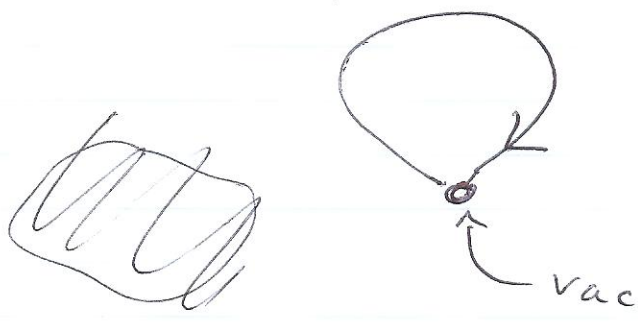
IX

Recall:

$A_0 = 0$ gauge: Disconnected vacuum sectors, winding #'s N, can tunnel between

Gauge where $F=0 \Longrightarrow A=0$
Tunneling path is



θ-vac: linear comb. of $|N\rangle$

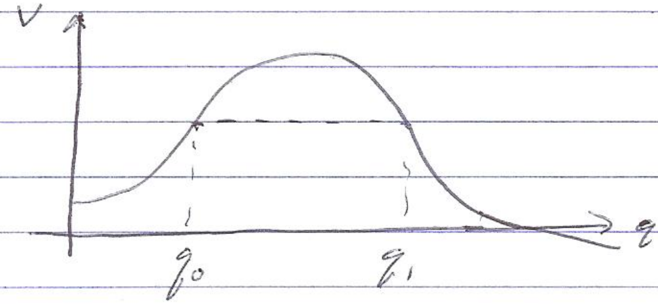
Effect same as $\cancel{\Delta \mathcal{L} \sim \theta \, \mathrm{Tr}\, F\tilde{F}}$

$$\Delta \mathcal{L} = \frac{\theta g^2}{16\pi^2} \mathrm{Tr}\, F\tilde{F}$$

QM tunneling

a) 1 deg of freedom

$$L = \frac{1}{2}\dot{q}^2 - V(q), \qquad H = \frac{1}{2}\dot{q}^2 + V(q)$$



$$V(q_0) = V(q_1) = E$$

WKB: tunneling amplitude $\sim e^{-B/2}$

$$\frac{1}{2}B = \int_{q_0}^{q_1} dq\, \sqrt{2[V(q) - E]}$$

Many d.o.f.: $L = \sum \frac{1}{2}\dot{q}_j^2 - V(q_1, q_2 \ldots)$

For each path through barrier, calc. B as in 1-dim problem

Path with minimum $B \rightarrow$ dominates

$$\Rightarrow 0 = \delta \int ds \sqrt{2[V(q_j(s)) - E]} \qquad ds^2 = \sum dq_j^2$$

Classical mech:

$$0 = \delta \int ds \sqrt{2[E - U(s)]} \Longleftrightarrow 0 = \delta \int dt\, L$$

$$= \delta \int dt \left[\frac{1}{2}\dot{q}_j^2 - U(q)\right]$$

$$\Rightarrow \ddot{q} = -\frac{\partial U}{\partial q}$$

Our problem: Like mechanics problem, but $V \longleftrightarrow -U$

$$\Rightarrow \text{solve } \ddot{q} = V'(q)$$

Suppose $q_0 =$ (local) min., calc B

Usual sign: $L = \frac{1}{2}\dot{q}^2 - U \Rightarrow \frac{1}{2}\dot{q}^2 + U = \text{const}$

Our sign: $L_{Euc} = \frac{1}{2}\dot{q}^2 + V \Rightarrow \frac{1}{2}\dot{q}^2 - V = \text{const}$

If $E = V(q_0)$, $\quad V - E = V(q) - V(q_0)$

But $V - \frac{1}{2}\dot{q}^2 = \text{const} = V(q_0)$

$$\Rightarrow V - E = \frac{1}{2}\dot{q}^2$$

$$\int dq \sqrt{2[V(q) - E]} = \int dq\, \dot{q} = \int dt\, \dot{q}^2$$

$$= \cancel{\tfrac{1}{2}} \int dt \left\{ \frac{1}{2}\dot{q}^2 + V(q) - V(q_0) \right\}$$

$$= S_{Euc}(\text{sol'n}) - S_{Euc}(\text{static } q_0)$$

Field theory

~~Bp ...~~ eg) Scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi) = \frac{1}{2}\dot{\varphi}^2 - \left[\frac{1}{2}(\nabla\phi)^2 + V(\varphi)\right]$$

$$\mathcal{L}_{Euc} = \cancel{\tfrac{1}{2}\dot{\phi}^2}\ \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$$

$$\frac{1}{2}B = \int d\tau \int d^3x \left\{ \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right\}$$

$\rightarrow$ Euc eq of motion

Note: τ = dummy variable $\leftrightarrow$ parameterizes path

NOT real time

Solution = Instanton

Note: Can derive by using Euc. path integral

Vac tunneling in Yang-Mills

Want to solve Euc. eqs

$$D_\mu F_{\mu\nu} = 0$$

Bdy cond: $t_0 = -T/2$: $F=0$, $A = -\frac{i}{g} G_0^{-1} \partial G_0$

$t_0 = T/2$: $F=0$, $A = -\frac{i}{g} G_1^{-1} \partial G_1$

→ G_0, G_1 have different winding #'s

→ Don't fix $A_0 = 0$; once have a solution, can put it in desired gauge

→ Solution will determine T
⇒ will find that $T = \infty$

Suppose $\Delta(\text{winding } \#) = \pm 1$

⇒ We want solution with

$$\int d^4x \, \mathrm{Tr}\, F\tilde{F} = \pm \frac{16\pi^2}{g^2}$$

Dominant effect from solution with smallest S_{Euc}

$$S_{Euc} = \int d^4x \, \mathrm{Tr}\left\{ \frac{1}{2} F^2 \right\}$$

$$= \int d^4x \, \mathrm{Tr}\left\{ \frac{1}{4} F^2 + \frac{1}{4} \tilde{F}^2 \right\}$$

$$= \int d^4x \, \mathrm{Tr}\left\{ \frac{1}{4} (F \mp \tilde{F})^2 \pm \frac{1}{2} F\tilde{F} \right\}$$

$$\geq \left| \frac{1}{2} \int d^4x \, \mathrm{Tr}\, F\tilde{F} \right|$$

with equality if

self-dual $\quad F = \tilde{F} \quad \left(\frac{1}{2} \int d^4x \, \mathrm{Tr}\, F\tilde{F} > 0 \right)$ instanton

anti-self-dual $\quad F = -\tilde{F} \quad (\quad " \quad < 0)$ anti-inst.

In fact, have solutions that saturate bound

$$S_{instanton} = \frac{8\pi^2}{g^2}$$

Explicit solution: (SU(2))

Define
$$\eta^a_{\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & (\mu,\nu = 1,2,3 \\ -\delta_{a\nu} & (\mu = 4) \\ \delta_{a\mu} & (\nu = 4) \\ 0 & \mu = \nu = 4 \end{cases}$$

$\bar{\eta}^a_{\mu\nu} \to$ signs reversed if μ or $\nu = 4$

$$A^a_\mu(x) = \frac{2}{g}\,\eta_{a\mu\nu}\,\frac{(x-z)_\nu}{[(x-z)^2 + \lambda^2]}$$

Note: 1) $|x| \to \infty \Rightarrow A \sim \frac{1}{x}$, but $F \sim \frac{1}{x^4}$

2) $z \to$ translation collective coord
$\vec{z} \to$ spatial trans
$z_4 \to$ artifact of calc.

$\vec{z} \to$ region where effects are concentrated

3) $\lambda \to$ "instanton size"
Classically, Y.M is scale invariant
$\Rightarrow$ all λ allowed
all give equal action

QM: one-loop effects ['t Hooft]

$$\Rightarrow g^2 = g^2(\lambda)$$

$$S = \frac{8\pi^2}{g^2(\lambda)}$$

λ large $\Rightarrow g^2(\lambda) \uparrow \Rightarrow S \downarrow \Rightarrow$ larger effect

4) $\eta \to \bar{\eta}$ $\Rightarrow$ anti-instanton

Multinstanton/anti-instanton sol'ns

$\Rightarrow$ approx. sol'ns with widely separated instantons, anti-instantons

$\rightarrow$ describe processes with local winding number changing in many places

Counting instanton effects:

i) Sum over positions ~~(& SU(2) orientations)~~

$\rightarrow$ Dilute gas approx

ii) Sum over scales

$\Rightarrow$ breakdown of DGA

iii) Sum over SU(2) orientations

Fermions

Recall anomaly:

$$\mathcal{L} = \sum_{i=1}^{N_f} \bar{\psi}_i (i\not{\partial} - m_i)\psi_i$$

Let $m_i \to 0$

Classically, $\Rightarrow U(N) \times U(N)$ chiral sym.

$U(1)$ currents:

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$
$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

$$\partial_\mu j^\mu = 0 \quad , \quad \partial_\mu j_5^\mu = 0$$

QM: Ward Identities involving j_5^μ false

As if

$$\partial_\mu j_5^\mu \propto \operatorname{tr} F\tilde{F}$$

If N_f flavors,

$$\partial_\mu j_5^\mu = \frac{2N_f}{16\pi^2} g^2 \operatorname{Tr} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Interpret in terms of chirality

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi$$
$$\psi_R = \tfrac{1}{2}(1+\gamma_5)\psi$$

~~j_5^μ~~ Def $j_{\mu L} = \frac{1}{2}(j_\mu - j_{\mu 5})$

$j_{\mu R} = \frac{1}{2}(j_\mu + j_{\mu 5})$

$$\partial_\mu j^\mu_L = -\frac{N_f}{16\pi^2} g^2 \operatorname{tr} F\tilde{F}$$
$$\partial_\mu j^\mu_R = +\frac{N_f}{16\pi^2} g^2 \operatorname{tr} F\tilde{F}$$