated classically. Consider a classical oscillator, which is made of a point positive charge e surrounded by a uniform rigid charge distribution of negative charge $-e$. The radius of the negative charge distribution is the same as the Bohr radius a_0. Such a classical oscillator is placed in a cavity, which is externally probed by a classical field. We want to calculate the transmittance of the oscillator-cavity system.

A. Susceptability of a Classical Oscillator

Suppose the charge distribution is displaced by x. The Coulomb force is then given by

$$F = -\frac{e^2(x/a_0)^3}{x^2} = -\frac{e^2}{a_0^3}x = -k_{\text{eff}}x \tag{39}$$

which is a restoring force with an effective spring constant k_{eff}. The harmonic oscillation frequency ω_0 of this classical oscillator is thus

$$\omega_0 = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{e^2}{m a_0^3}}. \tag{40}$$

Suppose this oscillator is driven by an electric field of frequency ω. The equation of motion is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = -\frac{eE(t)}{m} = -\frac{eE_0}{m}e^{-i\omega t}, \tag{41}$$

where 2γ is the damping constant (full width) and the direction of the displacement x is the same as the direction of the applied electric field. The solution is

$$x(t) = -\frac{eE(t)/m}{\omega_0^2 - 2i\gamma\omega - \omega^2}. \tag{42}$$

Suppose there are N such oscillators in a volume V. The polarization $P(t)$ is then

$$P(t) = -Nex(t)/V = \frac{(Ne^2/mV)E(t)}{\omega_0^2 - 2i\gamma\omega - \omega^2} \equiv \chi E(t), \tag{43}$$

where χ is the electrical susceptability given by

$$\chi = \frac{(Ne^2/mV)}{\omega_0^2 - 2i\gamma\omega - \omega^2} \tag{44}$$

Dielectric constant is $1 + 4\pi\chi$ and its square root is the index of refraction n.

$$n = \sqrt{1 + 4\pi\chi} = \sqrt{1 + \frac{(4\pi Ne^2/mV)}{\omega_0^2 - 2i\gamma\omega - \omega^2}} \tag{45}$$

We define the plasma frequency ω_p as

$$\omega_p^2 \equiv = \frac{4\pi Ne^2}{mV}. \tag{46}$$

In our consideration, because of low density, the plasma frequency ω_p is much smaller than ω_0, and thus we can expand the square root in n as

$$n \simeq 1 + \frac{1}{2}\frac{\omega_p^2}{\omega_0^2 - 2i\gamma\omega - \omega^2} \tag{47}$$

For near resonance, it is further simplified as

$$\begin{aligned} n &\simeq 1 - \frac{\omega_p^2/4\omega_0}{(\omega - \omega_0) + i\gamma} \\ &= \left[1 - \frac{\omega_p^2}{4\omega_0}\frac{\Delta_p}{\Delta_p^2 + \gamma^2}\right] + i\frac{\omega_p^2}{4\omega_0}\frac{\gamma}{\Delta_p^2 + \gamma^2} \\ &\equiv n_r + in_i, \end{aligned} \tag{48}$$

where n_r, n_i are real and imaginary parts of n.

B. Cavity Transmission

Suppose a Fabry-Perot-type cavity with a reflectance $R = |r|^2$ on each mirror and with a separation of L. The cavity is filled with a medium (*i.e.*, collection of classical oscillators) of refractive index n given by Eq. (48). Cavity transmission is obtained by considering multiple reflections between the mirrors and summing up all multiply reflected and transmitted components.

$$\begin{aligned} E_t &= E_0(1-R)e^{ikL}\left[1 + Re^{2ikL} + R^2e^{4ikL} + \cdots\right] \\ &= \frac{E_0(1-R)e^{ikL}}{1 - Re^{2ikL}}, \end{aligned} \tag{49}$$

where $k = n\omega/c$, which is complex. Tansmittance is given by

$$\begin{aligned} \mathcal{T} &= \left|\frac{E_t}{E_0}\right|^2 = \left|\frac{(1-R)e^{ikL}}{1 - Re^{2ikL}}\right|^2 \\ &= \frac{(1-R)^2 e^{i(k-k*)L}}{1 + R^2 e^{2i(k-k*)L} - R(e^{2ikL} + e^{-2ik*L})} \\ &= \frac{(1-R)^2 e^{-2k_iL}}{1 + R^2e^{-4k_iL} - 2Re^{-2ik_iL}\cos 2k_rL} \\ &= \frac{(1-R)^2 e^{-2k_iL}}{(1 - Re^{-2k_iL})^2 + 4Re^{-2ik_iL}\sin^2 k_rL}. \end{aligned} \tag{50}$$

C. Normal Mode Splitting in the Cavity Transmission

Now suppose the resonance frequency of the empty cavity is ω_c satisfying the resonance condition.

$$\frac{\omega_c}{c}L = q\pi, \tag{51}$$

where q is an integer. Then the argument of sine function at near resonance ($\omega \sim \omega_c$) can be expanded as

$$\begin{aligned} k_rL &= \frac{\omega}{c}L\left[1 - \frac{\omega_p^2}{4\omega_0}\frac{\Delta_p}{\Delta_p^2 + \gamma^2}\right] \\ &= (\Delta_c + \omega_c)\frac{L}{c}\left[1 - \frac{\omega_p^2}{4\omega_0}\frac{\Delta_p}{\Delta_p^2 + \gamma^2}\right] \\ &\simeq q\pi + \frac{\Delta_cL}{c} - \frac{\omega_cL}{c}\frac{\omega_p^2}{4\omega_0}\frac{\Delta_p}{\Delta_p^2 + \gamma^2} \end{aligned} \tag{52}$$

or

$$k_rL - q\pi \simeq \frac{\Delta_cL}{c} - \frac{\omega_p^2L}{4c}\frac{\Delta_p}{\Delta_p^2 + \gamma^2}, \tag{53}$$

where $\Delta_c = \omega - \omega_c$ and $\Delta_p = \omega - \omega_p$. The transmittance in Eq. (50) has maxima when the righthand side of Eq. (53) vanishes. Of particular interest is when $\Delta_c = \Delta_p = \Delta$ (zero atom-cavity detuning case). In this case,

$$\Delta - \frac{\omega_p^2}{4}\frac{\Delta}{\Delta^2 + \gamma^2} = 0, \tag{54}$$

or

$$\Delta = 0, \quad \Delta = \pm\sqrt{\frac{\omega_p^2}{4} - \gamma^2} \tag{55}$$

In the limit of negligible damping ($\omega_p \gg \gamma$), the nonvanishing zeros of the sine function becomes

$$\Delta \simeq \pm\frac{\omega_p}{2} = \pm\sqrt{\frac{\pi Ne^2}{mV}} \tag{56}$$

The resulting transmittance has only two peaks at these detunings. The detuning value $\Delta = 0$ does not give rise to a peak because of strong absorption at the resonance frequency of the oscillator. Specifically, the first exponential factor $e^{-2k_i L}$ in the denominator in Eq. (50) becomes small at zero detuning, making the transmittance become negligibly small.

Exercise

In the classical analysis, the separation of two peaks in the transmittance is given by $2\sqrt{\pi N e^2/mV}$ whereas in the semiclassical analysis it is given by $2\sqrt{N}g_0$. We can show these seemingly different two expressions are equivalent in fact. The proof is left as an exercise for the winter camp participants.