V

Topology

Want to ~~gener~~ formulate more precisely, then generalize, args for solitons so far.

Fundamental gp = First homotopy gp

Assume – Manifold M
pt x_0

Consider maps $\overbrace{\text{from}}^{f(s)}$ S^1 to M with

$$f(0) = x_0$$
$$f(1) = x_0$$

Maps $f_1(s)$, $f_2(s)$ homotopic if $\exists\ g(s,t)$ with

$$g(s,0) = f_1(s)$$
$$g(s,1) = f_2(s)$$

$\Leftrightarrow$ f_1 can be continuously deformed to f_2

Given ~~fu~~ $h_1(s)$, $h_2(s)$, define

$$h_1 \circ h_2(s) = \begin{cases} h_1(2s), & 0 \le s \le 1/2 \\ h_2(2s-1), & 1/2 \le s \le 1 \end{cases}$$

Define equiv classes ~~(d)~~ with homotopic maps equiv.

Clearly mult extends unambig. to equiv classes:
If $k_1 \cong h_1$, $k_2 \cong h_2$,
then
$$k_1 \circ k_2 \cong h_1 \circ h_2$$

Equiv classes, with this product $\rightarrow$ Group

[identity = const. map

inverse : $f^{-1}(s) = f(1-s)$]

$\pi_1(M, x_0)$

$\rightarrow$ If M connected, $\pi_1(M, x_0)$ for different x_0 are isomorphic

$\rightarrow$ usually then write $\pi_1(M)$

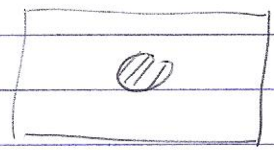
Ex

$M = \mathbb{R}^3$ $\quad \pi_1(\mathbb{R}^3) = 0$

$M = S^1$ $\quad \pi_1(S^1) = \mathbb{Z}$

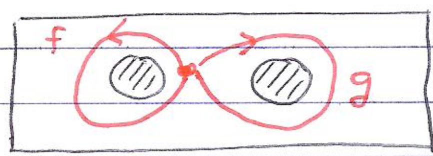
$M = S^2$ $\quad \pi_1(S^2) = 0$

M :



$\rightarrow \pi_1(M) = \mathbb{Z}$ (as for circle)

M :



$\pi_1(M)$ non-Abelian

$fgf^{-1} \neq g$

$\pi_1(M) = 0 \longleftrightarrow M$ is "simply-connected"

Groups

$SU(2)$: matrices $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $\underbrace{|a|^2 + |b|^2 = 1}$

$(\mathrm{Re}\,a)^2 + \cdots (\mathrm{Im}\,b)^2 = 1$

$\Rightarrow S^3$

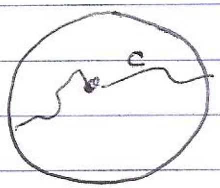
$\rightarrow \pi_1(SU(2)) = 0$

$SO(3)$: Ball of rad π : $\theta, \varphi \rightarrow$ dir. of axis of rot

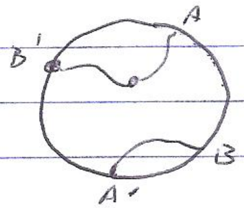
$r \rightarrow$ angle of rot

But : π about $\hat{n} \cong -\pi$ about $-\hat{n}$

$\Rightarrow$ Ball with antipodal pts identified



C $\neq$ homotopic to const. map



C homotopic to const. map [$A' \rightarrow B$]

$\Rightarrow \pi_1(SO(3)) = \mathbb{Z}_2$

$SU(2)$ = "covering group"

$\rightarrow$ Any compact Lie gp. has simply connected covering gp

R Hatian

Gp G; Center K = {el'ts that commute with all of G}

~~[illegible]~~ $G/K = G'$ → non-simply connected

e.g.: $SU(N)$: $K = e^{2\pi i n/N} I$, $n = 0, 1, \ldots N-1$

→ Z_N

~~[illegible]~~

$SU(3)/Z_3$ → adjoint, all triality 0 rep'ns are true rep'ns

Spin(N) → generators are $\sigma^{ij} = \frac{1}{2i}[\Gamma^i, \Gamma^j]$

→ Covering gp for $SO(N)$

$SO(N) = \text{Spin}(N)/Z_2$

$U(1)$ → covering gp = $\mathbb{R}$ = additive gp of real #'s

$U(1) = \mathbb{R}/\mathbb{Z}$ [i.e., ~~[illegible]~~ $x \cong x+n$]

Homotopy & solitons

2-dim vortices $\rightarrow$ continuous family of vacua $\rightarrow$ manifold M

Fields at $r \to \infty$ give map from S^1 to M = space of vacua

$\pi_1(M) \neq 0 \Rightarrow \exists$ maps that can't be deformed into constant map

Min. E within homotopy class $\longleftrightarrow$ static sol'n

[Caveats $\rightarrow$ min. may not exist
$\rightarrow$ two-vortex states]

If $G \longrightarrow H$, $\phi_\infty = g(\theta)\phi_0$
But $h\phi_0 = \phi_0$
$\Rightarrow M = G/H$

$\rightarrow$ vortices if $\pi_1(G/H) \neq 0$

Ex. (1) complex ϕ, $U(1)$ gauge sym
$\langle\phi\rangle \neq 0 \Rightarrow U(1)$ completely broken

$$\pi_1(G/H) = \pi_1(G) = \pi_1(U(1)) = \mathbb{Z}$$

(2) W-S model : $G = SU(2) \times U(1)$
$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ [φ_i cpx]

$$\text{vac} \Rightarrow |\varphi_1|^2 + |\varphi_2|^2 = v^2 \Rightarrow M = S^3$$

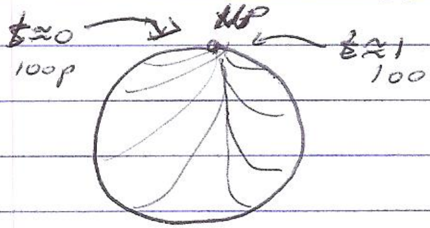
$$\pi_1(M) = \pi_1(S^3) = 0$$

$\Rightarrow$ No vortices

3-dim

Map of S^2 at $r=\infty$ to space of vacua

$\Rightarrow$ Define ~~$\pi_2(M)$~~ $\pi_2(M, x_0)$



$t: 0 \to 1 \Rightarrow$ labels loops

$s: 0 \to 1 \Rightarrow$ distance along loop

$f(s,t)$ with: $f(0,t) = f(1,t) = f(s,0) = f(s,1) = x_0$

$\Rightarrow$ Map of sphere to M

f_1 & f_2 homotopic if $\exists\ g(s,t,u)$,

$$g(s,t,0) = f_1(s,t)$$
$$g(s,t,1) = f_2(s,t)$$

$$f_1 \circ f_2 = \begin{cases} f_1(s, 2t), & 0 \le t \le \frac{1}{2} \\ f_2(s, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$\Rightarrow$ 2nd homotopy group $\pi_2(M, x_0)$

$\rightarrow$ Usually drop x_0 $\rightarrow \pi_2(M)$

$\pi_2(M)$ always Abelian

Ex 1) $G = SU(2)$, ϕ = triplet, $G \to U(1)$

$\langle \phi \rangle = v \Rightarrow \vec{\phi}^{\,2} = v^2$

$M = S^2$

$\pi_2(M) = \pi_2(S^2) = \mathbb{Z} \Rightarrow$ monopoles

2) W-S, $M = S^3$

$\pi_2(M) = \pi_2(S^3) = 0$

$\to$ no monopoles

$[\,\pi_k(S^n) = 0$ if $n > k\,]$

Hard to visualize π_2 for more complicated manifolds. Use theorem [see Coleman's book:

$\Rightarrow$ If $\pi_2(G) = \pi_1(G) = 0$

$$\boxed{\pi_2(G/H) = \pi_1(H)}$$

But: 1) Cartan $\rightarrow$ for any compact connected Lie group, $\pi_2(G) = 0$

2) $\mathcal{L}$ only determines Lie algebra $\rightarrow$ always free to take G to be the covering gp

$\Rightarrow \pi_1(G) = 0$

Ex: 1) $SO(3) \rightarrow SO(2) = U(1)$

Take $G = SU(2)$

$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}$

$\Rightarrow$ monopoles

2) Any GUT

$$G \xrightarrow{M_X} SU(3) \times U(1)$$

$G \downarrow$ simple (bec. GUT) & compact; $SU(3) \downarrow$ QCD; $U(1) \downarrow$ E&M

$\Rightarrow \pi_2(G/SU(3)\times U(1)) = \pi_1(SU(3)\times U(1)) = \mathbb{Z}$

$\Rightarrow$ <u>Any</u> GUT has monopoles

$$M_{Mon} \sim \frac{M_X}{g^2}$$