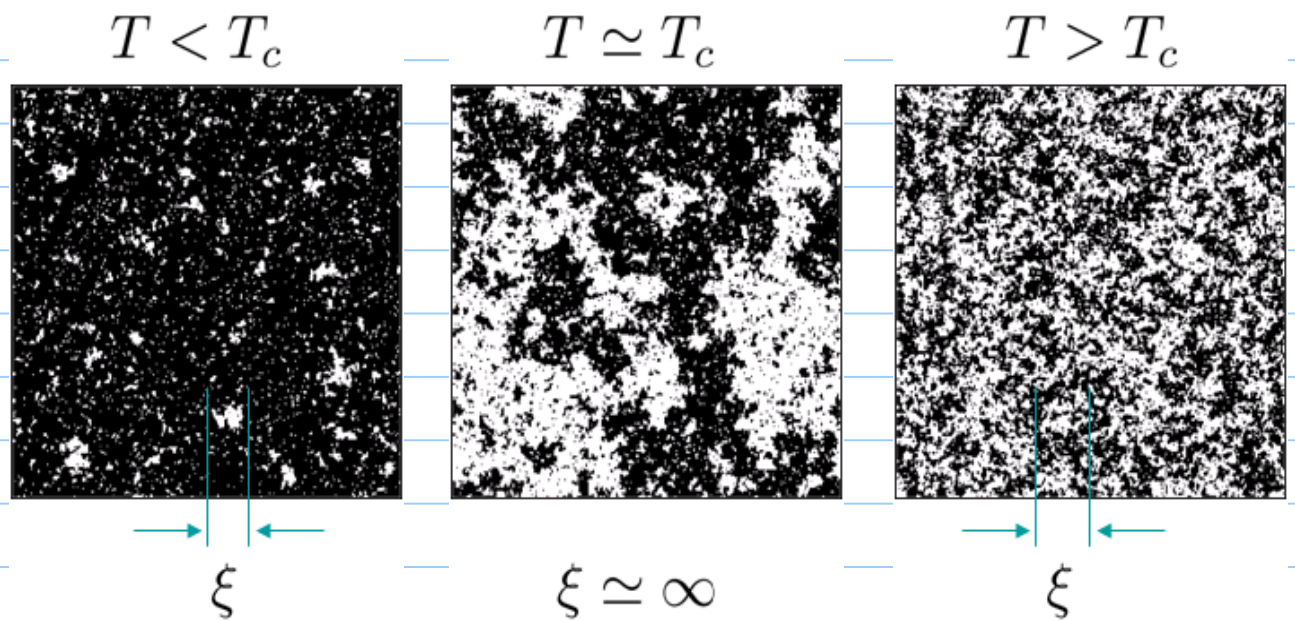


III. Basic Idea of Renormalization Group

Critical Phenomena : scale invariance ($\xi \rightarrow \infty$ as $T \rightarrow T_c$)

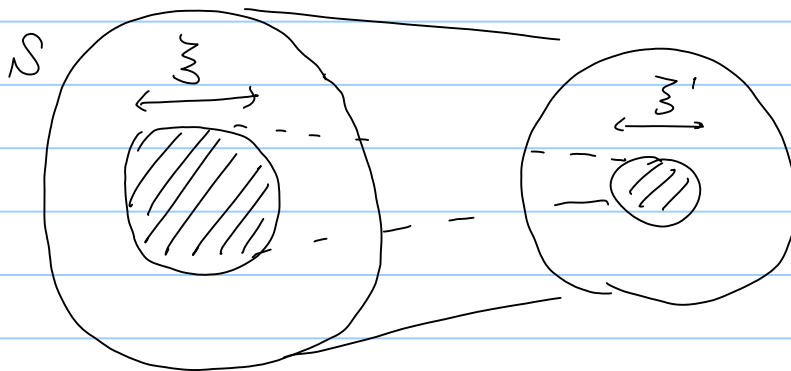


system S

translational invariance : $S' = \mathcal{T}S \sim S$

rotational invariance : $S' = \mathcal{O}S \sim S$

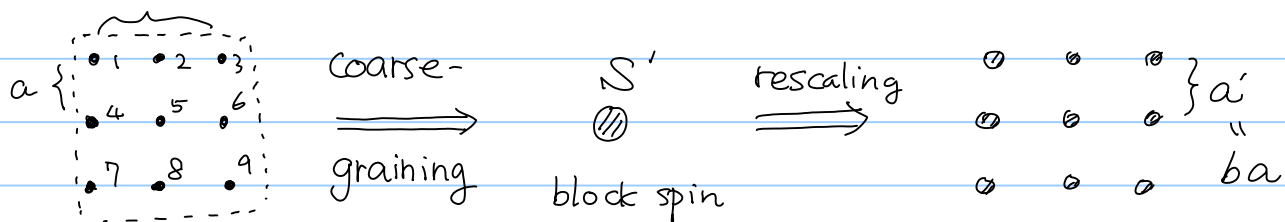
scale invariance : $S' = \mathcal{R}S \sim S$



with scale factor b , $\begin{cases} S \rightarrow S' \\ \xi \rightarrow \xi' = \xi/b \end{cases}$

(1) Block spin transformations (Kadanoff)

b : scale factor



• How to ?

① Decimation : $S' = S_5$

② Majority rule :

$$S' = \text{sign} \left\{ \sum_{i=1}^{b^d} S_i \right\}$$

③ Momentum shell

In general, one can do the coarse-graining

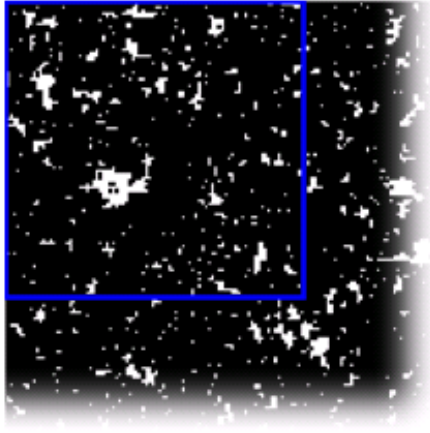
in a probabilistic way :

$$W(S' | \{S_1, S_2, \dots, S_{b^d}\})$$

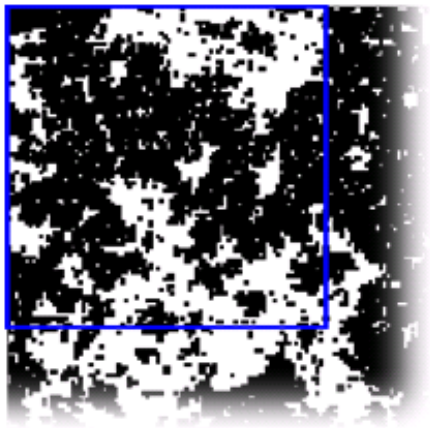
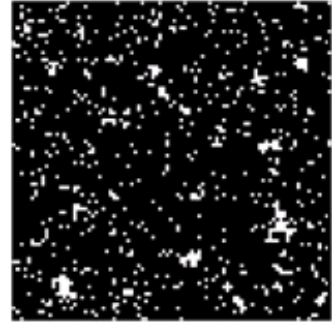
	before	after
spins	$N = L^d$	$N' = (L/b)^d$
lattice constant	a	$a' = ba$
correlation length	ξ	$\xi' = \xi/b$

scale invariance : $\xi = 0, \infty$
↙ trivial
↘ non-trivial

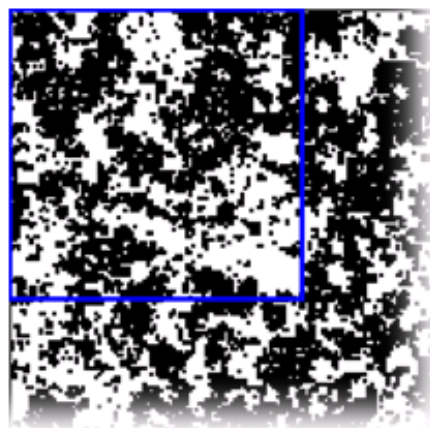
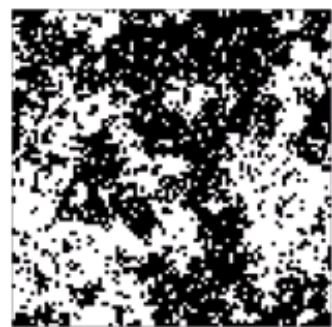
(example : decimation with $b=3$)



$$T < T_c$$



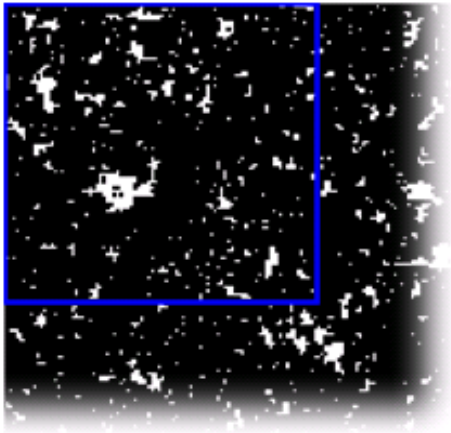
$$T = T_c$$



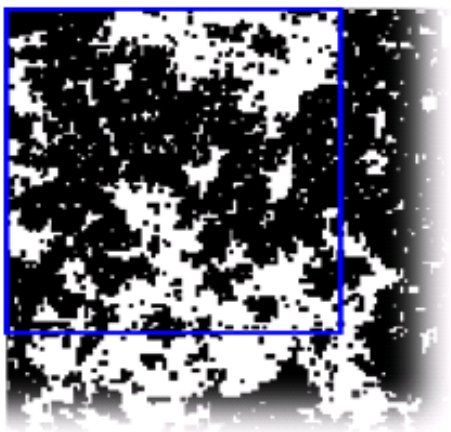
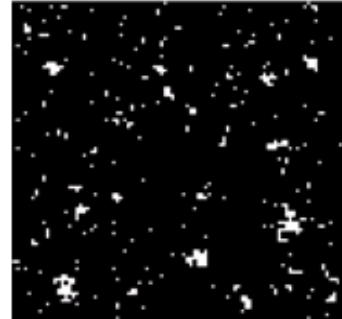
$$T > T_c$$



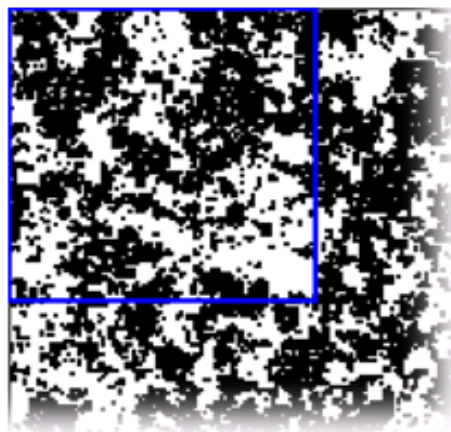
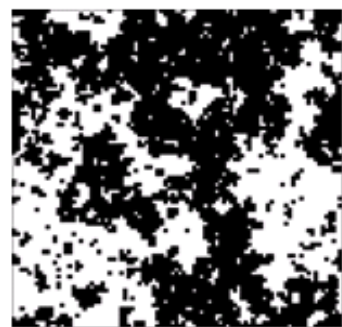
(example: majority rule with $b=3$)



$$T > T_c$$



$$T = T_c$$



$$T > T_c$$



(2) Renormalization group transformation

original spins $\{s_1, \dots, s_N\}$ with Hamiltonian

$$\begin{aligned} \mathcal{H}[\underline{s}] &= -\beta J \sum_{\langle i,j \rangle} s_i s_j - \beta B \sum_i s_i \\ &= - \sum_{\alpha} \underbrace{K_{\alpha}}_{\text{parameter}} \underbrace{\Theta_{\alpha}(\underline{s})}_{\text{operator}} \end{aligned}$$

coarse graining with $W(s' | s_1, \dots, s_{b^d})$

effective Hamiltonian $\mathcal{H}'[\underline{s}']$ for block spins $\{\underline{s}'\}$

$$e^{-\mathcal{H}'[\underline{s}']} = \text{Tr}_{\{\underline{s}'\}} \prod_{\text{block } i} W(s'_i | \{s_{i\alpha}\}) e^{-\mathcal{H}[\underline{s}]}$$

$$\mathcal{H}'[\underline{s}'] = - \sum_{\alpha} K'_{\alpha} \Theta_{\alpha}(\underline{s}')$$

$$\Rightarrow \boxed{K'_{\alpha} = K'_{\alpha}(K_1, K_2, \dots)} : \text{RG equation}$$

1) invariance of the partition function

$$\begin{aligned} Z' &= \sum_{\{\underline{s}'\}} e^{-\mathcal{H}'[\underline{s}']} = \sum_{\{\underline{s}'\}} \sum_{\{\underline{s}\}} \left(\prod_i W(s'_i | \{s_{i\alpha}\}) \right) e^{-\mathcal{H}[\underline{s}]} \\ &= \sum_{\{\underline{s}\}} e^{-\mathcal{H}[\underline{s}]} = Z \end{aligned}$$

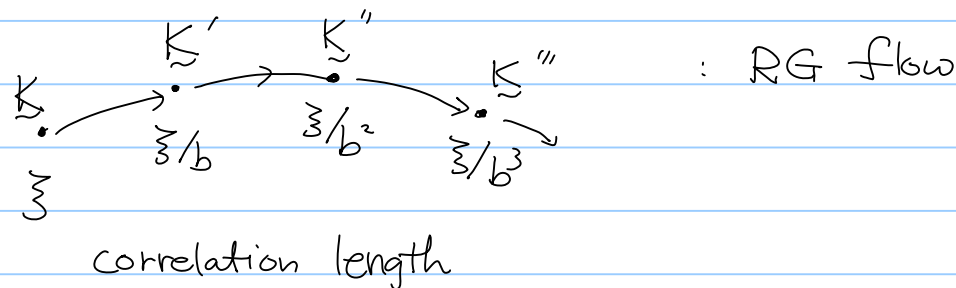
2) correlation length $\xi(\{K\})$

$$\boxed{\xi(\{K'\}) = \frac{1}{b} \xi(\{K\})}$$

(3) General theory on RG

• RG flow

under RG, $K_\alpha \rightarrow K'_\alpha(K_1, K_2, \dots)$ or $\underline{K}' = \mathcal{R}_b(\underline{K})$



• Fixed point \underline{K}^*

$$\mathcal{R}_b(\underline{K}^*) = \underline{K}^* \leftarrow \text{scale invariance}$$

linearized R.G. equation near \underline{K}^*

$$\underline{K} = \underline{K}^* + \delta \underline{K}$$

$$\Rightarrow \delta K'_\alpha = K'_\alpha(\underline{K}^* + \delta \underline{K}) - K'_\alpha$$

$$\approx \sum_\beta \left(\frac{\partial K'_\alpha}{\partial K_\beta} \right)_* \delta K_\beta \equiv R_{\alpha\beta}^* \delta K_\beta$$

$$\text{where } R_{\alpha\beta} \equiv \left(\frac{\partial K'_\alpha}{\partial K_\beta} \right)_*$$

Let $\{e^i\}$ be the i -th left eigenvector of $R_{\alpha\beta}$ with

$$\text{the eigenvalue } \lambda^i \quad \left(\sum_\alpha e^i_\alpha R_{\alpha\beta} = \lambda^i e^i_\beta \right)$$

Define scaling variables $U_i \equiv \sum_{\alpha} e_{\alpha}^i (K_{\alpha} - K_{\alpha}^*)$

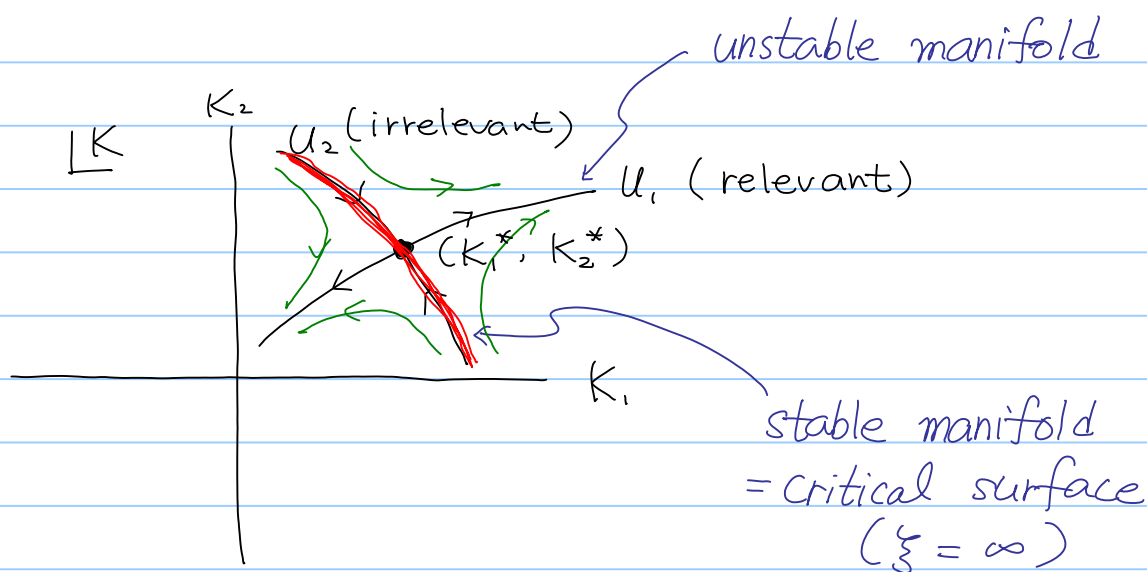
$$\Rightarrow U_i' = \sum_{\alpha} e_{\alpha}^i (K_{\alpha}' - K_{\alpha}^*) = \sum_{\alpha, \beta} e_{\alpha}^i R_{\alpha\beta} (K_{\beta} - K_{\beta}^*)$$

$$U_i' = \lambda_i U_i$$

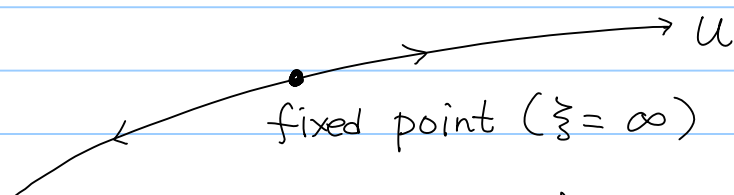
↗ R.G. eigenvalue

One can write $\lambda_i = b^{y_i}$ (why?)

- $y_i > 0$: U_i is said to be relevant
- $y_i = 0$: marginal
- $y_i < 0$: irrelevant



• Correlation length along the unstable manifold



$$\begin{aligned} \xi(u) &= b \xi(b^{\lambda} u) = b^2 \xi(b^{2\lambda} u) = \dots \\ &= b^n \xi(b^{n\lambda} u) \end{aligned}$$

set $b^{n\lambda} u = u_0$ where $\xi(u_0) = \xi_0 = O(1)$

$$\Rightarrow \xi(u) = \left(\frac{u_0}{u}\right)^{\frac{1}{\lambda}} \xi_0 \sim u^{-1/\lambda} = u^{-\nu}$$

$$\therefore \nu = \frac{1}{\lambda}$$

• Scaling theory

Systems with two relevant scaling variables

$$\begin{array}{ccc} u_t \sim t & \text{and} & u_h \sim h \\ \uparrow & & \uparrow \\ y_t & & y_h \end{array}$$

Since RG preserves the partition ftn.

$$Z_N(t, h) = Z_{N/b^d}(b^{y_t} t, b^{y_h} h)$$

free energy density

$$f(t, h) = \frac{1}{N} (-k_B T \ln Z_N(t, h))$$

$$\Rightarrow \boxed{f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h)}$$

• specific heat (at $h=0$)

$$C \sim \frac{\partial^2 f}{\partial t^2}$$

$$b \sim t^{-\frac{1}{y_t}}$$

$$\begin{aligned} C(t, 0) &= b^{2y_t - d} C(b^{y_t} t, 0) \sim t^{\frac{d}{y_t} - 2} \\ &\sim t^{-d} \end{aligned}$$

$$\Rightarrow \alpha = 2 - \frac{d}{y_t}$$

- Spontaneous magnetization ($h=0$)

$$m(t, 0) \sim \left. \frac{\partial}{\partial h} f(t, h) \right|_{h=0}$$

$$m(t, 0) \sim b^{-d+y_h} m(b^{y_t} t, 0)$$

$$\sim t^{\frac{d-y_h}{y_t}} \sim t^{\beta} \Rightarrow \beta = (d-y_h)/y_t$$

- zero-field susceptibility

$$\chi(t, 0) \sim \frac{\partial^2}{\partial h^2} f \propto |t|^{(d-2y_h)/y_t} \sim t^{-\gamma}$$

$$\Rightarrow \gamma = \frac{2y_h - d}{y_t}$$

Scaling for the correlation functions

spin-spin 2 point correlation function in the Ising model.

$$g(\vec{r}_1 - \vec{r}_2, H) \equiv \langle S(r_1) S(r_2) \rangle_H - \langle S(r_1) \rangle_H \langle S(r_2) \rangle_H$$

with the Hamiltonian

$$H = H_0(S) - \sum_r h(r) S(r)$$

\uparrow
 non-uniform mag. field

then

$$g(r_1, r_2) = \frac{\partial^2}{\partial h(r_1) \partial h(r_2)} \ln Z\{h\} \Big|_{h=0}$$

$h(r)$: varies significantly only over distances much larger than the block size $b a$
 \uparrow lattice const.

$\sim h(r)$ behaves as uniform field within the block

$$\Rightarrow H'(s') = H_0(s') - \sum_{r'} h'(r') S'(r')$$

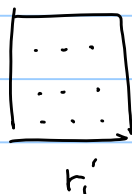
where $h'(r') = b^{y_h} h(r)$

Since Z is invariant under the RG

$$\frac{\partial^2 \ln Z'(h')}{\partial h'(r'_1) \partial h'(r'_2)} = \frac{\partial^2 \ln Z(h)}{\partial h'(r'_1) \partial h'(r'_2)}$$

(L.H.S.) = $g(r_1 - r_2)_b; H'$: correlation function after RG

(R.H.S.)



$$\begin{aligned}
 & \left[\text{changing } h'(r_i) \rightarrow h'(r_i) + \delta h'(r_i) \right] \\
 & = \left[\text{changing } h(r_i) \rightarrow h(r_i) + \delta h(r_i) \right. \\
 & \quad \left. \text{with } \delta h(r_i) = b^{-y_h} \delta h'(r_i) \text{ for} \right. \\
 & \quad \left. \text{all spins within the block} \right]
 \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial h'(r_i)} = b^{-y_h} \sum_{i \in \text{block}} \frac{\partial}{\partial h(r_i)} = b^{d-y_h} \frac{\partial}{\partial h(r)}$$

$$\therefore (\text{R.H.S.}) = b^{2(d-y_h)} \mathcal{G}(r, -r_2; \mathcal{H})$$

$$\therefore \begin{cases} \mathcal{G}((r, -r_2)/b, \mathcal{H}') = b^{2(d-y_h)} \mathcal{G}(r, -r_2, \mathcal{H}) \\ \text{or} \\ \mathcal{G}(r, t, h) = b^{-2(d-y_h)} \mathcal{G}(r/b, b^{y_t} t, b^{y_h} h) \end{cases}$$

$(h=0)$ iterating many times in such a way that $b^{y_t} t = t_0$.

$$\mathcal{G}(r, t) = \left| \frac{t}{t_0} \right|^{2(d-y_h)/y_t} \Psi \left(r / |t/t_0|^{-1/y_t} \right)$$

$$\Rightarrow \zeta \sim |t|^{-1/y_t} \Rightarrow \nu = 1/y_t$$

$(\text{at } t=0)$
 $(h=0)$

let $r/b = r_0$

$$\mathcal{G}(r) \sim r^{-2(d-y_h)} \sim r^{-(d-2+\eta)}$$

$$\eta = d + 2 - 2y_h$$

(thermodynamic \leftrightarrow correlation) exponents

$$\alpha = 2 - d/\gamma_t, \quad \nu = 1/\gamma_t \Rightarrow \alpha = 2 - d\nu$$

$$\gamma = (2\gamma_h - d)/\gamma_t, \quad \eta = d + 2 - 2\gamma_h \Rightarrow \gamma = \nu(2 - \eta)$$

: hyperscaling relation

- may fail for long range interaction

above upper critical dimension

dangerous irrelevant variables

(4) Scaling operators and scaling dimensions

Hamiltonian near the fixed point.

$$H = \sum_{\alpha} (K_{\alpha} - K_{\alpha}^*) S_{\alpha} = \sum_i u_i \phi_i$$

$S_{\alpha} = \sum_r$ (function of local spins) : operator

ex) $S_1 = \sum_r S(r)$

$$S_2 = \sum_{r,r'} J(r-r') S(r) S(r')$$

\vdots

ϕ_i : suitable linear combination of $\{S_{\alpha}\}$
"scaling operator"

previous

Generalizing the argument

$$\begin{aligned} \langle \phi_i(r_1) \phi_i(r_2) \rangle_H &= \frac{\delta^2}{\delta u_i(r_1) \delta u_i(r_2)} = \frac{\delta^2}{b^{2\gamma_i} \delta u_i(r_1) \delta u_i(r_2)} \\ &= b^{2(d-\gamma_i)} \frac{\delta^2}{\delta u_i(r_1) \delta u_i(r_2)} \\ &= b^{2(d-\gamma_i)} \langle \phi_i(r_1) \phi_i(r_2) \rangle_H \end{aligned}$$

at the critical point,

$$\langle \phi_i(r_1) \phi_i(r_2) \rangle \sim \frac{1}{|r_1 - r_2|^{2x_i}}$$

with $x_i = d - y_i$: scaling dimension of the scaling operator ϕ_i .

in general, for operator E ,

$$\langle E(r_1) E(r_2) \rangle = \sum_{ij} \frac{A_{ij}}{|r_1 - r_2|^{x_i + x_j}}$$

↑
all scaling operators are contributing.

N-point correlation fn.

$$\langle \phi_1(r_1) \phi_2(r_2) \dots \phi_N(r_N) \rangle = R^{-x_1 - x_2 - \dots - x_N} \langle \phi_1(r_1/R) \dots \phi_N(r_N/R) \rangle$$

(exercise)

1) Derive the RG eq. for the 1D Ising spin chain

2) Derive the transformation rule for the correlation function.