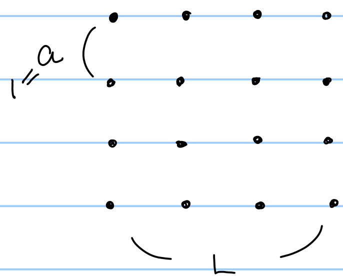
IV-1. Momentum shell RG - Gaussian model

(1) Gaussian model



d-dim. hypercubic lattice

$-\infty < S(\vec{r}) < \infty$: continuous field at site $\vec{r}$ ← in units of a

$$\beta H_0 = K \sum_{\langle r,r' \rangle} \left(S(r) - S(r') \right)^2 + \frac{m}{2} \sum_r S^2(r)$$

$$= -\frac{1}{2} \sum_r \sum_{r'} S(r) A(r,r') S(r')$$

$$= -\frac{1}{2} \tilde{S} A S$$

where $S = (S_1, \cdots, S_{L^d})^t = \tilde{S}^t$

$$A(r,r') = \begin{cases} m & \text{if } r = r' \\ -K & \text{if } r \text{ and } r' \text{ are neighboring} \\ 0 & \text{otherwise} \end{cases}$$

- Partition ftn. and correlation ftn.

identity : $\underbrace{\int [DS]\, e^{-\frac{1}{2}\tilde{S}AS + \tilde{H}S}}_{Z[H]} = \underbrace{\frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}}}_{C\,:\ \text{constant}} e^{\frac{1}{2}\tilde{H}A^{-1}H}$

$$\Rightarrow \quad Z_G = Z[H=0] = C$$

$$G(r,r') \equiv \langle S(r) S(r') \rangle$$

$$= \frac{1}{Z_G} \int [DS]\, S(r) S(r')\, e^{-\frac{1}{2}\tilde{S}AS}$$

$$= \frac{1}{Z_G} \frac{\partial}{\partial H(r)} \frac{\partial}{\partial H(r')} Z[H] \Big|_{H=0}$$

$$= (A^{-1})(r, r') \quad \Leftarrow \text{Wick theorem}$$

- Fourier transformation

$$S(r) = \frac{1}{L^d} \sum_k e^{ik\cdot r} S_k$$

where $\vec{k} = (k_1, \cdots, k_d)$

$$k_\alpha = \frac{2\pi m_\alpha}{La} \quad \left(-\frac{L}{2} < m_\alpha \le \frac{L}{2} \right)$$

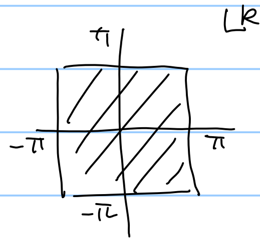
$-\pi < k_\alpha \le \pi$: in unit of $\left(\frac{1}{a}\right)$

$$S_k = \sum_r S(r) e^{-ik\cdot r} \quad (S_{-k} = S_k^*)$$

in continuum notation,

$$S(r) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot r} S_k$$

$$S_k = \int d^d r \, e^{-ik\cdot r} S(r)$$



$$H_G = \frac{1}{2} \sum_r \sum_{r'} S(r) A(r\, r') S(r') = \frac{1}{2} \sum_k S_k^* A(k) S_k$$

with $A(k) = \sum_r e^{-ik\cdot r} A(r)$

$$= m - 2K \sum_{\alpha=1}^{d} \cos k_\alpha$$

- Critical phenomena

at small $\vec{k}$,

$$A(k) \simeq (m - 2dK) + K k^2$$

$$\Rightarrow \quad A^{-1}(\vec{r}) = \frac{1}{L^d} \sum_k e^{ik\cdot r} A(k)^{-1}$$

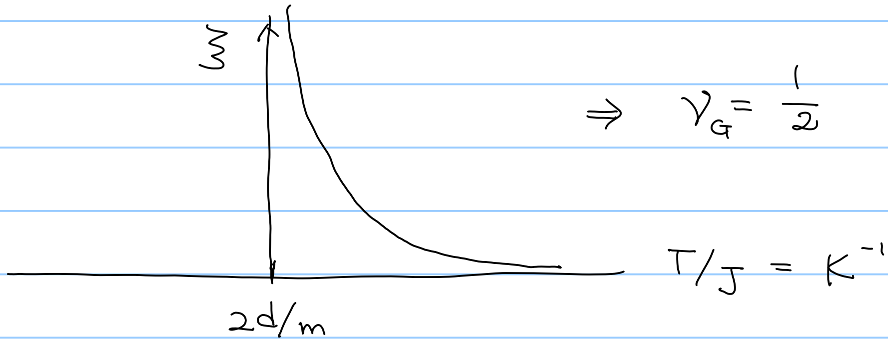
$\varepsilon = (m-2dK)/K$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot r}}{K k^2 + \tilde{r}} = \frac{1}{K} \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot r}}{k^2 + \varepsilon}$$

$$\sim e^{-|r|/\xi}$$

$$\therefore \quad G(\vec{r}) \sim e^{-|\vec{r}|/\xi}$$

with $\quad \xi = \varepsilon^{-\frac{1}{2}} \sim (m - 2dK)^{-\frac{1}{2}}$



$$\Rightarrow \quad \nu_G = \frac{1}{2}$$

- correlation function at $K = K_c$

$$G(r) \sim \int d^dk \, \frac{e^{ik\cdot r}}{k^2} \sim r^{-(d-2)} \sim r^{-(d-2+\eta)}$$

$$\Rightarrow \quad \eta_G = 0.$$

(2) Momentum shell RG.

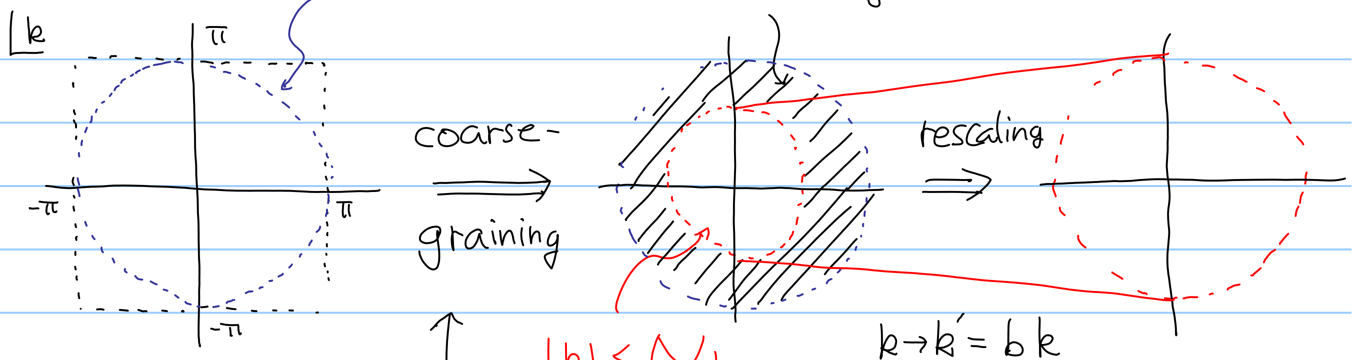
$$S(\vec{r}) = \frac{1}{V} \sum_{\vec{k}} S_{\vec{k}} \, e^{i\vec{k}\cdot\vec{r}}$$

Fourier mode

ultraviolet cutoff $\sim O(\frac{1}{a})$

$|k| < \Lambda$

coarse graind out



$|k| < \Lambda/b$

$k \to k' = bk$

$$\Delta k = \frac{2\pi}{L}$$

$$\Delta k' = \frac{2\pi b}{L} = \frac{2\pi}{(L/b)}$$

decimation in $\vec{k}$ space

- Momentum shell RG for the Gaussian model

effective Hamiltonian $\mathcal{H} = -\int d^d\vec{r} \left(\frac{1}{2}|\nabla S|^2 + \frac{r_0}{2} S^2 \right)$

in Fourier space,

$$\begin{cases} S(r) = \frac{1}{V}\sum_k \hat{S}_k \, e^{ik\cdot r} \Rightarrow \int \frac{d^dk}{(2\pi)^d} \hat{S}_k \, e^{ik\cdot r} \\ \hat{S}_k = \int_V d^d\vec{r}\, S(r) e^{-ik\cdot r} \end{cases}$$

$$\begin{cases} \frac{1}{V}\sum_k e^{ik\cdot(r-r')} = \delta(r-r') \\ \int d^d\vec{r}\ e^{ik\cdot r} = V\delta_{k,0} \end{cases}$$

$$\mathcal{H} = \int d^dr \left\{ \frac{1}{2}\frac{1}{V^2}\sum_{k,k'} \hat{S}_k \hat{S}_{k'} e^{i(k+k')\cdot r}\left[(ik)\cdot(ik') + r_0\right]\right\}$$

$$= \frac{1}{V}\sum_k \frac{1}{2}|\hat{S}_k|^2 (r_0 + k^2)$$

$\hat{S}_{-k} = \hat{S}_k^*$

ultra-violet cutoff

The sum is over $0 < |k| < \Lambda$

1) Coarse graining: integrating out the short wavelength degrees of freedom with $\Lambda/\ell < |k| < \Lambda$

scale factor.

introduce a notation

$$\hat{S}_k = \begin{cases} \hat{S}'_\ell(k) & \text{for } 0 < |k| < \Lambda/\ell \\ \hat{\sigma}_\ell(k) & \text{for } \Lambda/\ell < |k| < \Lambda \end{cases}$$

$$\mathcal{H} = \frac{1}{V}\sum_{k<\Lambda/\ell} \frac{1}{2}|\hat{S}_\ell'(k)|\,(r_0+k^2) + \frac{1}{V}\sum_{\Lambda/\ell<k<\Lambda}\frac{1}{2}|\hat{\sigma}_\ell(k)|\,(r_0+k^2)$$

The partition function

$$Z(r_0) = Z_s \cdot Z_\sigma$$

where $$Z_\sigma = \int \prod_k d\sigma_\ell \, e^{-\frac{1}{V}\sum \frac{1}{2}|\hat{\sigma}_\ell(k)|\,(r_0+k^2)}$$

$$= \prod_k \sqrt{\frac{2\pi V}{r_0+k^2}} = \exp\left[\frac{1}{2}\sum_{\Lambda/\ell<k<\Lambda} \ln \frac{2\pi V}{r_0+k^2}\right]$$

$\therefore$ does not renormalize the coupling constant
contributes only to regular part of free energy

$$Z_s = \int \prod_k dS_\ell' \exp\left[-\frac{1}{2V}\sum_{k<\Lambda/\ell}|\hat{S}_\ell'(k)|\,(r_0+k^2)\right]$$

2) Rescaling

$$k_\ell \equiv \ell k \quad \text{and} \quad \hat{S}_\ell(k_\ell) \equiv z^{-1}\hat{S}_\ell'(k)$$

wave function renormalization

$$Z_s(r_0) = \int D\hat{S}_\ell \exp\left[-\frac{1}{2}\int^\Lambda \frac{d^dk_\ell}{(2\pi)^d}\cdot \ell^{-d}\,(r_0+\ell^{-2}k_\ell^2)\,z^2\,|\hat{S}_\ell(k_\ell)|^2\right]$$

$$= \int D\hat{S}_\ell \exp\left[-\frac{1}{2}\int^\Lambda \frac{d^dk_\ell}{(2\pi)^d}\,(r_0'+k_\ell^2)\,|\hat{S}_\ell(k_\ell)|^2\right]$$

the renormalized system has the same form of the Hamiltonian
by choosing $\ell^{-d-2}z^2 = 1 \Rightarrow z = \ell^{1+\frac{d}{2}}$

r_0 renormalizes to $r_0' = \ell^{-d}z^2 r_0 = \ell^2 r_0$

$r_0' = \ell^2 r_0$: RG recursion relation

$$\mathcal{H} = \frac{1}{V}\sum_{k<\Lambda/\ell} \frac{1}{2}|\hat{S}_\ell'(k)| (r_0 + k^2) + \frac{1}{V}\sum_{\Lambda/\ell<k<\Lambda} \frac{1}{2}|\hat{\sigma}_\ell(k)| (r_0+k^2)$$

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$$= \int D\hat{S}_\ell \exp\left[-\frac{1}{2}\int^\Lambda \frac{d^d k_\ell}{(2\pi)^d} (r_0' + k_\ell^2) |\hat{S}_\ell(k_\ell)|^2\right]$$

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