IV-2. Momentum shell RG of the 2D XY model

(1) XY model

planar spin at site i : $\vec{S}_i = (\cos\theta_i, \sin\theta_i)$

XY model Hamiltonian

$$H_{XY} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

U(1) symmetry

Partition ftn.

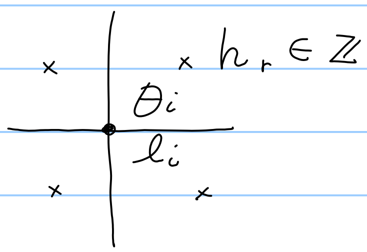
$$Z = \prod_i \int_{-\pi}^{\pi} d\theta_i \; \exp\left\{ K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right\}$$

$$e^{-K(1-\cos\Delta\theta)} \approx \cancel{e^{-\frac{K}{2}(\Delta\theta)^2}}$$

Villain transformation

$$e^{-K(1-\cos\Delta\theta)} \cong \sum_{l=-\infty}^{\infty} e^{il(\Delta\theta)} \frac{1}{\sqrt{2\pi K}} e^{-l^2/2K}$$

Fourier series



$$\Rightarrow \quad Z = \sum_{h_1=-\infty}^{\infty} \cdots \sum_{h_N=-\infty}^{\infty} e^{-\frac{1}{2K}\sum_{\langle rr' \rangle}(h_r - h_{r'})^2}$$

: solid-on-solid model on the dual lattice

Poisson summation formula

$$\sum_{h=-\infty}^{\infty} g(h) = \sum_{m=-\infty}^{\infty} \int d\phi \; g(\phi)\, e^{2\pi i m\phi}$$

$$\Rightarrow Z = \prod_r \int d\phi_r \sum_{m_r=-\infty}^{\infty} \exp\left\{ -\frac{1}{2K} \sum_r |\nabla\phi|^2 + 2\pi i \sum_r m(r)\phi(r) \right\}$$

$m(r)$: voticity

after Gaussian integration over $\{\phi\}$,

$$Z = \sum_{\{m(r)\}}{}' \exp\left\{ -\frac{1}{2}\pi^2 K \sum_r m(r)^2 - 2\pi^2 K \sum_{r,r'} m(r)\, G^{(0)}(r-r')\, m(r') \right\}$$

where
$$G^{(0)}(r) = G(r) - G(0) \simeq -\frac{1}{2\pi} \ln r$$

$$G(r) = \int \frac{d^2 q}{(2\pi)^2} \frac{e^{iq\cdot r}}{4 - 2\cos q_x - 2\cos q_y}$$

: lattice Green's ftn.

$\Leftarrow$ Coulomb gas

$$Z = \prod_r \int d\phi_r \sum_{m_r} \exp\left\{ -\frac{1}{2K} \sum_r |\nabla\phi|^2 + 2\pi i \sum_r m_r \phi_r + (\ln y) \sum_r m_r^2 \right\}$$

y : vortex fugacity

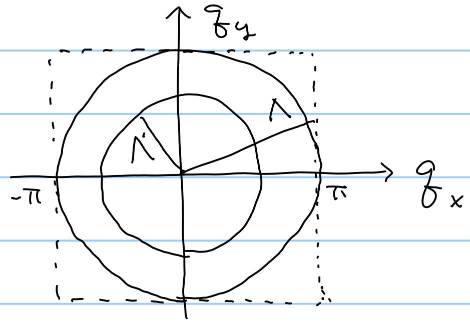
dilute limit ($y \ll 1$) : $\sum_{m_r=-\infty}^{\infty} \to \sum_{m_r=0,\pm1}$

$$\Rightarrow Z = \prod_r \int d\phi_r \exp\left\{ -\frac{1}{2} \int d^2 r |\nabla\phi|^2 + 2y \int d^2 r \cos(2\pi\sqrt{K}\phi) \right\}$$

: sine-Gordon model

2) Momentum shell R.G.

Strategy : ① Fourier tr. $\phi(r) = \int \frac{d^2q}{(2\pi)^2} \phi(q) e^{iq\cdot r}$



$$\phi(r) = \phi_<(r) + \phi_>(r)$$

where $\phi_<(r) = \int_{\Lambda'} \frac{d^2q}{(2\pi)^2} \phi(q) e^{iq\cdot r}$

: long wavelength part

$$\phi_>(r) = \phi(r) - \phi_<(r)$$

: short wavelength part

② Coarse graining

$$Z = \int D\phi_< D\phi_> \, e^{-H(\phi_<, \phi_> ; K, y)}$$

partial integration over $\phi_>$

$\hookrightarrow \quad Z = \int D\phi_< \exp\{-H'(\phi_< ; K, y)\}$

③ Rescaling (scale factor $b = \Lambda/\Lambda'$)

$$q \to q' = bq \quad (\text{or } r \to r' = r/b)$$

$$\phi_<(q) \to \phi'(q') = z\,\phi_<(q)$$

④ RG eq.

$$K' = f_K(K, y; b) \;, \quad y' = f_y(K, y; b)$$

- Coarse-graining

$$Z = \int [D\phi_<] \exp\left\{ -\frac{1}{2}\int d^2r\, |\nabla \phi_<|^2 \right\} Z'(\phi_<)$$

$$Z'(\phi_<) = \int [D\phi_>] \exp\left\{ -\frac{1}{2}\int d^2r\, |\nabla\phi_>|^2 + \underbrace{2y\int d^2r \cos\left(2\pi\sqrt{K}(\phi_< + \phi_>)\right)}_{\text{perturbative expansion}} \right\}$$

$$\propto 1 + 2y\int d^2r \left\langle \cos 2\pi\sqrt{K}(\phi_< + \phi_>) \right\rangle_>$$

$$+ \frac{(2y)^2}{2}\int d^2r \int d^2r' \left\langle \cos 2\pi\sqrt{K}(\phi_<(r) + \phi_>(r)) \times \cos(r \to r') \right\rangle_>$$

$$+ \cdots$$

$$\langle (\cdots) \rangle_> \equiv \int [D\phi_>] (\cdots) e^{-\frac{1}{2}\int d^2r\, |\nabla\phi_>|^2} \Big/ \int [D\phi_>] e^{-\frac{1}{2}\int d^2r\, |\nabla\phi_>|^2}$$

$$\left\langle \cos 2\pi\sqrt{K}(\phi_< + \phi_>) \right\rangle_> = \mathrm{Re} \left\langle e^{2\pi i\sqrt{K}\phi_<} \, e^{2\pi i \sqrt{K}\phi_>} \right\rangle_>$$

$$= e^{-2\pi^2 K G_h(0)} \cos\left(2\pi\sqrt{K}\phi_<\right)$$

$$G_>(r) = \int_{\Lambda' < q < \Lambda} \frac{d^2q}{(2\pi)^2} \frac{e^{iq\cdot r}}{q^2} = \frac{1}{2\pi} J_0(\Lambda |r|) \frac{d\Lambda}{\Lambda}$$

setting $\quad A(r) \equiv e^{-2\pi^2 K G_>(r)}$

$$Z = C \int [D\phi_<] \exp\left\{ -\frac{1}{2}\left(1 + \frac{a_2}{2}\pi^2 K A^2(0)(2y)^2\right)\int d^2x\, |\nabla\phi_<|^2 \right.$$

$$\left. + 2y A(0) \int d^2r \cos\left(2\pi\sqrt{K}\phi_<\right) \right\}$$

$$a_2 \equiv \int d^2\xi\, \xi^2 \left[A^{-2}(\xi) - 1\right]$$

- rescaling

$$\begin{cases} r \to r' = r/b, \quad q \to q' = bq \\ \phi_<(r) \to \phi'(r') = z\,\phi_<(r) \end{cases}$$

$$Z = C\int[D\phi']\exp\Big\{-\frac{1}{2}z^{-2}\Big(1+\frac{1}{2}\pi^2K(2y)^2A^2(0)\Big)\int|\nabla\phi'|d^2r'$$

$$+ 2yA(0)b^2\int d^2r'\cos\big(2\pi\sqrt{K}z^{-1}\phi'\big)\Big\}$$

- RG eqs.

$$z = \Big(1+\frac{a_2}{2}\pi^2K(2y)^2A^2(0)\Big)^{1/2}$$

$$\begin{cases} K' = \dfrac{K}{1+\frac{a_2}{2}\pi^2K(2y)^2A^2(0)} \\ y' = yA(0)b^2 \end{cases}$$

$$a_2 = \int d^2\xi\;\xi^2\big(A^{-2}(\xi)-1\big)$$

$$= \int d^2\xi\;\xi^2\big(e^{4\pi^2KG_>(\xi)}-1\big)$$

$$= 4\pi^2K\underbrace{\Big(\int d\xi\;\xi^3J_0(\Lambda\xi)\Big)}_{\text{"}\alpha_2\text{ : constant}}\frac{d\Lambda}{\Lambda}$$

$$A(0) = e^{-2\pi^2KG_>(0)} = e^{-\pi KJ_0(0)\frac{d\Lambda}{\Lambda}}$$

$$\sim 1-\pi K\frac{d\Lambda}{\Lambda}$$

$$b = \frac{\Lambda}{\Lambda'} = 1+\frac{d\Lambda}{\Lambda}$$

$$\begin{cases} dy = y' - y = y\left(A(0)\, b^2 - 1\right) = y\left(2 - \pi K\right) \dfrac{d\Lambda}{\Lambda} \\ dK = -\dfrac{1}{2} \pi^2 K^2 (2y)^2 A^2(0)\, a_2 \\ \quad\;\; = -2\pi^4 \alpha_2 K^3 (2y)^2 \dfrac{d\Lambda}{\Lambda} \end{cases}$$

for small $x \equiv \pi K - 2$

$$\begin{cases} dy = -xy \dfrac{d\Lambda}{\Lambda} \\ dx = -\underbrace{64\pi^2 \alpha_2}_{\Rightarrow 1} y^2 \dfrac{d\Lambda}{\Lambda} \end{cases}$$

$$\Lambda' = e^{-dl} \Lambda \quad \rightarrow \quad d\Lambda/\Lambda = dl$$

$$\begin{cases} d(y^2) = -2xy^2\, dl \\ dx = -y^2\, dl \end{cases}$$

$$\Rightarrow \quad x^2 - y^2 = c$$

- RG flow