V. ϕ^4 theory : ε - expansion

1. Basic idea of the ϵ - expansion

effective Hamiltonian of Landau theory for the Ising universality class

$$-\mathcal{H}\{S\} = \int d^d\vec{r}\left[\frac{1}{2}(\nabla S)^2 + \frac{1}{2}r_0 S^2 + \frac{1}{4}u_0 S^4 - h_0 S\right]$$

Gaussian fixed point at $r^* = h^* = u^* = 0$

: power counting shows that the G FP is stable for $d > 4$.

power counting

$$r = br' \quad , \quad S = b^{-x} S'$$

$$-\mathcal{H}\{S, r_0, u_0, h_0\} = b^d \int d^d r' \Big[\frac{1}{2}(\nabla' S')^2 \cdot b^{-2-2x} + \frac{1}{2} r_0 b^{-2x} S'^2 + \frac{1}{4}u_0 b^{-4x} S'^4 - h_0 b^{-x} S'\Big]$$

$$= -\mathcal{H}\{S', r_0', u_0', h_0'\}$$

if $b^{d-2-2x} = 1 \Rightarrow x = \frac{d}{2} - 1$

$$\begin{cases} r_0 \to r_0' = b^{d-2x} r_0 = b^2 r_0 & \Rightarrow y_{r_0} = 2 \\ u_0 \to u_0' = b^{d-4x} u_0 = b^{4-d} u_0 & \Rightarrow y_{u_0} = 4-d \\ h_0 \to h_0' = b^{d-x} h_0 = b^{1+d/2} h_0 & \Rightarrow y_h = 1 + \frac{d}{2} \end{cases}$$

#

For $d<4$, the G fp. becomes unstable and the new f.p., the so-called Wilson-Fisher f.p. emerges.

2. RG beyond the Gaussian Model

1) setting up perturbation theory

effective hamiltonian

$$H\{S\} = \int d^d\vec{r}\left[\frac{1}{2}(\nabla S)^2 + \frac{1}{2}r_0 S^2 + \frac{1}{4}u_0 S^4 \left(-h_0 S\right)\right]$$

in Fourier space

$$\begin{cases} S(\vec{r}) = \dfrac{1}{V}\sum_{\vec{k}} \hat{S}_k e^{ik\cdot r} = \int^{\Lambda} \dfrac{d^d\vec{k}}{(2\pi)^d} \hat{S}_k e^{ik\cdot r} & \dfrac{d^d\vec{k}}{(2\pi)^d} \equiv \text{đ}k \\ \hat{S}_k = \int d^d\vec{r}\, S(r)\, e^{-ik\cdot r} \end{cases}$$

$$\int d^d r\, S(r)^4 = \int d^d\vec{r} \int \text{đ}k_1 \text{đ}k_2 \text{đ}k_3 \text{đ}k_4\, \hat{S}_{k_1}\hat{S}_{k_2}\hat{S}_{k_3}\hat{S}_{k_4}\, e^{i(k_1+\cdots+k_4)\cdot r}$$

$$= \int \text{đ}k_1 \text{đ}k_2 \text{đ}k_3 \text{đ}k_4\, (2\pi)^d \delta(k_1+\cdots+k_4)\, \hat{S}_{k_1}\cdots\hat{S}_{k_4}$$

$$\Rightarrow H\{\hat{S}\} = \frac{1}{2}\int \text{đ}k\, (r_0+k^2)|\hat{S}_k|^2 + \frac{1}{4}u_0 \int \text{đ}k_1\cdots\text{đ}k_4 (2\pi)^d \delta(\,)\, SSSS$$

Decompose $\hat{S}_k$ into two parts

$$\hat{S}_k = \begin{cases} \hat{S}'_\ell(k), & 0 < k < \Lambda/\ell \\ \hat{\sigma}_\ell(k), & \Lambda/\ell < k < \Lambda \end{cases}$$

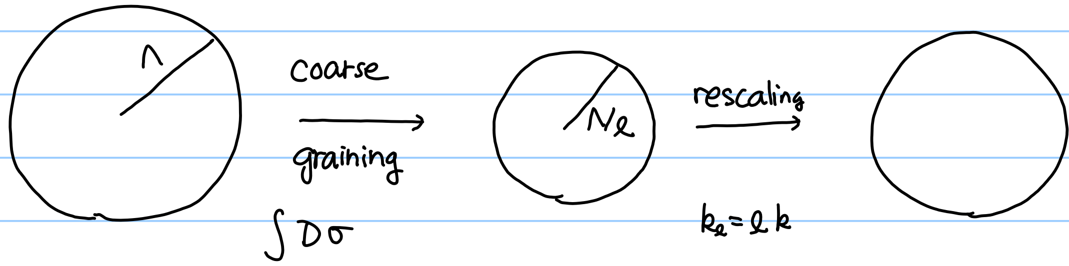
With this decomposition,

$$H\{\hat{S}\} = H_s\{\hat{S}'_\ell\} + H_\sigma\{\hat{\sigma}_\ell\} + V\{\hat{S}'_\ell, \hat{\sigma}_\ell\}$$

The partition function

$$Z(r_0, u_0) = \int D\hat{S}\, e^{-H} = \int D\hat{S}'_\ell\, e^{-H_s} \int D\hat{\sigma}\, e^{-H_\sigma - V}$$

Strategy :



Let us define $\langle A(S_\ell')\rangle_0 \equiv \dfrac{\int D\sigma_\ell \, e^{-\mathcal{H}_\sigma} A(S_\ell', \sigma_\ell)}{\int D\sigma_\ell \, e^{-\mathcal{H}_\sigma}}$

Then the partition function can be written as

$$Z = Z_\sigma(r_0) \int D S_\ell' \, e^{-\mathcal{H}_{S'}} \langle e^V \rangle_0$$

where $Z_\sigma = \int D\sigma_\ell \, e^{-\mathcal{H}_\sigma}$: contributes to the regular part of the free energy $\to$ ignore !

Cumulant expansion

$$\langle e^V \rangle_0 = \langle 1 + V + \tfrac{1}{2} V^2 + \tfrac{1}{3!} V^3 + \cdots \rangle_0$$

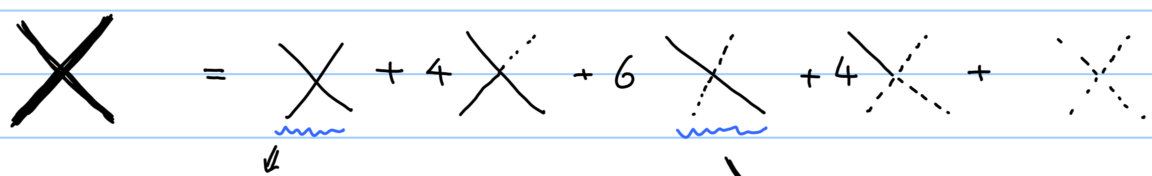
$$= e^{\langle V \rangle_0 + \frac{1}{2}\left[\langle V^2\rangle_0 - \langle V\rangle_0^2\right] + O(V^3)}$$

2) Calculation of $\langle V \rangle_0$: Strategy

$$\hat{S}_{k_1} \hat{S}_{k_2} \hat{S}_{k_3} \hat{S}_{k_4} = \left(S_\ell'(k_1) + \sigma_\ell(k_1)\right) \cdots \left(S_\ell'(k_4) + \sigma_\ell(k')\right)$$

$$= S'S'S'S' + 4 S'S'S'\sigma + 6 S'S'\sigma\sigma + 4 S'\sigma\sigma\sigma + \sigma\sigma\sigma\sigma$$

or



$\dfrac{u_0}{4} \int dk_1 \cdots dk_4 (2\pi)^d \delta(k_1 + \cdots + k_4) S_{k_1}' \cdots S_{k_4}'$

$\dfrac{u_0}{4} \int dk_1 \cdots dk_4 (2\pi)^d \delta(\,) S_{k_1}' S_{k_2}' \langle \sigma_{k_3} \sigma_{k_4} \rangle_0$

$\Rightarrow$ we need to evaluate $\langle \hat{\sigma}_\ell(k_1) \cdots \hat{\sigma}_\ell(k_m) \rangle_0$

3) Correlation Functions of $\hat{\sigma}_\ell(k)$: Wick's theorem

$$\mathcal{H}_\sigma = \int_{N_\ell}^{\Lambda} \bar{d}k \, (r_0 + k^2) \, |\hat{\sigma}_\ell(k)|^2$$

It is obvious that

$$\langle \hat{\sigma}_\ell(k_1) \cdots \hat{\sigma}_\ell(k_m) \rangle_0 = 0 \quad \text{for odd } m$$

for $m=2$

$$\langle \hat{\sigma}_\ell(k_1) \hat{\sigma}_\ell(k_2) \rangle_0 = \frac{\int D\sigma_\ell \; \hat{\sigma}_\ell(k_1)\hat{\sigma}_\ell(k_2) \, e^{-\frac{1}{V}\sum' A_k |\sigma_k|^2}}{Z_\sigma}$$

$$= \delta_{k_1+k_2,0} \left(-V \frac{\partial}{\partial A_{k_1}} \underbrace{\ln Z_\sigma}_{= \frac{1}{2}\sum_k \ln \frac{2\pi V}{A_k}} \right)$$

$$= \frac{V}{2} \delta_{k_1+k_2,0} \frac{1}{A_{k_1}}$$

$$= \delta_{k_1+k_2,0} \, V \, G_0(k_1) \xrightarrow{V\to\infty} (2\pi)^d \delta(k_1+k_2) \, G_0(k_1)$$

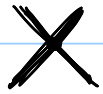
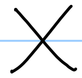
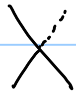
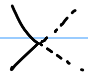
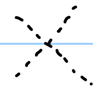
where $G_0(k) = \dfrac{1}{r_0 + k^2}$: propagator

higher order correlations are factorized into the product of the two point correlation $\Leftarrow$ Wick's theorem

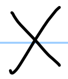
$$\langle 1234 \rangle_0 = \langle 12 \rangle_0 \langle 34 \rangle_0 + \langle 13 \rangle_0 \langle 24 \rangle_0 + \langle 14 \rangle_0 \langle 23 \rangle_0$$

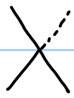
where $\langle ij \rangle_0 = (2\pi)^d \, \delta(k_i + k_j) \, G_0(k_i)$

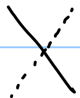
4) Evaluation of $\langle V_0 \rangle$

 $=$  $+ 4$  $+ 6$  $+ 4$  $+$ 

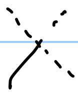
- coarse graining

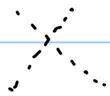
 $= \frac{u_0}{4} \int^{\Lambda/\ell} dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4) \, S'_\ell(k_1) \cdots S'_\ell(k_4)$

 $= \frac{u_0}{4} \int dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4) \, S'_\ell(k_1) \cdots S'_\ell(k_3) \cancel{\langle \sigma_\ell(k_4) \rangle_0}$

 $= \frac{u_0}{4} \int dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4) \, S'_\ell(k_1) S'_\ell(k_2) \underbrace{\langle \sigma_\ell(k_3) \sigma_\ell(k_4) \rangle_0}_{= (2\pi)^d \delta(k_3 + k_4) \frac{1}{r_0 + k_3^2}}$

$$= \frac{u_0}{4} \int^{\Lambda/\ell} dk_1 dk_2 \, S'_\ell(k_1) S_\ell(k_2) \, \delta(k_1 + k_2) \times \int_{\Lambda/\ell}^{\Lambda} \frac{dk_3}{r_0 + k_3^2}$$

 $= 0$

 : free energy $\rightarrow$ ignore

after coarse graining

$$Z(r_0, u_0) \propto \int DS' \, e^{-\mathcal{H}'(S')}$$

where
$$H'(S') = \frac{1}{2} \int^{\Lambda/\ell} dk \left(r_0 + k^2 + 6 \cdot \frac{u_0}{2} \int_{\Lambda/\ell}^{\Lambda} \frac{dk'}{r_0 + k'^2} \right) |\hat{S}_\ell(k)|^2$$
$$+ \frac{1}{4} u_0 \int^{\Lambda/\ell} dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4) \, \hat{S}'_\ell(k_1) \cdots \hat{S}'_\ell(k_4)$$

- rescaling : $k_\ell = \ell k$, $\hat{S}_\ell(k_\ell) = z^{-1}\hat{S}_\ell'(k)$

$$H(\hat{S}_\ell) = \frac{1}{2}\int^\Lambda dk_\ell \; \ell^{-d}\left(r_0+\ell^{-2}k_\ell^2+3u_0\int_{\Lambda/\ell}^\Lambda \frac{dk'}{r_0+k'^2}\right) z^2 \,|\hat{S}_\ell(k_\ell)|^2$$

$$+\frac{u_0}{4}\int^\Lambda dk_1\cdots dk_4 \; \ell^{-4d}(2\pi)^d \ell^d \,\delta(k_{\ell_1}+\cdots+k_{\ell_4})\, z^4 \,\hat{S}_\ell(k_{\ell_1})\cdots\hat{S}_\ell(k_{\ell_4})$$

$$\Rightarrow \begin{cases} u_2' = z^2\ell^{-d}\left(r_0+\ell^{-2}k_\ell^2+3u_0\int_{\Lambda/\ell}^\Lambda \frac{dk'}{r_0+k'^2}\right) \\ u_4' = z^4\ell^{-3d}u_0 \end{cases}$$

fixing the coefficient of k^2 term in u_2 : $\Rightarrow$ $z^2\ell^{-d-2} = 1$

$$z = \ell^{1+d/2}.$$

$$\therefore \begin{cases} u_{2,\ell}^{(1)} = \ell^2 r_0 + k_\ell^2 + 3u_0\ell^2\int_{\Lambda/\ell}^\Lambda \frac{dk}{r_0+k^2} \\ u_{4,\ell}^{(1)} = \ell^{4-d}u_0 = \ell^\varepsilon u_0 \end{cases}$$

3. Feynman Diagram

perturbation series $\rightarrow$ diagram

each term in V^m : $\left(\frac{u_0}{4}\right)^m \int dk_1\cdots dk_4\,(2\pi)^d\delta(k_1+\cdots+k_4)\,(S_\ell' \text{ or } \sigma_\ell)$

$$\cdots\int dk_{4m-3}\cdots dk_{4m}\,(2\pi)^d\delta(k_{4m-3}+\cdots+k_{4m})\,(S_\ell' \text{ or } \sigma_\ell)$$

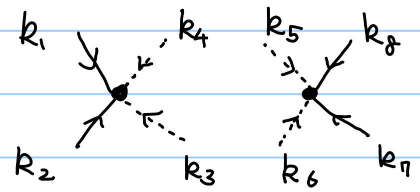
graphical representation : primitive diagram

$\left(\frac{u_0}{4}\right)\int dk\,dk'\,dk''\,dk'''\,(2\pi)^d\delta(k+k'+k''+k''') \Rightarrow$ • vertex

$S_\ell'(k) \Rightarrow$ edge with wavevector $\xrightarrow{k}$

$\sigma_\ell(k) \Rightarrow$ dotted edge with wavevector $\overset{k}{\dashrightarrow}$

e.g.) a term in V^2 may have

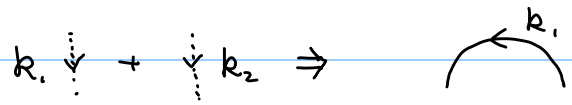


What happens when we take the average $\langle\ \rangle_0$?

There's no change for the solid line edge.

$\Rightarrow$ external line attached to $S'(k)$ with $0 < k < \Lambda/\ell$.

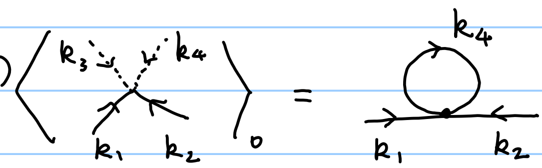
Wick's theorem $\Rightarrow$ pairing two broken lines to make an internal line



the internal line has the weight $(2\pi)^d \delta(k_1+k_2)\, G_0(k_1)$

and the momentum $\Lambda/\ell < k_1 < \Lambda$

ex)



from the vertex.

$$= \frac{u_0}{4}\int dk_1 dk_2 dk_3 dk_4\, (2\pi)^d \delta(k_1+k_2+k_3+k_4)$$

$$\times\ \underbrace{S'(k_1)\, S'(k_2)}_{\text{external line}} \times \underbrace{(2\pi)^d \delta(k_3+k_4)\ G_0(k_4)}_{\text{paired internal line}}$$

$$= \frac{u_0}{4}\int dk_1 dk_2\, (2\pi)^d \delta(k_1+k_2)\ S'(k_1) S'(k_2) \int_{\Lambda/\ell}^{\Lambda} dk_4\, \frac{1}{r_0+k_4^2}$$

$$= \frac{u_0}{4}\int^{\Lambda/\ell} dk_1\, |\hat{S}'_\ell(k_1)|^2 \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{r_0+k^2}$$

cumulant

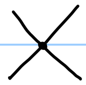
In summary, Feynman diagram for $\langle V^m \rangle_0^c$

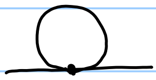
- Using m vertices with 4 legs per each vertex, draw all possible "connected" diagrams obtaind by pairing legs.

- each diagram is assigned to weights :
  1. label the momenta in the incoming sense at each vertex
  2. internal momenta range from Λ/ℓ to Λ, external momenta from 0 to Λ/ℓ
  3. associate a propagator $(2\pi)^d \delta(k_1+k_2) / (r_0+k_1^2)$ with each internal line
  4. associate a factor $\frac{u_0}{4}(2\pi)^d \delta(k_1+\cdots+k_4)$ with each four-point vertex

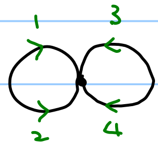
- degeneracy of a diagram $\leftarrow$ combinatorics

1) Feynman diagram for $\langle V \rangle_0$

$1 \times$  : $\dfrac{u_0}{4} \displaystyle\int_0^{\Lambda/\ell} dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4) \, \hat{S}_\ell'(k_1) \cdots \hat{S}_\ell'(k_4)$

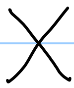
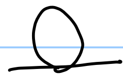
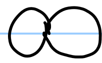
${}_4C_2 \times$  : $\dfrac{u_0}{4} \displaystyle\int_0^{\Lambda/\ell} dk_1 \, |S_\ell'(k_1)|^2 \int_{\Lambda/\ell}^{\Lambda} \dfrac{dk_2}{(r_0 + k_2)^2}$

↑ # of possible pairing

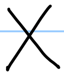
$\dfrac{{}_4C_2}{2} \times$  : $\dfrac{u_0}{4} \left(\displaystyle\int_{\Lambda/\ell}^{\Lambda} \dfrac{dk}{(r_0+k)^2} \right)^2$

why?) $\dfrac{u_0}{4} \displaystyle\int d_{1234} \, (2\pi)^d \delta(1+2+3+4) \, \dfrac{(2\pi)^d \delta(1+2)}{r_0 + k_1^2} \, \dfrac{(2\pi)^d \delta(3+4)}{r_0 + k_3^2}$

$(2\pi)^d$ factor?

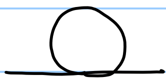
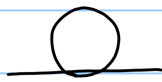
$\Rightarrow \langle V \rangle_0 =$  $+ 6$  $+ 3$ 

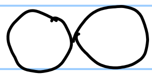
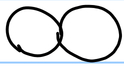
upon rescaling $k_\ell = \ell k$, $\hat{S}_\ell'(k) = z \hat{S}_\ell(k_\ell)$

 $\longrightarrow z^{4} \ell^{-4d} \ell^{1d}$  $= z^4 \ell^{-3d}$ 

momentum conservation

four external line   four momentum integral

 $\rightarrow z^2 \ell^{-d}$ 

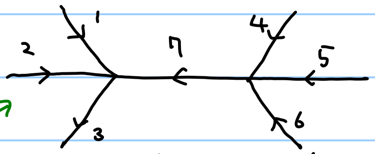
 $\rightarrow$ 

$(\text{diagram}) \longrightarrow z^{[\#\text{ of external line}]} \, \ell^{-d[\#\text{ of indep. mom. integral}]}$ (diagram)

2) Feynman Diagrams for $\langle V^2\rangle_0 - \langle V\rangle_0^2$

$$\langle V^2\rangle_c \equiv \langle V^2\rangle_0 - \langle V\rangle_0^2$$

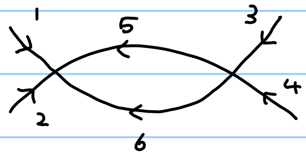
diagrams (connected only ← linked cluster theorem)



S^6 term

$$: \quad 4 \times 4 = 16$$

$$\left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \cdots dk_6 \, (2\pi)^d \delta(k_1 + \cdots + k_6)\, S_\ell'(k_1)\cdots S_\ell'(k_6) \int_{\Lambda/\ell}^{\Lambda} \frac{dk_7}{r_0^2 + (k_1+k_2+k_3)^2}$$

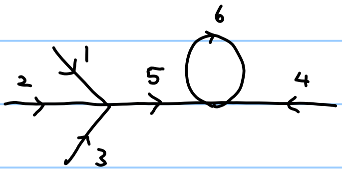


renormalize u_0

$$: \quad {}_4C_2 \times {}_4C_2 \times 2 = 72$$

$$\left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \cdots dk_4 \, (2\pi)^d \delta(k_1 + \cdots + k_4)\, S_\ell'(k_1)\cdots S_\ell'(k_4)$$

$$\times \int_{\Lambda/\ell}^{\Lambda} \frac{dk_5}{(r_0 + k_5^2)(r_0 + (k_1+k_2 \pm k_5)^2)}$$



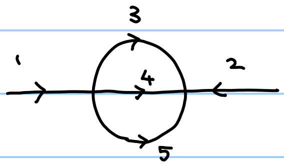
zero

$$: \quad 2 \times 4 \times 4 \times 3 = 96$$

좌우비대칭

momentum con. ⇒ $\delta(k_4 + k_5) \to 0$!

internal (k_5), external (k_4)

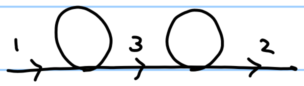


renormalize r_0

$O(u^2)$

$$: \quad 4 \times 4 \times 3 \times 2 = 96$$

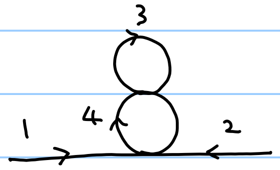
$$\left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \, |S_\ell'(k_1)|^2 \int_{\Lambda/\ell}^{\Lambda} \frac{dk_3\, dk_4}{(r_0+k_3^2)(r_0+k_4^2)(r_0 + (k_1 \pm k_3 \pm k_4)^2)}$$



zero

$: \quad 4 \times 3 \times 4 \times 3 = 144$

$\delta(k_1 + k_3) \to 0$

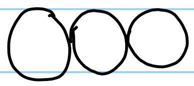


renormalize r_0

$O(u_0^2)$

$: \quad 2 \times {}_4C_2 \times {}_4C_2 \times 2 = 144$

$$\left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \, |S_\ell'(k_1)|^2 \left(\int_{\Lambda/\ell}^{\Lambda} \frac{dk_3}{r_0 + k_3^2} \right)^2$$

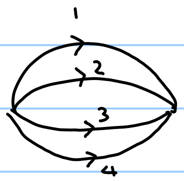


free energy

regular parts

$: \quad {}_4C_2 \times {}_4C_2 \times 2 = 72$

$$\left(\frac{u_0}{4}\right)^2 \left(\int_{\Lambda/\ell}^{\Lambda} \frac{dk}{r_0 + k^2} \right)^3$$



$: \quad 4 \times 3 \times 2 \times 1 = 24$

$$\left(\frac{u_0}{4}\right)^2 \int_{\Lambda/\ell}^{\Lambda} \frac{dk_1 \, dk_2 \, dk_3}{(r_0 + k_1^2)(r_0 + k_2^2)(r_0 + k_3^2)(r_0 + (k_1 + k_2 + k_3)^2)}$$

4. The RG recursion relation

higher order

ignore

the coarse grained Hamiltonian

$\langle V \rangle_0$ $\quad \frac{1}{2}\langle V^2 \rangle_c$

$$\mathcal{H}(S_\ell') = \left[\frac{1}{2}(r_0 + k^2) \quad + 6 \quad + 72 \quad + 48 \right]$$

$\langle V^0 \rangle_0$

$$+ \left[\qquad - 36 \right]$$

$\frac{1}{2}\langle V^2 \rangle_c$

rescaling

$$\mathcal{H}(S_\ell) = \left[\frac{1}{2}\left(r_0 + \frac{k_\ell^2}{\ell^2}\right) z^2 \ell^{-d} \quad + 6 z^2 \ell^{-d} \right]$$

$$+ \left[z^4 \ell^{-3d} \quad - 36 \, z^4 \ell^{-3d} \right]$$

RG recursion relation

$$\begin{cases} r_0' + k_\ell^2 = z^2 \ell^{-d} \left(r_0 + 3u_0 \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{r_0 + k^2} \right) + z^2 \ell^{-d-2} k_\ell^2 \\ \frac{u_0'}{4} = z^4 \ell^{-3d} \left(\frac{u_0}{4} - 36 \left(\frac{u_0}{4}\right)^2 \int_{\Lambda/\ell}^{\Lambda} \underbrace{\frac{dk}{(r_0+k^2)(r_0 + (k_1+k_2-k)^2)}} \right) \end{cases}$$

k-dependent coupling terms are generated. Consider only zero momentum contribution

fixing z from the coefficient of k_ℓ^2 $\Rightarrow$ $z^2 = \ell^{d+2}$.

$\Rightarrow$ $y_h = 1 + \frac{d}{2}$

$$\begin{cases} r_0' = \ell^2 (r_0 + 3u_0 I_1) \\ u_0' = \ell^{4-d} u_0 (1 - 9u_0 I_2) \end{cases}$$

where $I_1 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{r_0 + k^2}$, $I_2 = \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{(r_0 + k^2)^2}$.

Since $u_0 = O(\varepsilon)$, we only need to evaluate I_1 and I_2 up to the leading order in $\epsilon = 4-d$, that is, $d=4$.

$$I_1 = \frac{1}{(2\pi)^d} \int_{\Lambda/\ell}^{\Lambda} \frac{d^d\vec{k}}{r_0 + k^2} \simeq \frac{1}{(2\pi)^d} \int_{\Lambda/\ell}^{\Lambda} \frac{1}{k^2}\left(1 - \frac{r_0}{k^2} + O(r_0^2)\right) d^d\vec{k}$$

$$= \int_{\Lambda/\ell}^{\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k^2} - r_0 \int_{\Lambda/\ell}^{\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k^4} + O(r_0^2)$$

$$\overset{d=4}{=} K_4 \int_{\Lambda/\ell}^{\Lambda} k\,dk - r_0 K_4 \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{k} + O(r_0^2, \epsilon)$$

$$= \frac{K_4 \Lambda^2}{2} \left(1 - \frac{1}{\ell^2}\right) - r_0 K_4 \ln \ell + O(r_0^2, \epsilon)$$

where $K_4 = S_4 / (2\pi)^4 = 1/8\pi^2$ (S_4: surface area of unit sphere in 4d.)

$$I_2 = \int_{\Lambda/\ell}^{\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{(r_0+k^2)^2} = K_4 \ln \ell + O(r_0, \epsilon)$$

Hence, the recursion relation becomes

$$\begin{cases} r_0' = \ell^2\left[r_0 + \frac{3}{4\pi^2}\left(\frac{u_0}{4}\right)\Lambda^2\left(1-\frac{1}{\ell^2}\right) - \frac{3}{2\pi^2}\left(\frac{u_0}{4}\right) r_0 \ln \ell\right] \\ \qquad + O(\epsilon u_0, r^2 u) \\ \frac{u_0'}{4} = \ell^{\varepsilon}\left[\frac{u}{4} - \frac{9}{2\pi^2}\left(\frac{u_0}{4}\right)^2 \ln \ell\right] + O(\epsilon^3, r_0\epsilon^2) \end{cases}$$

Notice that $\ell^{\varepsilon} = e^{\varepsilon \ln \ell} \simeq 1 + \varepsilon \ln \ell + \cdots$

$$\Rightarrow \begin{cases} r_0' = \ell^2\left(r_0 + \frac{3}{4\pi^2}\left(\frac{u_0}{4}\right)\Lambda^2\left(1-\frac{1}{\ell^2}\right) - \frac{3}{2\pi^2}\left(\frac{u_0}{4}\right) r_0 \ln \ell\right) \\ \frac{u_0'}{4} = \frac{u_0}{4} + \frac{u_0}{4}\left(\epsilon - \frac{9}{2\pi^2}\frac{u_0}{4}\right)\ln \ell \end{cases}$$

- Fixed points up to $O(\varepsilon)$.

i) Gaussian fixed point (G f.p.)

$$r_0^* = u_0^* = 0$$

ii) Wilson-Fisher fixed point (WF f.p.)

$$(u_0' = u_0) \Rightarrow u_0^* = \frac{8\pi^2}{9}\epsilon$$

$$(r_0' = r_0) \Rightarrow r_0^* = -\frac{\Lambda^2}{6}\epsilon$$

linearize RG equations near those fixed points:

$$\begin{pmatrix} \delta r_0' \\ \delta u_0' \end{pmatrix} = M \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix}$$

where

$$M = \begin{pmatrix} \partial r_0'/\partial r_0 & \partial r_0'/\partial u_0 \\ \partial u_0'/\partial r_0 & \partial u_0'/\partial u_0 \end{pmatrix}_*$$

i) at the G f.p.

$$M_G = \begin{pmatrix} \ell^2 & \frac{3}{16\pi^2}\Lambda^2(\ell^2-1) \\ 0 & 1+\varepsilon \ln \ell = \ell^{\varepsilon} \end{pmatrix}$$

eigen system

$$\begin{cases} \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ with } \Lambda_1 = \ell^{y_1} \text{ with } y_1 = 2 \\ \vec{v}_2 = \begin{pmatrix} -3\Lambda^2/16\pi^2 \\ 1 \end{pmatrix} \text{ with } \Lambda_2 = \ell^{y_2} \text{ with } y_2 = \epsilon \end{cases}$$

ii) at the WF f.p.

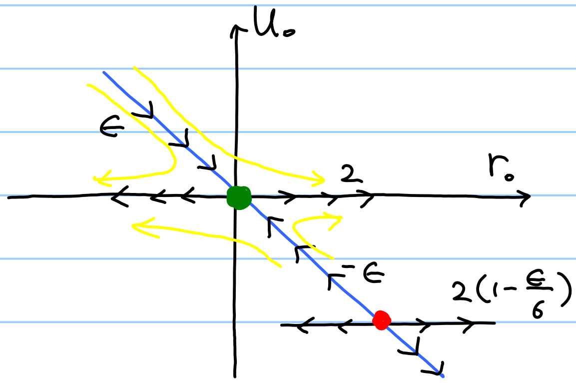
$$M_{WF} = \begin{pmatrix} \ell^2\left(1 - \frac{3}{8\pi^2}u_0^* \ln \ell\right) & \frac{3}{16\pi^2}\Lambda^2(\ell^2-1) - \cancel{\frac{3r_0^*}{8\pi^2}\ell^2 \ln \ell} \\ 0 & 1 + \left(\epsilon - \frac{9}{4\pi^2}u_0^*\right)\ln \ell \end{pmatrix}$$

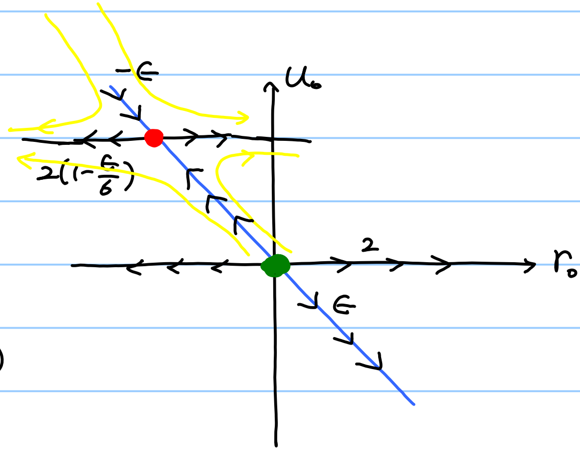
$$= \begin{pmatrix} \ell^{(2-\frac{\epsilon}{3})} & \frac{3\Lambda^2}{16\pi^2}(\ell^2-1) \\ 0 & \ell^{-\varepsilon} \end{pmatrix}$$

eigensystem: $\vec{v}_1^{WF} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Lambda_1 = \ell^{y_1}$ with $y_1 = 2\left(1 - \frac{\epsilon}{6}\right)$

$\vec{v}_2^{WF} = \begin{pmatrix} -3\Lambda^2(\ell^2-1)/16\pi^2 \\ 1 \end{pmatrix}$ and $\Lambda_2 = \ell^{y_2}$ with $y_2 = -\epsilon$

- RG flows





$d>4$ or $\epsilon<0$ $\qquad\qquad$ $d<4$ or $\epsilon>0$

Critical properties are governed by the Gaussian fixed point. $\leftrightarrow$ Critical properties are governed by the Wilson-Fisher fixed point.

- Critical exponents for $d<4$ $(\epsilon = 4-d > 0)$ $\qquad (O(\epsilon))$

$$\nu = \frac{1}{y_t} = \frac{1}{2}\left(1+\frac{\epsilon}{6}\right)$$

$1+\frac{d}{2}$ from wave function renormalization

$$\beta = \frac{d-y_h}{y_t} = \frac{\frac{d}{2}-1}{2(1-\frac{\epsilon}{3})} = \frac{1-\frac{\epsilon}{2}}{2(1-\frac{\epsilon}{3})} = \frac{1}{2}\left(1-\frac{\epsilon}{3}\right)$$

$$\gamma = \frac{2y_h-d}{y_t} = 1+\frac{\epsilon}{6}$$

$$\eta = 0$$

$$\alpha = 2-\frac{d}{y_t} = 2-\frac{4-\epsilon}{2(1-\frac{\epsilon}{6})} = 2-2\left(1-\frac{\epsilon}{4}\right)\left(1+\frac{\epsilon}{6}\right)$$
$$= \frac{\epsilon}{6}$$

cf) ϕ^4 theory with $\epsilon=1$ $\Leftrightarrow$ 3-d Ising model

	ϕ^4 ($O(\epsilon)$)	3D Ising
α	0.167	0.110(5)
β	0.333	0.3250(15)
γ	1.167	1.2405(15)
δ	4.0	4.82(4)
ν	0.583	0.630(2)
η	0	0.032(3)

5. RG recursion relation for infinitesimal scale tr. $l = e^s$

recursion relation $\begin{cases} r_0' = l^2 (r_0 + 3u_0 I_1) \\ u_0' = u_0 l^\epsilon (1 - 9u_0 I_2) \end{cases}$

where $$I_n \equiv \int_{\Lambda/l}^{\Lambda} \frac{đk}{(r_0+k^2)^n}$$

for $s \ll 1$,

$$I_n = \int_{\Lambda(1-s)}^{\Lambda} \frac{đk}{(r_0+k^2)^n} = \frac{\Lambda^d K_4}{(r_0+\Lambda^2)^n} s$$

redefining $r_0 \equiv r\Lambda^2$ and $u_0 = u$

$$r' = e^{2s}\left(r + 3u \frac{K_4}{1+r} s\right)$$

$$= r + s\left[2r + K_4 \frac{3u}{1+r}\right]$$

$$u' = e^{\epsilon s} u \left(1 - K_4 \frac{9u}{(1+r)^2} s\right)$$

$$= u + su\left[\epsilon - K_4 \frac{9u}{(1+r)^2}\right]$$

in the differential form,

$$\begin{cases} \dfrac{dr}{ds} = 2r + \dfrac{Au}{1+r} \equiv p(r,u) \\ \dfrac{du}{ds} = u\left(\epsilon - \dfrac{Bu}{(1+r)^2}\right) \equiv q(r,u) \end{cases}$$

where $A = 3K_4 = \dfrac{3}{8\pi^2}$, $B = 9K_4 = \dfrac{9}{8\pi^2}$

fixed point $p = q = 0$

$$(q=0) \Rightarrow \quad u^* = 0 \quad \text{or} \quad \frac{(1+r)^2\epsilon}{B} \simeq \frac{8\pi^2}{9}\epsilon$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$r^* = 0 \qquad\qquad r = -\frac{A\epsilon}{2B} = -\frac{\epsilon}{6}$$

$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

Gaussian fixed point Wilson-Fisher fixed point

derivatives at the fixed points

$$T = \begin{pmatrix} \dfrac{\partial p}{\partial r} = 2 - \dfrac{Au}{(1+r)^2} & \dfrac{\partial p}{\partial u} = \dfrac{A}{1+r} \\ \dfrac{\partial q}{\partial r} = \dfrac{2Bu^2}{(1+r)^3} & \dfrac{\partial q}{\partial u} = \epsilon - \dfrac{2Bu}{(1+r)^2} \end{pmatrix}$$

at the fixed points

i) G. f.p

$$T = \begin{pmatrix} 2 & A \\ 0 & \epsilon \end{pmatrix} \Rightarrow y_1 = 2, \; y_2 = \epsilon$$

ii) W-F. f.p

$$T = \begin{pmatrix} 2 - \frac{\epsilon}{3} & \frac{3}{8\pi^2} \\ 0 & -\epsilon \end{pmatrix} \Rightarrow y_1 = 2\left(1 - \frac{\epsilon}{6}\right), \; y_2 = -\epsilon$$

in the differential form,

$$\begin{cases} \dfrac{dr}{ds} = 2r + \dfrac{Au}{1+r} \equiv p(r,u) \\ \dfrac{du}{ds} = u\left(\epsilon - \dfrac{Bu}{(1+r)^2}\right) \equiv q(r,u) \end{cases}$$

where $A = 3K_4 = \dfrac{3}{8\pi^2}$, $B = 9K_4 = \dfrac{9}{8\pi^2}$

fixed point $p = q = 0$

$(q=0) \Rightarrow \quad u^* = 0 \quad$ or $\quad \dfrac{(1+r)^2\epsilon}{B} \simeq \dfrac{8\pi^2}{9}\epsilon$

$\downarrow \qquad\qquad \downarrow$

$r^* = 0 \qquad\qquad r = -\dfrac{A\epsilon}{2B} = -\dfrac{\epsilon}{6}$

$\Uparrow \qquad\qquad \Uparrow$

Gaussian fixed point $\qquad$ Wilson-Fisher fixed point

derivatives at the fixed points

$$T = \begin{pmatrix} \dfrac{\partial p}{\partial r} = 2 - \dfrac{Au}{(1+r)^2} & \dfrac{\partial p}{\partial u} = \dfrac{A}{1+r} \\ \dfrac{\partial q}{\partial r} = \dfrac{2Bu^2}{(1+r)^3} & \dfrac{\partial q}{\partial u} = \epsilon - \dfrac{2Bu}{(1+r)^2} \end{pmatrix}$$

at the fixed points

i) G. f.p

$$T = \begin{pmatrix} 2 & A \\ 0 & \epsilon \end{pmatrix} \Rightarrow y_1 = 2, \; y_2 = \epsilon$$

ii) W-F. fp

$$T = \begin{pmatrix} 2 - \frac{\epsilon}{3} & \frac{3}{8\pi^2} \\ 0 & -\epsilon \end{pmatrix} \Rightarrow y_1 = 2\left(1 - \frac{\epsilon}{6}\right), \; y_2 = -\epsilon$$