Cosmology – lightning review

R-W metric

$$ds^2 = dt^2 - R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right]$$

$k = 1, 0, -1$ (closed, flat, open)

$R(t) \rightarrow$ scale factor

$\rightarrow$ rad. of curv. ($k = \pm 1$)

$\rightarrow$ relative scale ($k=0$)

$\rightarrow$ coord vs physical distance

$\rightarrow$ describe expansion

Friedmann eq: $\left(\frac{\dot{R}}{R}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2}$

Early univ: 1) k/R^2 negligible

2) ρ dominated by ρ_{rad}

System at equil with temp. T

$\rho_{rad} \sim \mathcal{N} T^4$

3) Adiabatic expansion: $R^3 S \sim R^3 \mathcal{N} T^3$ constant

(3) $\Rightarrow T \sim 1/R$

$\Rightarrow$ can solve for expansion

$$\Rightarrow \left(\frac{\dot{T}}{T}\right)^2 = C^2 \frac{1}{M_p^2} T^4 \qquad [C = \mathcal{O}(1)]$$

$$\dot{T} = -\frac{C}{M_p} T^3$$

$$\Rightarrow T = \frac{1}{\sqrt{2C}} \left(\frac{M_{pl}}{t}\right)^{1/2}$$

$$R \sim t^{1/2}$$

[$t = 0$ nominally = "big bang"]

Horizons

Emit light from $r=0$ at $t=t_0$

Trajectory given by $ds=0$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{R(t)}$$

$$r(t) = \int_{t_0}^{t} dt' \, \frac{1}{R(t')}$$

This corresponds to a physical distance

$$\cancel{R}\, d(t) = R(t) \int_{t_0}^{t} dt' \, \frac{1}{R(t')}$$

(assuming C constant) $\quad = t^{1/2} \int_{t_0}^{t} \frac{dt'}{(t')^{1/2}}$

$$= 2\sqrt{t}\,(\sqrt{t} - \sqrt{t_0})$$

$t_0 \to 0 \Rightarrow$ "Horizon distance" $\quad \cancel{d_H = 2t}$

$$\boxed{d_H = 2t \sim \frac{M_{pl}}{T^2}}$$

Symmetry "restoration"

Familiar systems: spont. broken sym restored at higher temp:

ex: 1) Crystals – broken trans, rot sym
– restored above melting pt.

2) Ferromagnets – $\vec{B}$ breaks rot. sym.
– restored above Curie temp.

Field theory: look at $V_{eff}(\phi, T)$

$\underline{T=0}$ $V_{eff}(\bar{\phi})$: min. energy density among states with $\langle\phi\rangle = \bar{\phi}$.

$T \neq 0$ $V_{eff}(\bar{\phi}, T)$ = min free energy density among states with $\langle\phi\rangle = \bar{\phi}$

$T=0$: leading contrib → tree level $V(\phi)$
1-loop → • zero-point energies of fields depend on particle masses.
These affected by $\langle\phi\rangle$
$\Rightarrow$ contrib. to $V_{eff}(\phi)$

$T \neq 0$: Similarly, ϕ ~~modified~~ modifies masses
$\Rightarrow$ modifies contrib. to free energy

Find: 1) Dominant correction near $\phi = 0$ is

$$m^2 \phi^2 \rightarrow (m^2 + bT^2)\phi^2 = m^2_{eff}\phi^2$$

where $b \sim g^2, G^2_{Yuk}, \lambda$

$\Rightarrow$ if $m^2 < 0$, have $m^2_{eff} < 0$ ($\Rightarrow$ SSB)
if $T < m/\sqrt{b}$
but $m^2_{eff} > 0$ if $T > T_c = m/\sqrt{b}$

2) Effects at larger ϕ can change min from sym. to asym., get T_c of similar order of mag

(1) $\Rightarrow$ 2^{nd} order transition
(2) $\Rightarrow$ 1^{st} order transition

In either case, expect $\boxed{T_c \sim v}$

Kibble mechanism

$T > T_c \rightarrow$ unbroken sym, $\langle \phi \rangle = 0$

T falls below $T_c \Rightarrow$ SSB, $\langle \phi \rangle \neq 0$

$\rightarrow$ production of "domains", size l_{dom}

$\langle \phi \rangle \approx$ uniform within domain

l_{dom} ? $\rightarrow$ depends on detailed dynamics, but certainly

$$l_{dom}(T_c) < d_H(T_c)$$

As time goes on, ~~domains grow~~, ϕ smooths out, domains grow, expect

$$l_{dom}(T) \sim d_H(T)$$

Topological defects $\longleftrightarrow$ places where adjacent domains can't be smoothly joined

discrete vacua $[\pi_0(G/H) \neq 0] \rightarrow$ domain walls

$\pi_1(G/H) \neq 0 \rightarrow$ vortices $\longleftrightarrow$ cosmic strings

$\pi_2(G/H) \neq 0 \rightarrow$ magnetic monopoles

Domain walls

Kibble mech → If spont. broken discrete sym, expect $\mathcal{O}(1)$ domain wall in observed universe

Consider its mass:

mass/area = (1+1)-dim kink mass

$\Rightarrow$ mass $\sim (\mu^3/\lambda)$ [Area]

Take area $\sim [10^{10}\, lyr]^2 \sim [10^{28}\, cm]^2$

$1\, cm \sim 5 \times 10^{13}\, GeV^{-1}$

$\Rightarrow$ area $\sim 2 \times 10^{83}\, GeV^{-2}$

Wall mass $\sim \left(\frac{1}{\lambda}\right)\left(\frac{\mu}{GeV}\right)^3 (10^{83})\, GeV$

For comparison, present horizon includes $\sim 10^{78}$ baryons $\Rightarrow \sim 10^{78}$ GeV in baryon mass

$\Rightarrow$ Grav effects of dom. wall would be manifest unless $\mu \ll 1$ GeV

→ Could new physics with such a low mass scale be still undiscovered

→ Wouldn't associated phase transition disturb successful results of nucleosynthesis [at $T \sim 100$ keV]

$\Rightarrow$ Domain walls to be avoided

Strings

Form if $U(1)$ broken
Orig., $\gtrsim 1$ / hor volume

Evolution — studied
Possible ~~e~~ origin of structure?
— doesn't fit data
$\Rightarrow$ upper limit on scale

Old estimates: $\frac{\delta T}{T} \sim 10\, G\mu$

$\mu = \frac{\text{mass}}{\text{length}} \sim v^2$

$\Rightarrow$ To get $\frac{\delta T}{T} \sim 10^{-5}$, would want $\frac{v}{M_{pl}} \sim 10^{-3}$

Since now interpret $\frac{\delta T}{T}$ as being from inflation, need

$$v \ll 10^{-3} M_{pl}$$

Could observe strings as grav. lens.

Monopoles

Expectation from thermal equil?

$M \sim \frac{4\pi v}{g}$, but $T_c \sim v$

$\Rightarrow$ equil density small as soon as they can exist

BUT $\rightarrow$ look at formation by Kibble mech.

Expect $n_{mon}(T_c) \sim p\, n_{Dom} \sim p\,[l_{dom}(T_c)]^{-3}$

prob. of non-trivial winding at domain intersection

$n_{mon}(T_c) \gtrsim p\,[l_{hor}(T_c)]^{-3}$

$\sim p\,\cancel{\frac{M_p^3}{T_c^6}}\quad p\,\frac{T_c^6}{M_p^3}$

Define $r = \frac{n_{mon}}{s} \sim \frac{n_{Mon}}{T^3}$

$r(T_c) \gtrsim \left(\frac{T_c}{M_p}\right)^3$

Evolution of r: $s \sim 1/R^3$ [adiabatic]
if no $M\text{-}\bar{M}$ annih., $n_M \sim 1/R^3$ $\Rightarrow r = \text{const}$.

Annih: M & $\bar{M}$ capture, then annihilate
$\rightarrow$ Must lose energy to be bound
High T $\rightarrow$ friction with charged plasma
Low T $\rightarrow$ radiation of Brehmsstrahlung

Annih. in expanding univ:

$$\frac{dn_M}{dt} = \underbrace{-\langle\sigma v\rangle n_M^2}_{\text{std. annih rate}} \quad \underbrace{-3\frac{\dot{R}}{R}n_M}_{\text{expansion}}$$

$\Rightarrow$ Once $\underbrace{\langle\sigma v\rangle n_M}_{n_M \sim \cancel{\ } T^3} \ll \underbrace{H}_{\sim \frac{T^2}{M_P}}$

annih. effectively ceases

$\rightarrow$ "freeze-out"

$\rightarrow$ $r \approx$ const.

Preskill: detailed calc $\Rightarrow$ for GUT scale monopoles,

$$r_{asym} = \min\begin{cases} 10^{-10} \\ r_{initial} \end{cases}$$

Assuming $T_c \sim 10^{15}$ GeV, (low est.) might get r_{init} as low as $\underline{\underline{10^{-12}}}$

Monopole bounds

1) Density bound

Mon mass density bounded by total mass density

$$\Rightarrow m_{Mon}\, n_{Mon} \lesssim \cancel{(\text{GeV}) \times n_{prot}\, n_B} \lesssim 6\, m_{prot}\, n_B$$

$$\frac{m_{Mon}}{m_{prot}}\, r_{Mon} \lesssim 6 \frac{n_B}{s} \sim \cancel{10^{-10}}\ 10^{-9}$$

$$r_{Mon} \lesssim \left(\frac{10^{16}\,\text{GeV}}{m_{Mon}}\right) \times 10^{-25}$$

2) Bounds on flux

a) Monopoles not clustered in galaxies:

$$\text{Flux} \sim n_M\, v_{gal} \sim \cancel{10^{-3}\,\text{cm}}\ (10^{-3} c)\, n_M$$

b) Monopoles clustered in galaxy: For given flux, same n_M in galaxy

$\Rightarrow$ lower average n_M

$\Rightarrow$ Use (a)

$$\Rightarrow r = \frac{n_M}{s} \sim \frac{1}{s} \frac{1}{(10^{-3} c)}\ \text{Flux}$$

But $s \sim 3 \times 10^{3}\ \text{cm}^{-3}$

$$\Rightarrow r \sim 10^{-10}\ \text{Flux}(\text{cm}^{-2}\,\text{sec}^{-1})$$

Flux bounds

→ Direct searches : $\sim \begin{cases} 10^{-15} \\ 10^{-16} \end{cases}$ $cm^{-2} sr^{-1} sec^{-1}$

→ Parker bound $\sim 10^{-16}$

→ Callan-Rub. effect $\sim 10^{-23}$ [only some monos]

$\Rightarrow$ $r \lesssim 10^{-26}$ (without C-R effect)

or $r \lesssim 10^{-33}$ (assuming C-R effect)

$\Rightarrow$ Estimated production by Kibble mechanism inconsistent with bounds

Solutions : ⟶ inflationary univ. scenario

⟶ Langacker - Pi

⟶ no GUT

BPS limit

Return to $SU(2)$ case [only for simplicity]

$$\varphi^a = \hat{r}^a h(r)$$
$$A_i^a = \epsilon_{iam} \hat{r}^m \left[\frac{1-u(r)}{er} \right]$$

$$0 = u'' - \frac{u(u^2-1)}{r} - e^2 u h$$

$$0 = h'' + \frac{2}{r} h' - \frac{2u^2 h}{r^2} + \lambda(v^2 - h^2) h$$

Prasad & Sommerfield:

1) Elim e by rescaling r, u
$$\Rightarrow \lambda \to \lambda/e^2$$

2) Let $\lambda, \mu^2 \to 0$; $v^2 = \frac{\mu^2}{\lambda}$ fixed
$\Rightarrow$ ~~sim~~ No parameters in eqs,
GUESS (trial & error)

$$u = \frac{evr}{\sinh(evr)}$$
$$h = v \coth(evr) - \frac{1}{er}$$

Bogomolny:

a) Suppose $\vec{E} = 0$, $A_0 = 0$, static

$$E = \int d^3x \left\{ \tfrac{1}{2} \vec{B}_i^2 + \tfrac{1}{2} (\vec{D_i \phi})^2 + V(\phi) \right\}$$
$$= \int d^3x \left\{ \tfrac{1}{2} (B_i - D_i \phi)^2 + V(\phi) \right\}$$
$$+ \int d^3x \, \vec{B}_i \cdot \vec{D_i \phi}$$

Last term: $\int d^3x \, \vec{B}_i \cdot D_i \vec{\phi} = \cancel{\int d^3}$

$$= \int d^3x \left\{ \partial_i (\vec{B}_i \cdot \vec{\phi}) - \underbrace{\vec{D_i B_i}}_{\text{identically } = 0} \right\}$$

$$= \int d^2S_i \, \vec{B}_i \cdot \vec{\varphi}$$
$$= v \, Q_M$$

$$\Rightarrow E = v Q_M + \int d^3x \left\{ \tfrac{1}{2}(\vec{B}_i - D_i\vec{\phi})^2 + V(\phi) \right\}$$

If $\lambda \to 0$, $V \to 0$

1) Minimize E if

$$\vec{B}_i = D_i \vec{\phi} \qquad \text{[Bogomolny eq]}$$

2) $\Rightarrow$ sol'n to 2^{nd}-order eqs of motion from a 1st-order eq

2) For such "BPS" solutions,

$$E = v Q_M = \frac{4\pi}{e} v$$

b) Remove restriction:

$$E = \int d^3x \left\{ \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 + \tfrac{1}{2}(D_0\vec{\phi})^2 + \tfrac{1}{2}(D_i\vec{\phi})^2 + V(\phi) \right\}$$

$$= \int d^3x \left\{ \tfrac{1}{2}(\vec{E} - \sin\alpha\, D_i\vec{\phi})^2 + \tfrac{1}{2}(\vec{B} - \cos\alpha\, D_i\vec{\phi})^2 + \tfrac{1}{2}(D_0\vec{\phi})^2 + V(\phi) \right\}$$
$$+ \int d^3x \left\{ \sin\alpha\, \vec{E}\cdot\vec{D}\phi + \cos\alpha\, \vec{B}\cdot\vec{D}\phi \right\}$$

* α arbitrary

Last line:

$$\int d^3x\, E\cdot D\phi = \int d^3x \left\{ \partial_i(\vec{\phi}\cdot\vec{E}) + \vec{\phi}\cdot\vec{D}\vec{E} \right\}$$

$$= v Q_E + \int d^3x\, \vec{\phi}\cdot \underbrace{(D_0\vec{\phi})\times\vec{\phi}}_{\text{Gauss}}$$

$$= v Q_E$$

Set $V=0$

$$\Rightarrow E = \int d^3x \left\{ \tfrac{1}{2}(E - \sin\alpha\, D\phi)^2 + \tfrac{1}{2}(B - \cos\alpha\, D\phi)^2 + \tfrac{1}{2}(D_0\phi)^2 \right\} + v\cos\alpha\, Q_M + v\sin\alpha\, Q_E$$

$$\geq (\cos\alpha\, Q_M + \sin\alpha\, Q_E)\, v$$

Vary w.r.t. α to get strongest bound:

$$\Rightarrow \sin\alpha = \frac{Q_E}{\sqrt{Q_E^2+Q_M^2}} \ , \quad \cos\alpha = \frac{Q_M}{\sqrt{}}$$

$$E \geq v\sqrt{Q_M^2+Q_E^2}$$

Bound saturated if:

$$B_i = \cos\alpha \, D_i\phi$$
$$E_i = \sin\alpha \, D_i\phi$$
$$D_0\phi = 0$$

Explicit solution:

$$u = \frac{e\hat{v}r}{\sinh(e\hat{v}r)}$$

$$h = \frac{\sqrt{Q_M^2+Q_E^2}}{Q_M}\left[v\coth(e\hat{v}r) - \frac{1}{er}\right]$$

$$j = -\frac{Q_E}{Q_M}\left[\hat{v}\coth(e\hat{v}r) - \frac{1}{er}\right]$$

$$\hat{v} = \frac{Q_M}{\sqrt{Q_M^2+Q_E^2}} \ v$$

Montonen-Olive duality

Look at spectrum:

	M	Q_E	Q_M
γ	0	0	0
$W^\pm$	ev	$\pm e$	0
ϕ	0	0	0
Mon.	$\frac{4\pi}{e}v$	0	$\pm\frac{4\pi}{e}$
Dyon	$v\sqrt{Q_E^2+Q_M^2}$	Q_E	Q_M

Note: invariant under $E \longleftrightarrow M$, with change of names.

But: Spins don't match: $W^\pm$ spin 1, how does Mon. have spin?

→ Extend theory, make it supersym.
add adjoint fermions
→ each Dirac fermion ⇒ 2 zero modes about Monopole

One Dirac fermion: 2^2 states

$$S_z = \underbrace{0}_{\text{neither occupied}},\ \underbrace{0}_{\text{both occupied}},\ \underbrace{1/2,\ -1/2}_{\text{one occupied}}$$

Two Dirac fermions: 4 zero modes, $2^4 = 16$ states

Two modes ↑, two modes ↓

	S_z
4 modes filled	0
3 " "	$\frac{1}{2}, -\frac{1}{2}$ (each twice)
2 " "	$1, -1, 0$ (four times)
1 " "	$\frac{1}{2}, -\frac{1}{2}$ (each twice)
0 " "	0

$\Rightarrow$ $S = 1$, $\frac{1}{2}$ (four times), 0 (5 times)

Matches elem. particle sector if 5 extra scalars
$\rightarrow$ This is just field content of $N=4$ SUSY-YM

Also, $V(\varphi)$ for SUSY-YM is

$$V = \frac{e^2}{4}\sum \vec{\varphi}_a \times \vec{\varphi}_b \qquad (a, b = 1, 2 \cdots 6)$$

$\Rightarrow$ ~~aa~~ If all $\langle\vec{\varphi}_a\rangle$ are parallel, V naturally vanishes

$\rightarrow$ Evades issues of quantum corrections in $\lambda \to 0$ limit

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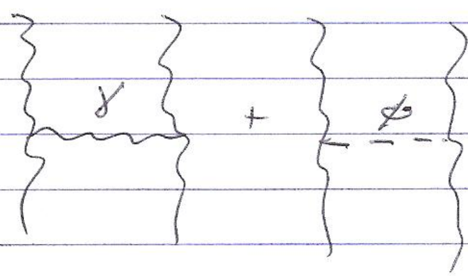
Another check: low-E scattering
$W^+(p)\, W^+(q) \longrightarrow W^+(p')\, W^+(q')$

Assume $|\vec{p}|, |\vec{q}| \ll m$, take static limit in end:



$$\mathcal{M} = \mathcal{O}(\vec{p}^{\,2}, \vec{q}^{\,2})$$

$\Rightarrow$ no force in static limit
$\rightarrow$ agrees with absence of interaction energy for mono's.

Note: No cancellation in $W^+W^- \to W^+W^-$

Duality conjecture

(Theory I, coupling g_I) $\Longleftrightarrow$ (Theory II, coupling g_{II})

g_I weak $\Longleftrightarrow$ g_{II} strong

g_I strong $\Longleftrightarrow$ g_{II} weak

Ex Massive Thirring & Sine-Gordon

Massive Thirring:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - M)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2$$

Spectrum: elem: M

bound states: lowest $M_{bound} = M\left[2 - g^2 + \frac{4g^3}{\pi} + \mathcal{O}(g^4)\right]$

Sine-Gordon:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right]$$

~~Spectrum kinks~~ Let $\beta^2 = \frac{\lambda}{m^2}$

Spectrum: Kink: $\frac{8m}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right) = M_{kink}$

"breathers": $M_n = 2M_{kink}\sin\left[\frac{n\pi}{2}\,\frac{\beta^2/8\pi}{1-\beta^2/8\pi}\right]$

$$= 2M_{kink}\sin\left[\frac{\pi}{2}\,\frac{n}{\frac{8\pi}{\beta^2} - 1}\right]$$

$\Rightarrow$ Require $n < \frac{8\pi}{\beta^2} - 1$

$n=1$ breather $\longleftrightarrow$ elem ϕ

small β: $M_1 \approx 2\left(\frac{8m}{\beta^2}\right)\left[\left(\frac{\pi}{2}\right)\left(\frac{\beta^2}{8\pi}\right)\right] = m$

As $\beta \uparrow$: a) $M_{kink} \downarrow$

b) $M_n \uparrow$

c) $n_{max} \downarrow$

At $\beta = \sqrt{4\pi}$, $n=1$ state disappears

Near $\beta = \sqrt{4\pi}$, let $\frac{\beta^2}{4\pi} = \frac{1}{1+\delta/\pi}$

$$M_1 = M_{kink}\left[2 - \delta^2 + \frac{4\delta^3}{\pi} + O(\delta^4)\right]$$

→ agrees with lowest bound state of massive Thirring, if $\delta \leftrightarrow g$

or $\frac{4\pi}{\beta^2} = 1 + g/\pi$

small β (small λ) $\longleftrightarrow$ large g

kink $\longleftrightarrow$ elem. ψ

anti kink $\longleftrightarrow$ elem $\bar{\psi}$

elem ϕ $\longleftrightarrow$ $\bar{\psi}\psi$ bound state

top. charge $\longleftrightarrow$ fermion number

Exact equiv:

$$\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi = -\bar{\psi}\gamma^\mu\psi$$

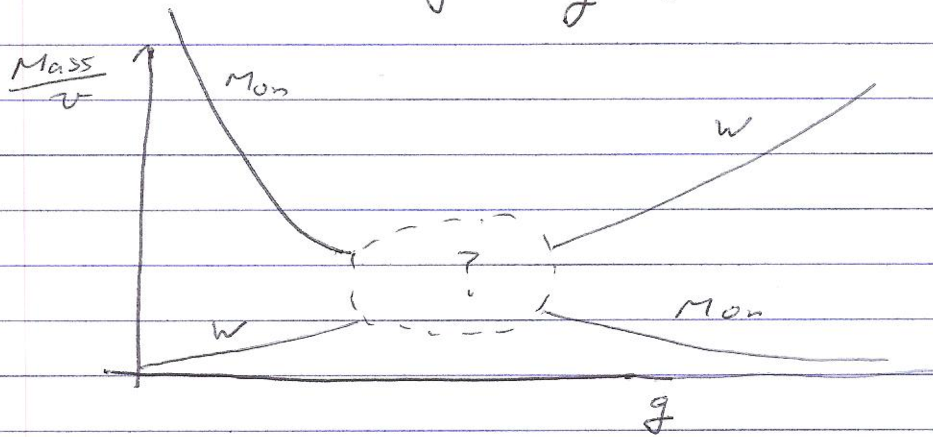
$$\frac{m^2}{\beta^2}\cos\beta\phi = -M\bar{\psi}\psi$$

(with proper normal ordering, etc.)

Conjecture for $\mathcal{N}=4$ Y-M

Self-duality: Both theories of same form, but

$$g \to \frac{4\pi}{g}, \quad \text{elec} \longleftrightarrow \text{mag.}$$



Large g elem W $\longleftrightarrow$ Small g' monopole
Large g Monopole $\longleftrightarrow$ Small g' elem. boson

Instantons

Pure Y-M :

in components,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

or, $A_\mu = A^a_\mu t^a$ $\longrightarrow$ A_μ as matrix

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$\mathrm{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$ [e.g, $t^a = \frac{1}{2}\sigma^a$ for $SU(2)$]

$$\mathcal{L} = -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

Under gauge trans,

$$A_\mu \to G^{-1} A_\mu G - \frac{i}{g} G^{-1} \partial_\mu G$$

Vacuum: $F = 0$, **But** can have $A \neq 0$

$$A_\mu = -\frac{i}{g} G^{-1} \partial_\mu G$$

Two approaches:

1) [Most common treatment]

Set $A_0 = 0$; $\vec{A} \to 0$ as $r \to \infty$

$\to$ still allows t-indep gauge trans.

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \frac{1}{4}F_{ij}^2$$

$\Rightarrow$ $\begin{cases} \text{too many deg. of freedom} \\ \text{No } A_0 \text{ eq. of motion} \Rightarrow \text{must impose} \\ \quad \text{Gauss's law as add'l constraint} \end{cases}$

QM : Gauss is $0 = D_j F^{j0}$
$= D_j \Pi^j$

Suppose $\Lambda(x) \to 0$ as $r \to \infty$

$$0 = \int dx\, \Lambda(x)\, D_j \Pi^j = -\int dx\, D_j \Lambda(x)\, \Pi^j(x)$$

$=$ generator of local g.t.

QM Gauss law $\longleftrightarrow$ wave fn. invariant under "local" ["small"] g.t.

2) Gauge choice that fixes A_μ uniquely

e.g., Axial
$A_3 = 0$
$A_2(z_0) = 0$
$A_1(y_0, z_0) = 0$
$A_0(x_0, y_0, z_0) = 0$

$$F = 0 \Longrightarrow A = 0$$

For (1), consider $A_j = \frac{-i}{g} G^{-1} \partial_j G$

$\Rightarrow$ Map of $\mathbb{R}^3$ onto Group

But, $A_j \to 0 \Rightarrow G \to 1$ as $r \to \infty$

$\Rightarrow$ Map of S^3 onto Group

Suppose Group $= SU(2) = S^3$

Map $\to$ element of $\pi_3(S^3) = \mathbb{Z}$

In fact, for any simple & simply connected Lie gp, $\pi_3 = \mathbb{Z}$

$\Rightarrow$ Expect vacua to fall into disconnected sets.

Consider

$$J[G] = \epsilon_{ijk} \int d^3x \, \mathrm{Tr} \, G^{-1}\partial_i G \; G^{-1}\partial_j G \; G^{-1}\partial_k G$$

1) $G \to G(I + i\Lambda)$, $\Lambda \to 0$ as $r \to \infty$, Λ small

$\Rightarrow J[G]$ unchanged

($\mathcal{O}(\Lambda)$ term is total deriv)

2) $J[G_1 G_2] = J[G_1] + J[G_2]$

3) J invariant if $G(x) \to G_0 G(x)$

If $SU(2)$:, $G = \exp\{i \sigma_a \alpha_a(x)\}$

Near $G = I$, $G = I + i\sigma_a \alpha_a(x)$

Integrand $= \epsilon_{ijk} \mathrm{Tr} \, i^3 \sigma_a \partial_i \alpha_a \, \sigma_b \partial_j \alpha_b \, \sigma_c \partial_k \alpha_c$

$$= 2 \epsilon_{ijk} \epsilon_{abc} \partial_i \alpha_a \, \partial_j \alpha_b \, \partial_k \alpha_c$$

$\to$ essentially, Jacobian

To fix normalization, consider [SU(2)]

$$G(x) = \exp\{ i \vec{\sigma} \cdot \hat{r} \, f(r) \}$$

$$f(0) = -\pi, \quad f(\infty) = 0 \qquad [G(0) = -I, \; G(\infty) = +I]$$

$\Rightarrow$ Covers SU(2) once

$$G = \cos f + i \vec{\sigma} \cdot \hat{r} \sin f$$

$$\partial G = (-\sin f + i \vec{\sigma} \cdot \hat{r} \cos f)\, \partial f + i \vec{\sigma}^a (\partial \hat{r}^a) \sin f$$

$$G^{-1} \partial G = i \vec{\sigma} \cdot \hat{r}\, \partial f + \cos f \sin f \,(i \vec{\sigma} \cdot \partial \hat{r})$$
$$+ \sin^2 f \; \sigma^a \hat{r}^a \sigma^b \partial \hat{r}^b$$
$$= i \vec{\sigma} \cdot \hat{r}\, \partial f + i \cos f \sin f \; \sigma^a \partial \hat{r}^a$$
$$+ \sin^2 f \, [\underbrace{\hat{r}^a \partial \hat{r}^a}_{\to 0} + i \epsilon^{abc} \hat{r}^a \partial \hat{r}^b \sigma^c]$$
$$= i \sigma^c \{ \hat{r}^c \partial f + \cos f \sin f \, \partial \hat{r}^c + \sin^2 f \, \epsilon^{abc} \hat{r}^a \partial \hat{r}^b \}$$

For $tr(G^{-1}\partial G)^3$, use $i^3 \, Tr\, \sigma^c \sigma^d \sigma^e = 2 \epsilon^{cde}$

$$\epsilon_{ijk} \, Tr\, G^{-1}\partial_i G \; G^{-1}\partial G \; G^{-1}\partial G$$
$$= 2 \epsilon_{ijk} \epsilon_{cde} \{ \hat{r}^c \partial_i f + \cos f \sin f \, \partial_i \hat{r}^c + \sin^2 f \, \epsilon^{abc} \hat{r}^a \partial_i \hat{r}^b \}$$
$$\times \{ \hat{r}^d \partial_j f + \cdots \}$$
$$\times \{ \hat{r}^e \partial_k f + \cdots \}$$

① Must have $\hat{r}\partial f$ once $\Rightarrow$ put it in i term [factor of 3]

② $\partial_j \hat{r}^d = \frac{1}{r} [\delta_{jd} - \hat{r}^j \hat{r}^d] \longrightarrow \frac{1}{r} \delta_{jd}$

③ $\epsilon^{abd} \hat{r}^a \partial_j \hat{r}^b = \frac{1}{r} \epsilon^{abd} \hat{r}^a \delta_{jb} = \frac{1}{r} \epsilon^{ajd} \hat{r}^a$

$$\Rightarrow 6 \epsilon_{ijk} \epsilon_{cde} (\hat{r}_c \hat{r}_i f') (\cos f \sin f \, \delta_{jd} + \sin^2 f \, \epsilon^{ajd} \hat{r}^a)$$
$$(\cos f \sin f \, \delta_{ke} + \sin^2 f \, \epsilon^{bke} \hat{r}^b) \left(\frac{1}{r^2}\right)$$

$$= \frac{6}{r^2} \hat{r}_c \hat{r}_i f' \Big\{ \cos^2 f \sin^2 f \, (2\delta_{ic})$$
$$+ \cos f \sin^3 f \, [\, \epsilon_{idk} (\cancel{\delta_{bc}\delta_{dk}} - \delta_{bd}\delta_{kc}) \hat{r}^b$$
$$+ \epsilon_{ije} (\delta_{jc}\delta_{ae} - \cancel{\delta_{je}\delta_{ac}}) \hat{r}^a \,]$$
$$+ \sin^4 f \, [\, (\delta_{ai}\delta_{kd} - \delta_{ak}\delta_{id})(\delta_{bc}\delta_{kd} - \delta_{bd}\delta_{kc}) \hat{r}^a \hat{r}^b \,] \Big\}$$

$$\cancel{= \frac{6}{r^2} \hat{r}_c \hat{r}_i f' \{}$$

$$= \frac{6}{r^2} f' \Big\{ 2\cos^2 f \sin^2 f$$
$$+ \cos f \sin^3 f \, [0]$$
$$+ \sin^4 f \, [\, (\delta_{kd} - \hat{r}_k \hat{r}_d)(\delta_{kd} - \hat{r}_k \hat{r}_d) \,] \Big\}$$

$$= \frac{6}{r^2} f' \{ 2\cos^2 f \sin^2 f + 2\sin^4 f \} = \frac{12}{r^2} f' \sin^2 f$$

$$= \frac{6}{r^2} f' (1 - \cos 2f)$$

$$= \frac{6}{r^2} \frac{d}{dr} [\, f - \sin 2f \,]$$

$$J[G] = \int d^3x \, \frac{6}{r^2} \frac{d}{dr} [f - \sin 2f]$$

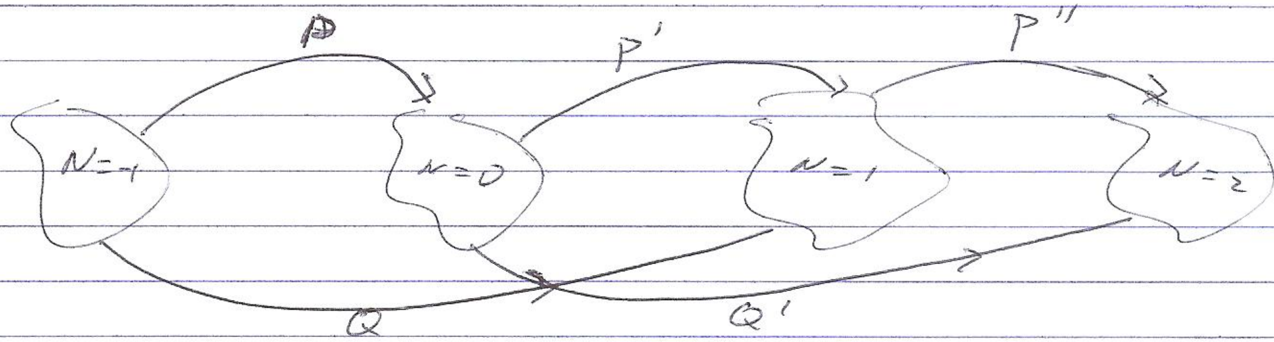
$$= (4\pi)(6) \int_0^\infty dr \, \frac{d}{dr} [f - \sin 2f]$$

$$= 24\pi \, [f - \sin 2f]_0^\infty$$

$$= 24\pi^2$$

$\Rightarrow$ Define $N[G] = \frac{1}{24\pi^2} J[G]$ = "winding #"

Vacua in space of config:



$P, P', P'', \cdots$ — gauge equiv.
$Q, Q', \cdots$ $\cdots$ — gauge equiv

Energy barriers between vac of $\Delta N \neq 0$?
$A \sim 1/g \Rightarrow F \sim 1/g \Rightarrow$ Energy density $F^2 \sim 1/g^2$

Define n-vac $|n\rangle \to$ vac. state with $N = n$ "periodic Vacua"
" θ-vac $|\theta\rangle = \sum_n e^{-in\theta} |n\rangle$

Def T by
$$T|n\rangle = |n+1\rangle$$
$$\Rightarrow T|\theta\rangle = \sum_n e^{-in\theta} T|n\rangle = e^{i\theta}|\theta\rangle$$

Finite g.t. $T \Longleftrightarrow$ Finite translations in periodic potential

$$\langle \theta' | e^{-iHt} | \theta \rangle = \sum_{m,n} e^{in\theta'} e^{-in\theta} \langle m | e^{-iHt} | n \rangle$$

$$= \sum_{m,n} e^{im(\theta'-\theta)} e^{i(m-n)\theta} \langle m | e^{-iHt} | n \rangle$$

→ depends only on $m-n$

$$= \sum_{m,k} e^{im(\theta'-\theta)} e^{ik\theta} \langle k | e^{-iHt} | 0 \rangle$$

$$= 2\pi\,\delta(\theta-\theta') \sum_{k} e^{ik\theta} \langle k | e^{-iHt} | 0 \rangle$$

⇒ $|n\rangle$ vacua may mix, but $|\theta\rangle$ vacua don't

⇒ World is in state built upon a particular θ-vac

↔ as if θ = const. of nature

Note: $\langle k | e^{-iHt} | 0 \rangle = \int [dA]\, e^{iS}$

→ effect of θ-vac is to ~~add to~~ insert $e^{ik\theta}$ for every path with $\Delta N = k$

Define a current

$$K_\mu = 4g^2 \epsilon_{\mu\nu\alpha\beta} \mathrm{Tr}\left\{A_\nu \partial_\alpha A_\beta + \frac{2ig}{3} A_\nu A_\alpha A_\beta\right\}$$

$$= 2g^2 \epsilon_{\mu\nu\alpha\beta} \mathrm{Tr}\left\{A_\nu F_{\alpha\beta} - \frac{2ig}{3} A_\nu A_\alpha A_\beta\right\}$$

→ Associated charge (not nec. conserved) is

$$Q \equiv \int d^3x\, K_0$$

$$= 2g^2 \epsilon_{ijk} \int d^3x\, \mathrm{Tr}\left\{A_i F_{jk} - \frac{2ig}{3} A_i A_j A_k\right\}$$

If $F=0$, $A = -\frac{i}{g} G^{-1}\partial G$,

$$Q = \frac{4}{3} \epsilon_{ijk} \int d^3x\, \mathrm{Tr}\; G^{-1}\partial_i G\; G^{-1}\partial_j G\;\; G^{-1}\partial_k G$$

$$= \frac{4}{3}(24\pi^2)\, N[G]$$

$$= 32\pi^2 \times [\text{winding number}]$$

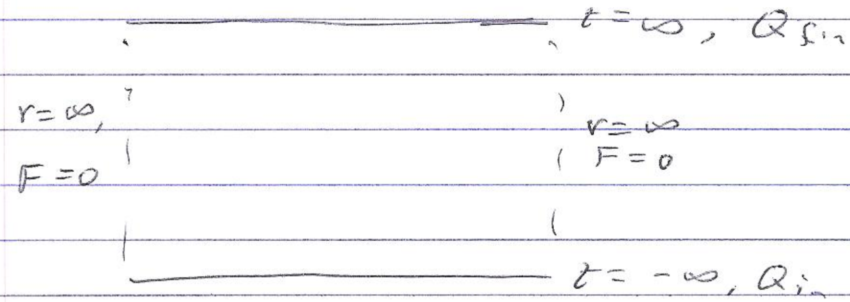
→ $\partial_\mu K_\mu = \epsilon_{\mu\nu\alpha\beta} g^2 \mathrm{Tr}\, F_{\mu\nu} F_{\alpha\beta}$

$\quad = 2g^2 \mathrm{Tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu}$ $\qquad [\sim \vec{E}\cdot\vec{B}]$

Note : K_μ gauge-dep

$\qquad \partial_\mu K_\mu$ gauge-indep.

Look at $\int d^4x\, \partial_\mu K_\mu$



$A_0=0$ gauge : surface terms at $r=\infty$ vanish

$$\Rightarrow \int d^4x\, \partial_\mu K_\mu = \int_{t=\infty} d^3x\, K^0 - \int_{t=-\infty} d^3x\, K^0 = \Delta Q$$

$$= 32\pi^2\, \Delta(\text{winding } \#)$$

$* \Rightarrow \int F\tilde{F}$ quantized

Suppose we added ~~$\alpha\mathcal{L}$~~

$$\Delta\mathcal{L} = \frac{\alpha\, g^2}{16\pi^2}\, Tr\, F\tilde{F} = \frac{\alpha}{32\pi^2 \cancel{g^2}}\, \partial_\mu K^\mu \rightarrow \cancel{\frac{\alpha}{32\pi^2 g^2}}$$

$\rightarrow$ in path integral, each path with $\Delta N = k$ gets factor $e^{ik\alpha}$

$\Rightarrow$ Choosing θ-vacuum has same effect as adding

$$\Delta\mathcal{L} = \frac{\theta\, g^2}{16\pi^2 \cancel{g^2}}\, tr\, F\tilde{F} \rightarrow \frac{\theta\, g^2}{16\pi^2}$$

Note: $\Delta\mathcal{L}$ = total divergence

$\Rightarrow$ (1) doesn't change classical eqs

(2) doesn't contrib to Feyn. rules

<u>BUT</u> CAN HAVE QM effects, non-perturbatively

[see below]