Chapter 3. Radiative Interactions

3.1 Review of Field Quantization

From classical electrodynamics we know that an electromagnetic field can be derived from scalar and vector potentials, $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$. We also know that with a proper choice of gauge the wave equations for the potentials can be greatly simplified. For a radiation field (far away from the source) it is convenient to choose the Coulomb gauge,

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) = 0.$$

The scalar potential then satisfies the Poisson equation as in a static case, and thus the electric field decays like $1/r^2$ from the source. Therefore, the scalar potential does not contribute to the electric field in the radiation zone. The electromagnetic field is then given by the vector potential alone:

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t), \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t), \tag{3.1.1}$$

where the vector potential satisfies a homogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = 0. \tag{3.1.2}$$

The solution of the above homogeneous equation in free space is a plane wave, which can be represented by a sum of two opposite traveling waves

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \hat{\mathbf{e}}_{\mathbf{k}\lambda}\left\{A_{\mathbf{k}\lambda}\exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)\right] + A^*_{\mathbf{k}\lambda}\exp\left[-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}}t)\right]\right\}, \tag{3.1.3}$$

where $\hat{\mathbf{e}}_{\mathbf{k}\lambda}$ is an unit vector representing the polarization. The frequency $\omega_{\mathbf{k}}$ is given by the dispersion relation, $|\mathbf{k}| = \omega_{\mathbf{k}}/c$. The Coulomb gauge ensures the transversality of the electromagnetic fields, $\mathbf{k}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda} = 0$. The possible wave vectors are determined by a boundary condition for a cubic cavity of side length of *L*. It is customary to employ a periodic boundary condition (alternatively, one can consider a field in a conducting rectangular box, of course),

$$e^{ik_x x} = e^{ik_x(x+L)}, \quad etc, \tag{3.1.4}$$

which result in an infinite set of wave vectors given by

$$\mathbf{k} = \frac{2\pi}{L}\left(N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}} + N_z\hat{\mathbf{k}}\right), \quad N_x, N_y, N_z = 0, \pm 1, \pm 2, \ldots \tag{3.1.5}$$

The number of modes per unit frequency interval is obtained by considering the number of modes in $d^3\mathbf{k}$:

$$2\frac{d^3\mathbf{k}}{(2\pi/L)^3} = 2\frac{k^2 dk d\Omega}{(2\pi/L)^3} = 2\frac{V\omega_{\mathbf{k}}^2 d\omega_{\mathbf{k}} d\Omega}{(2\pi c)^3}, \tag{3.1.6}$$

where a factor of 2 is needed for accounting for the two possible polarizations. The number of modes per unit frequency interval is then given by

$$\frac{V\omega_{\mathbf{k}}^2 d\Omega}{4\pi^3 c^3}.$$

The Hamiltonian of the electromagnetic field in the cavity is obtained by integrating the field energy density over the cavity volume.

$$W = \frac{1}{8\pi}\int_V \left(\mathbf{E}^2 + \mathbf{B}^2\right) dV \tag{3.1.7}$$

Exercise 1. Using the orthogonality of the plane waves with different **k**'s, show that

$$W = \frac{V}{2\pi c^2}\sum_{\mathbf{k}\lambda} \omega_{\mathbf{k}}^2 \left|A_{\mathbf{k}\lambda}(t)\right|^2. \tag{3.1.8}$$

where $A_{\mathbf{k}\lambda}(t) = A_{\mathbf{k}\lambda} e^{-i\omega_{\mathbf{k}} t}$.

We now introduce a new set of real variables $P_{\mathbf{k}\lambda}$ and $Q_{\mathbf{k}\lambda}$.

$$A_{\mathbf{k}\lambda} = \frac{c}{\omega_{\mathbf{k}}}\sqrt{\frac{\pi}{V}}\left(\omega_{\mathbf{k}} Q_{\mathbf{k}\lambda} + iP_{\mathbf{k}\lambda}\right), \quad A_{\mathbf{k}\lambda}^* = \frac{c}{\omega_{\mathbf{k}}}\sqrt{\frac{\pi}{V}}\left(\omega_{\mathbf{k}} Q_{\mathbf{k}\lambda} - iP_{\mathbf{k}\lambda}\right), \tag{3.1.9}$$

in terms of which the field energy becomes

$$W = \frac{1}{2}\sum_{\mathbf{k}\lambda}\left(P_{\mathbf{k}\lambda}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda}^2\right). \tag{3.1.10}$$

From the time dependence of $\dot{A}_{\mathbf{k}\lambda}(t) = -i\omega_{\mathbf{k}} A_{\mathbf{k}\lambda}(t), etc,$ one can show the equation of motion for $Q_{\mathbf{k}\lambda}$ and $P_{\mathbf{k}\lambda}$ to be

$$\dot{Q}_{\mathbf{k}\lambda} = P_{\mathbf{k}\lambda}, \quad \dot{P}_{\mathbf{k}\lambda} = -\omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda}, \tag{3.1.11}$$

which is exactly the Hamiltonian equation one would obtain from the above field energy while treating it a Hamiltonian with canonical variables $Q_{\mathbf{k}\lambda}$ and $P_{\mathbf{k}\lambda}$. Therefore, one identify the Hamiltonian for the electromagnetic field and associated *canonical variables*:

$$H = \frac{1}{2}\sum_{\mathbf{k}\lambda}\left(P_{\mathbf{k}\lambda}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda}^2\right). \tag{3.1.12}$$

Field quantization then proceeds as in the case of a harmonic oscillator. We impose commutation relations for the canonical variables as

$$\left[Q_{\mathbf{k}\lambda},P_{\mathbf{k}'\lambda'}\right]=i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'},\quad \left[Q_{\mathbf{k}\lambda},Q_{\mathbf{k}'\lambda'}\right]=\left[P_{\mathbf{k}\lambda},P_{\mathbf{k}'\lambda'}\right]=0. \tag{3.1.13}$$

We define a set of new variables as

$$a_{\mathbf{k}\lambda}=\frac{1}{\sqrt{2\hbar\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}Q_{\mathbf{k}\lambda}+iP_{\mathbf{k}\lambda}\right),\quad a_{\mathbf{k}\lambda}^{+}=\frac{1}{\sqrt{2\hbar\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}Q_{\mathbf{k}\lambda}-iP_{\mathbf{k}\lambda}\right), \tag{3.1.14}$$

which then obeys

$$\left[a_{\mathbf{k}\lambda},a_{\mathbf{k}'\lambda'}^{+}\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'},\quad \left[a_{\mathbf{k}\lambda},a_{\mathbf{k}'\lambda'}\right]=\left[a_{\mathbf{k}\lambda}^{+},a_{\mathbf{k}'\lambda'}^{+}\right]=0, \tag{3.1.15}$$

and the Hamiltonian becomes

$$H=\sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k}\lambda}^{+}a_{\mathbf{k}\lambda}+1/2\right). \tag{3.1.16}$$

Exercise 2. Using the commutation relations of $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^{+}$, show that $a_{\mathbf{k}\lambda}^{+}$ is a creation operator and $a_{\mathbf{k}\lambda}$ is an annihilation operator satisfying

$$a_{\mathbf{k}\lambda}\left|n_{k\lambda}\right\rangle=\sqrt{n_{k\lambda}}\left|n_{k\lambda}-1\right\rangle,\quad a_{\mathbf{k}\lambda}^{+}\left|n_{k\lambda}\right\rangle=\sqrt{n_{k\lambda}+1}\left|n_{k\lambda}+1\right\rangle, \tag{3.1.17}$$

where $\left|n_{k\lambda}\right\rangle$ is the eigenstate of the number operator $N_{\mathbf{k}\lambda}=a_{\mathbf{k}\lambda}^{+}a_{\mathbf{k}\lambda}$.

Exercise 3. Show that in the *Schrödinger picture* (meaning the time dependence is absorbed in the wave function) the vector potential can be written in terms of $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^{+}$ as

$$\mathbf{A}(\mathbf{r})=\sum_{\mathbf{k}\lambda}\sqrt{\frac{2\pi\hbar c^{2}}{\omega_{\mathbf{k}}V}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}\left\{a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}+a_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\cdot\mathbf{r}}\right\} \tag{3.1.18}$$

and the corresponding electric and magnetic field operators as

$$\mathbf{E}(\mathbf{r})=\sum_{\mathbf{k}\lambda}\left[\mathbf{E}_{\mathbf{k}\lambda}^{(-)}+\mathbf{E}_{\mathbf{k}\lambda}^{(+)}\right],\quad \mathbf{B}(\mathbf{r})=\sum_{\mathbf{k}\lambda}\left[\mathbf{B}_{\mathbf{k}\lambda}^{(-)}+\mathbf{B}_{\mathbf{k}\lambda}^{(+)}\right], \tag{3.1.19}$$

where

$$\begin{aligned}&\mathbf{E}_{\mathbf{k}\lambda}^{(-)}=i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}},\quad \mathbf{E}_{\mathbf{k}\lambda}^{(+)}=-i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\cdot\mathbf{r}},\\ &\mathbf{B}_{\mathbf{k}\lambda}^{(-)}=i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\left(\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}},\quad \mathbf{B}_{\mathbf{k}\lambda}^{(+)}=-i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\left(\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\cdot\mathbf{r}}.\end{aligned} \tag{3.1.20}$$

3.2 Interaction with the Radiation Fields

Total Hamiltonian can be written as

$$\begin{aligned}H&=H_{atom}+H_{field}+H_{int}\\&=\left(\frac{\mathbf{p}^{2}}{2m}+V\right)+\sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k}\lambda}^{+}a_{\mathbf{k}\lambda}+1/2\right)+H_{int},\end{aligned} \tag{3.2.1}$$

where the interaction Hamiltonian is given by

$$\begin{aligned} H_{int} &= \frac{e}{2mc}\left(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}\right)+\frac{e^2}{2mc^2}A^2+\frac{e\hbar}{2mc}\sigma\cdot\nabla\times\mathbf{A} \\ &= \frac{e}{mc}\mathbf{p}\cdot\mathbf{A}+\frac{e^2}{2mc^2}A^2+\frac{e\hbar}{2mc}\sigma\cdot\nabla\times\mathbf{A}. \end{aligned} \tag{3.2.2}$$

under the Coulomb gauge ($\nabla\cdot\mathbf{A}=0$). We shall concentrate on the leading term, which can be written as

$$H_1=\frac{e}{mc}\mathbf{p}\cdot\mathbf{A}=\underbrace{\frac{e}{mc}\sum_{\mathbf{k}\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}}\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}}_{H_1^{(-)}}+\underbrace{\frac{e}{mc}\sum_{\mathbf{k}\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}}\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\cdot\mathbf{r}}}_{H_1^{(+)}}. \tag{3.2.3}$$

Since $H_1^{(-)}$ is proportional to the annihilation operators, nonvanishing matrix elements corresponding to photon absorption are

$$\left\langle b;n_{k\lambda}-1\left|H_1^{(-)}\right|a;n_{k\lambda}\right\rangle=\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_{\mathbf{k}}V}}\left\langle b;n_{k\lambda}-1\left|\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\right|a;n_{k\lambda}\right\rangle=\frac{e}{m}\sqrt{\frac{2\pi\hbar n_{k\lambda}}{\omega_{\mathbf{k}}V}}\left\langle b\left|\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)e^{i\mathbf{k}\cdot\mathbf{r}}\right|a\right\rangle, \tag{3.2.4}$$

where *a* and *b* denote atomic lower and upper states, respectively. The field state should be the same before and after except the $\mathbf{k}\lambda$ mode, for which the number of photon has been decreased by 1 due to the absorption. Similarly, nonvanishing matrix elements for $H_1^{(+)}$ are in the form of

$$\left\langle a;n_{k\lambda}+1\left|H_1^{(+)}\right|b;n_{k\lambda}\right\rangle=\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_{\mathbf{k}}V}}\left\langle a;n_{k\lambda}+1\left|\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)a_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\cdot\mathbf{r}}\right|b;n_{k\lambda}\right\rangle=\frac{e}{m}\sqrt{\frac{2\pi\hbar(n_{k\lambda}+1)}{\omega_{\mathbf{k}}V}}\left\langle a\left|\left(\mathbf{p}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right)e^{-i\mathbf{k}\cdot\mathbf{r}}\right|b\right\rangle, \tag{3.2.5}$$

corresponding to photon emission.

Electric Dipole Approximation

If the atomic dimension is negligibly small compared to the wavelength (long wavelength limit), the exponential factor can be approximated by $e^{\pm i\mathbf{k}\cdot\mathbf{r}}\approx 1$. The higher order multipoles in a series expansion of the vector potential scale as d/λ with *d* the dimension of the atom. Therefore, the long wavelength limit is equivalent to the electric dipole approximation (E1). Since

$$\begin{aligned}\left[x,H_{atom}\right]&=\left[x,p^2/2m\right]=i\left(\hbar/m\right)p_x,\\ \left\langle b\left|\mathbf{p}\right|a\right\rangle&=\left(m/i\hbar\right)\left\langle b\left|\left[\mathbf{r},H_{atom}\right]\right|a\right\rangle=\left(m/i\hbar\right)\left(E_a-E_b\right)\left\langle b\left|\mathbf{r}\right|a\right\rangle\cong im\omega_{\mathbf{k}}\left\langle b\left|\mathbf{r}\right|a\right\rangle,\end{aligned}\tag{3.2.6}$$

where $E_b-E_a=\hbar\omega_{ba}\approx\hbar\omega_{\mathbf{k}}$ (a: lower, b: upper state) for photon absorption to occur, and therefore,

$$\left\langle b;n_{k\lambda}-1\left|H_1^{(-)}\right|a;n_{k\lambda}\right\rangle=ie\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}n_{k\lambda}}{V}}\left\langle b\left|\mathbf{r}\right|a\right\rangle\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}.\tag{3.2.7}$$

Likewise,

$$\left\langle a;n_{k\lambda}+1\left|H_1^{(+)}\right|b;n_{k\lambda}\right\rangle=-ie\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}\left(n_{k\lambda}+1\right)}{V}}\left\langle a\left|\mathbf{r}\right|b\right\rangle\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}.\tag{3.2.8}$$

Exercise 4. Show that

$$\left\langle f;n_{\mathbf{k}\lambda}\pm1\left|H_1^{(\pm)}\right|i;n_{\mathbf{k}\lambda}\right\rangle=\left\langle f;n_{\mathbf{k}\lambda}\pm1\left|\left(e\mathbf{r}\cdot\mathbf{E}_{\mathbf{k}\lambda}^{(\pm)}\right)\right|i;n_{\mathbf{k}\lambda}\right\rangle.\tag{3.2.9}$$

Unitary Transformation to the Form of Dipole Interaction

Under a unitary transformation the Hamiltonian H can be cast in a form of electric dipole interaction. Consider a unitary operator

$$O=\exp\left[-\left(ie/\hbar c\right)\mathbf{A}\left(\mathbf{r},t\right)\cdot\mathbf{r}\right].\tag{3.2.10}$$

Within a region of atomic dimension the position dependence in **A** *can be neglected*. This assumption is equivalent to the long wavelength limit. We perform the unitary operation on the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi=H\psi.\tag{3.2.11}$$

Since

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}O^+\psi&=i\hbar\frac{\partial O^+}{\partial t}\psi+i\hbar O^+\frac{\partial\psi}{\partial t}=-\frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r}O^+\psi+O^+i\hbar\frac{\partial\psi}{\partial t}\\ &=-\frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r}O^+\psi+O^+H\psi=\left[O^+HO-\frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r}\right]O^+\psi,\end{aligned}\tag{3.2.12}$$

the Schrödinger equation is transformed to

$$i\hbar\frac{\partial}{\partial t}\psi'=H'\psi'\tag{3.2.13}$$

where

$$\psi'=O^+\psi,\quad H'=O^+HO-\frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r}.\tag{3.2.14}$$

Then our Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2, \tag{3.2.15}$$

which corresponds to both H_{atom} and H_{int} before, is transformed to

$$H' = \frac{1}{2m}O^{+}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 O - \frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r}. \tag{3.2.16}$$

Consider

$$\mathbf{p}O = \frac{\hbar}{i}\nabla\exp\left[-\frac{ie}{\hbar c}\mathbf{A}(t)\cdot\mathbf{r}\right] = -\frac{\hbar}{i}\frac{ie}{\hbar c}\mathbf{A}(t)\exp\left[\frac{ie}{\hbar c}\mathbf{A}(t)\cdot\mathbf{r}\right] + O\mathbf{p} = -\frac{e}{c}\mathbf{A}O + O\mathbf{p}. \tag{3.2.17}$$

Thus

$$\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 O = \left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)O\mathbf{p} = O\mathbf{p}^2, \tag{3.2.18}$$

and therefore,

$$H' = \frac{\mathbf{p}^2}{2m} - \frac{e}{c}\dot{\mathbf{A}}(t)\cdot\mathbf{r} = \frac{\mathbf{p}^2}{2m} + e\mathbf{r}\cdot\mathbf{E}(t) = \frac{\mathbf{p}^2}{2m} - \mathbf{d}\cdot\mathbf{E}(t), \tag{3.2.19}$$

where $\mathbf{d}$=-$e\mathbf{r}$ is the electric dipole moment operator. This result is consistent with the result of Exercise 4. However, it should be noted that the Hamiltonian H contains not only the electric dipole term $\mathbf{p}\cdot\mathbf{A}$ but also higher order terms. This seeming contradiction is resolved by the fact that H' was obtained with the assumption that $\mathbf{A}$ is position-independent, which is consistent with the long wavelength limit or the dipole approximation, $e^{\pm i\mathbf{k}\cdot\mathbf{r}} \approx 1$. Although H seems to include higher order terms, those higher order terms were neglected in practice in the transformation process.

3.5 Transition Probabilities

Consider an atom interacting with a radiation field of frequency $\omega_{\mathbf{k}}$. Assume that only a transition between particular two states (let us call them level a and level b) is resonant with the radiation field. Let the transition frequency be $\omega_{ba} = (E_b - E_a)/\hbar > 0$. The approximation that we can neglect the other levels is called two-level approximation and the atom in this case is called a two level system.

The transition probability from an initial state to a final state can be calculated from the Fermi Golden Rule:

$$W_{fi} = \frac{2\pi}{\hbar}\left|\langle f|V|i\rangle\right|^2 \delta(E_f - E_i) \tag{3.5.1}$$

For E1 absorption process in which the two-level atom interacts with a field mode $\mathbf{k}\lambda$,

$$\begin{aligned} W_{abs} &= \frac{2\pi}{\hbar}\left|\langle b, n_{\mathbf{k}} - 1|e\mathbf{R}\cdot\mathbf{E}^{(-)}_{\mathbf{k}\lambda}|a, n_{\mathbf{k}}\rangle\right|^2 \delta(E_b - E_a - \hbar\omega_{\mathbf{k}}) \\ &= \frac{2\pi}{\hbar^2}\left(e\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}} n_{k\lambda}}{V}}\right)^2 \left|\langle b|\mathbf{R}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}|a\rangle\right|^2 \delta(\omega_{ba} - \omega_{\mathbf{k}}) \\ &= \frac{4\pi^2 e^2 \omega_{\mathbf{k}} n_{\mathbf{k}\lambda}}{\hbar V}\left|\langle b|\mathbf{R}|a\rangle\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right|^2 \delta(\omega_{ba} - \omega_{\mathbf{k}}). \end{aligned} \tag{3.5.2}$$

When we have a distribution of modes instead of the infinitely-narrow-frequency mode, the above expression should be multiplied by the number of modes in a frequency interval in vacuum and integrated over all frequencies. The number of modes in a frequency interval per polarization direction is

$$\frac{V\omega_{\mathbf{k}}^2 d\Omega}{(2\pi)^3 c^3}, \tag{3.5.3}$$

and thus

$$\begin{aligned} dW_{abs} &= \int_0^\infty d\omega_{\mathbf{k}} \frac{V\omega_{\mathbf{k}}^2 d\Omega}{(2\pi)^3 c^3} \frac{4\pi^2 e^2 \omega_{\mathbf{k}} n_{\mathbf{k}\lambda}}{\hbar V}\left|\langle b|\mathbf{R}|a\rangle\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right|^2 \delta(\omega_{ba} - \omega_{\mathbf{k}}) \\ &= \frac{e^2 \omega_{\mathbf{k}}^3 n_{\mathbf{k}\lambda}}{2\pi\hbar c^3}\left|\langle b|\mathbf{R}|a\rangle\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}\right|^2 d\Omega. \end{aligned} \tag{3.5.4}$$

which then needs be integrated over all angles accounting for two possible polarization directions. We choose the polarization directions as shown in **Fig. 1**.

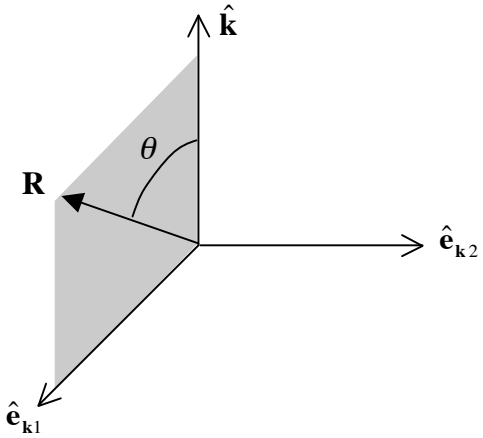

Fig. 1. Our choice of polarization vectors with respect to the position vector **R**.

Then,

$$W_{abs} = \int \frac{e^2\omega_{ba}^3 n_{\mathbf{k}\lambda}}{2\pi\hbar c^3}\left|\langle b|\mathbf{R}|a\rangle\right|^2 \sin^2\theta_k d\Omega = \frac{e^2\omega_{ba}^3 n}{2\pi\hbar c^3}\left|\langle b|\mathbf{R}|a\rangle\right|^2\left[2\pi\int_{-1}^{1}\sin^2\theta_k d(\cos\theta_k)\right]$$
$$= \frac{e^2\omega_{ba}^3 n}{2\pi\hbar c^3}\left|\langle b|\mathbf{R}|a\rangle\right|^2\left(\frac{8\pi}{3}\right) = \frac{4\omega_{ba}^3 e^2 n}{3\hbar c^3}\left|\langle b|\mathbf{R}|a\rangle\right|^2 = \left[\frac{4|\mu_{ab}|^2}{3\hbar}\left(\frac{\omega_{ba}}{c}\right)^3\right]n. \tag{3.5.5}$$

For E1 emission process, the procedure proceeds in the same way except that the result is proportional to n+1.

$$W_{em} = \left[\frac{4|\mu_{ab}|^2}{3\hbar}\left(\frac{\omega_{ba}}{c}\right)^3\right](n+1). \tag{3.5.6}$$

Emission can occur even in the absence of photons in the mode. Such emission is called the *spontaneous emission* whereas the part proportional to n is called the *stimulated emission*. The absorption with its rate proportional to n is by definition a stimulated process. The spontaneous emission rate

$$W_{sp} = \frac{4|\mu_{ab}|^2}{3\hbar}\left(\frac{\omega_{ba}}{c}\right)^3 \tag{3.5.7}$$

is also known as the Einstein A coefficient. The inverse of this rate is called the *radiative lifetime* of the excited state. As clearly demonstrated so far, the spontaneous emission is a true quantum mechanical effect, not explainable in the classical electromagnetism. It is remarkable that Einstein introduced the concept of spontaneous emission in his consideration of the black body radiation, solely based on the thermodynamic detailed balance argument.

Extension to Multilevels

Suppose that the level *a* consists of degenerate sublevels $|\alpha_a J_a M_a\rangle$ with degeneracy $g_a (= 2J_a + 1)$. Likewise, the level *b* consists of degenerate sublevels $|\alpha_b J_b M_b\rangle$ with degeneracy $g_b (= 2J_b + 1)$. When the sublevels of level *a* are equally populated initially, the absorption rate becomes

$$W_{abs} = \frac{1}{g_a} \sum_{M_a M_b} \frac{4\omega_{ba}^3 e^2 n}{3\hbar c^3} \left| \langle \alpha_b J_b M_b | \mathbf{R} | \alpha_a J_a M_a \rangle \right|^2, \tag{3.5.8}$$

Similarly, the emission rate becomes

$$W_{em} = \frac{1}{g_b} \sum_{M_a M_b} \frac{4\omega_{ba}^3 e^2 (n+1)}{3\hbar c^3} \left| \langle \alpha_b J_b M_b | \mathbf{R} | \alpha_a J_a M_a \rangle \right|^2 \tag{3.5.9}$$

if the sublevels of level *b* are equally populated initially.

3.8 Effects of Intense Laser Field

Dressed State Picture

When a two-state atom interacts with a near-resonant electromagnetic field, the atom can be easily excited to the upper state and thus the transition probability calculated before, which is valid only when the atom stays mostly in the lower state and the probability that atom is excited is negligibly small. For the near-resonant field, we need to sum up all orders of perturbative transition amplitudes. In the semiclassical approach, we need to use the optical Bloch equation for a correct description of the problem. In the quantum mechanical approach, we use the dressed picture to be discussed in this section.

Recall that the total Hamiltonian is composed of three parts.

$$\begin{aligned} H &= H_{atom} + H_{field} + H_{int} \\ &= \frac{\mathbf{p}^2}{2m} + V + \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}\lambda}^{+} a_{\mathbf{k}\lambda} + 1/2 \right) + H_{int}, \end{aligned} \tag{3.8.1}$$

For the E1 interaction

$$H_{int} = -\mathbf{d}\cdot\mathbf{E} = e\mathbf{r}\cdot\left[i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}\left(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^{+} \right) \right], \tag{3.8.2}$$

for an electromagnetic mode $\mathbf{k}\lambda$. We compose uncoupled (without H_{int}) atom-field states denoted by $|g,n+1\rangle$ and $|e,n\rangle$, where g (e) stands for the lower (upper) state and n indicates the number of photons in the electromagnetic field mode. Our task is to diagonalize the total Hamiltonian in this basis. We first calculate the interaction matrix elements between uncoupled states. Only nonzero element is

$$\langle e,n|H_{int}|g,n+1\rangle = ie\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\langle e|\mathbf{r}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}|g\rangle\sqrt{n_{\mathbf{k}\lambda}+1} \tag{3.8.3}$$

From now on we drop the subscript $\mathbf{k}\lambda$. If we define a coupling constant

$$g \equiv \frac{\mu_{eg}}{\hbar}\sqrt{\frac{2\pi\hbar\omega}{V}}, \tag{3.8.4}$$

where $\mu_{eg} \equiv ie\langle e|\mathbf{r}\cdot\hat{\mathbf{e}}_{\mathbf{k}\lambda}|g\rangle$ (we choose the atomic state phases in such a way that μ_{eg} is real), the interaction Hamiltonian can be written as

$$H_{\text{int}} = \begin{bmatrix} 0 & g\sqrt{n+1} \\ g\sqrt{n+1} & 0 \end{bmatrix}. \tag{3.8.5}$$

In the uncoupled basis, the atom and field Hamiltonians are given by

$$H_{atom} = \begin{bmatrix} \hbar\omega_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{field} = \begin{bmatrix} (n+1/2)\hbar\omega & 0 \\ 0 & (n+3/2)\hbar\omega \end{bmatrix}, \tag{3.8.6}$$

and thus the total Hamiltonian can be written as

$$H/\hbar = \frac{1}{2}\begin{bmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{bmatrix} + \begin{bmatrix} (n+1/2)\omega & 0 \\ 0 & (n+3/2)\omega \end{bmatrix} + \begin{bmatrix} 0 & g\sqrt{n+1} \\ g\sqrt{n+1} & 0 \end{bmatrix} + \frac{\omega_0}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.8.7}$$

The last term can be dropped by rescaling the origin of energy.

Exercise 6. Show that the following Hamiltonian has the same matrix form as Eq. (3.8.7) except for a constant term, and thus it can replace the total Hamiltonian, Eq. (3.8.1), when only a single radiation mode is involoved.

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\left(a^+a + 1/2\right) + \hbar g\left(\sigma_+ a + \sigma_- a^+\right). \tag{3.8.8}$$

where $\sigma_+ \equiv \left(\sigma_x + i\sigma_y\right)/2$, $\sigma_- \equiv \left(\sigma_x - i\sigma_y\right)/2$ and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

We can rescale the origin of energy further and can rewrite Eq. (3.8.7) as

$$H/\hbar = \begin{bmatrix} 0 & 0 \\ 0 & -\omega_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega \end{bmatrix} + \begin{bmatrix} 0 & g\sqrt{n+1} \\ g\sqrt{n+1} & 0 \end{bmatrix}. \tag{3.8.9}$$

The secular equation is then

$$\begin{vmatrix} -\nu & g\sqrt{n+1} \\ g\sqrt{n+1} & \delta - \nu \end{vmatrix} = 0, \tag{3.8.10}$$

where $\delta \equiv \omega - \omega_0$, the laser-atom detuning. The secular equation gives

$$\begin{aligned} &\nu^2 - \delta\nu - g^2(n+1) = 0, \\ &\nu_\pm = \delta/2 \pm \sqrt{\left(\delta/2\right)^2 + g^2(n+1)} = g\sqrt{n+1}\left(x \pm \sqrt{x^2+1}\right), \end{aligned} \tag{3.8.11}$$

where

$$x = \delta/\left(2g\sqrt{n+1}\right). \tag{3.8.12}$$

Let the plus (+) energy state is denoted by

$$\left|+,n\right\rangle = \alpha\left|e,n\right\rangle + \beta\left|g,n+1\right\rangle. \tag{3.8.13}$$

Then

$$0 = -\nu_+\alpha + g\sqrt{n+1}\beta = -\left[\delta/2 + \sqrt{\left(\delta/2\right)^2 + g^2(n+1)}\right]\alpha + g\sqrt{n+1}\beta.$$

(3.8.14)

$$\alpha = \frac{g\sqrt{n+1}}{\sqrt{g^2(n+1)+\left[\delta/2+\sqrt{(\delta/2)^2+g^2(n+1)}\right]^2}} = \frac{1}{\sqrt{1+\left(x+\sqrt{x^2+1}\right)^2}} \equiv \cos\theta,$$

$$\beta = \frac{\delta/2+\sqrt{(\delta/2)^2+g^2(n+1)}}{\sqrt{g^2(n+1)+\left[\delta/2+\sqrt{(\delta/2)^2+g^2(n+1)}\right]^2}} = \frac{x+\sqrt{x^2+1}}{\sqrt{1+\left(x+\sqrt{x^2+1}\right)^2}} \equiv \sin\theta.$$

(3.8.15)

The minus energy state, orthogonal to the plus state, is then given by

$$|-,n\rangle = \beta|e,n\rangle - \alpha|g,n+1\rangle. \tag{3.8.16}$$

i) For zero detuning (x=0), $\alpha = \beta = 1/\sqrt{2}$. $\nu_\pm = \pm g\sqrt{n+1}$.

$$|\pm,n\rangle \approx \frac{1}{\sqrt{2}}\left(|e,n\rangle \pm |g,n+1\rangle\right). \tag{3.8.17}$$

ii) For a large positive detuning (x>>1), $\alpha \approx 0$, $\beta \approx 1$. $\nu_\pm = \pm\delta/2$.

$$|+,n\rangle \approx |g,n+1\rangle, \quad |-,n\rangle \approx |e,n\rangle. \tag{3.8.18}$$

iii) For a large negative detuning (-x>>1), $\alpha \approx 1$, $\beta \approx 0$. $\nu_\pm = \mp\delta/2$.

$$|+,n\rangle \approx |e,n\rangle, \quad |-,n\rangle \approx -|g,n+1\rangle. \tag{3.8.19}$$

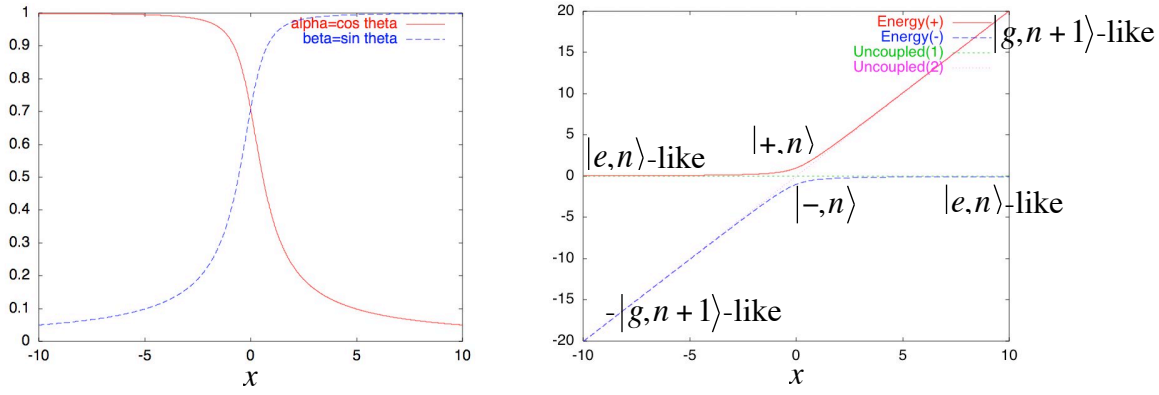

Fig. 5. The coefficients α and β and eigen-energies $\nu_\pm$ in the dressed-state picture are plotted.

iv) Uncoupled states (g=0), $x = \pm\infty$.

$$\nu_+ = \begin{cases}\delta, & \delta > 0\\ 0, & \delta < 0\end{cases}, \quad \nu_- = \begin{cases}0, & \delta > 0\\ \delta, & \delta < 0\end{cases}. \tag{3.8.20}$$

More natural eigenvalues are $\nu_g = \delta$ and $\nu_e = 0$ and corresponding eigen functions are $|e,n\rangle$ and $|g,n+1\rangle$. The eigenvalues of the uncoupled states cross at $\delta = 0$ whereas the coupled states exhibit anti-crossing (separated by $2g\sqrt{n+1}$ at $\delta = 0$) due to the atom-field coupling.

Jaynes-Cummings Model

The single mode considered above can be a single mode of a cavity with which a two-level atom is interacting. This situation is of particular interest in the field of cavity quantum electrodynamics. In this context, the Hamiltonian, Eq. (3.8.8), is called Jaynes-Cummings Hamiltonian. The energy eigenvalues for zero atom-field (or -cavity) detuning (δ=0) is given by $\nu_\pm = \pm g\sqrt{n+1}$, or

$$E_\pm = \hbar(n+1)\omega_0 \pm \hbar g\sqrt{n+1} \tag{3.8.21}$$

with corresponding eigenstates given by

$$|\pm,n\rangle = \frac{1}{\sqrt{2}}\left(|e,n\rangle \pm |g,n+1\rangle\right).$$

These doublets form the so-called Jaynes-Cumming Ladder structure.

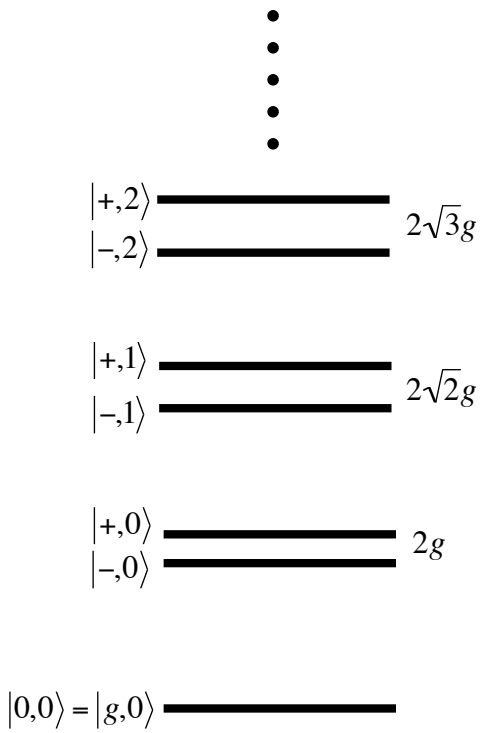

Fig. 6. Jaynes-Cumming Ladder structure.

The time evolution of atom-cavity system can be found by decomposing a given state into eigenstates $|+,n\rangle,|-,n\rangle$ and by evolving each eigenstate. For example, suppose the system is initially prepared in $|e,n\rangle$.

$$|\Psi(0)\rangle = |e,n\rangle = \frac{1}{\sqrt{2}}\left(|+,n\rangle + |-,n\rangle\right) \tag{3.8.22}$$

At time t, the state becomes

$$\begin{aligned}
|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[|+,n\rangle\exp\left(-\frac{i}{\hbar}E_{+}t\right) + |-,n\rangle\exp\left(-\frac{i}{\hbar}E_{-}t\right)\right] \\
&\rightarrow \frac{1}{\sqrt{2}}\left[|+,n\rangle\exp\left(-ig\sqrt{n+1}t\right) + |-,n\rangle\exp\left(ig\sqrt{n+1}t\right)\right] \\
&= \frac{1}{2}\left(|e,n\rangle + |g,n+1\rangle\right)\exp\left(-ig\sqrt{n+1}t\right) + \frac{1}{2}\left(|e,n\rangle - |g,n+1\rangle\right)\exp\left(ig\sqrt{n+1}t\right) \\
&= \cos\left(g\sqrt{n+1}t\right)|e,n\rangle - i\sin\left(g\sqrt{n+1}t\right)|g,n+1\rangle.
\end{aligned} \tag{3.8.23}$$

At the second line, we neglected a common phase factor. This oscillation is called the *enhanced* Rabi oscillation. Of particular interest is the case of n=0, which corresponds to the *vacuum* Rabi oscillation.