"Real Space Renomalization"

The 8th Summer Institute for Theoretical Physics

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Jae Dong Noh

(University of Seoul)

References

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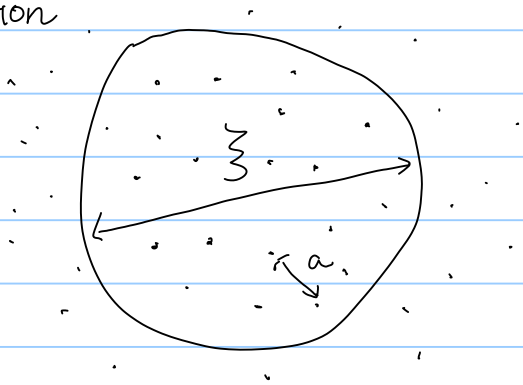
Contents

I. Introduction

interacting many degrees of freedom

- long range correlations even with short range interaction



- emergence of collective/cooperative phenomena

  phase transitions and critical phenomena

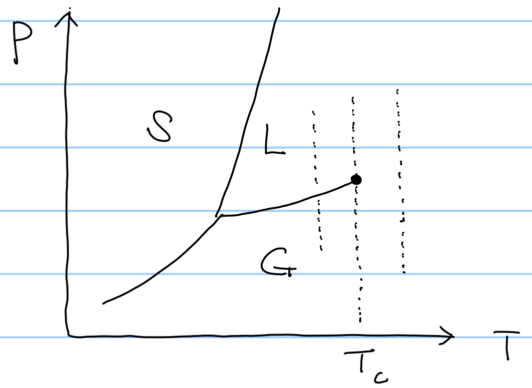
  in statistical physical systems

- theoretical framework

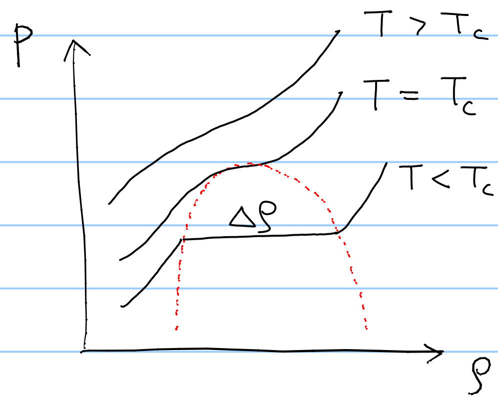
  ⇒ Renormalization Group

(1) Phase transitions and Critical Phenomena

∘ Liquid - gas transition



phase diagram



coexistence curve

• Singular behaviors

reduced temperature : $t \equiv (T - T_c)/T_c$

density difference $\Delta\rho \sim (-t)^{\beta}$

isothemal compressibility $\kappa \equiv -\frac{1}{V}\frac{\partial V}{\partial p}\Big)_T \sim |t|^{-\gamma}$

critical exponents

• Universality

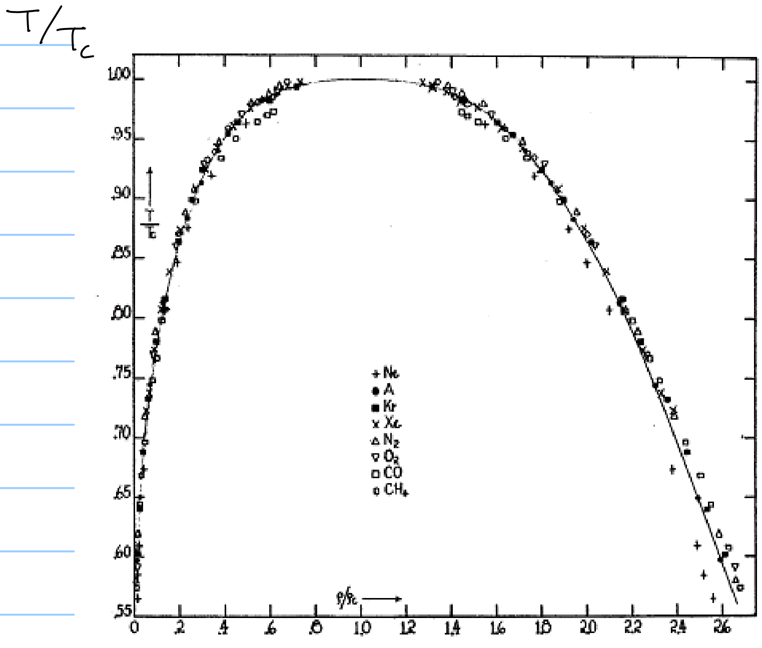


Fig. 2.

ρ/ρ_c [Guggenheim '45]

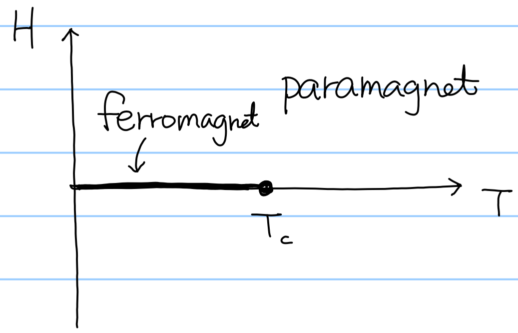
same exponents
$(\beta \approx \frac{1}{3})$

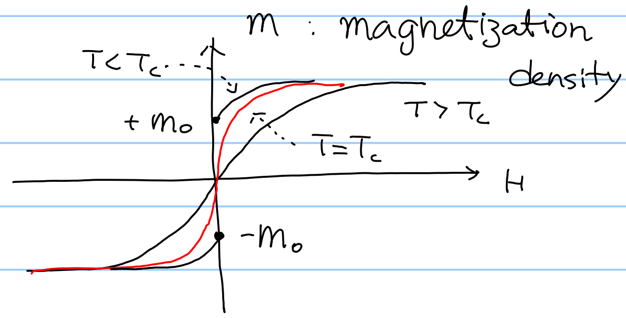
same curve
$T/T_c = f(\rho/\rho_c)$

for all materials

o Magnetic systems

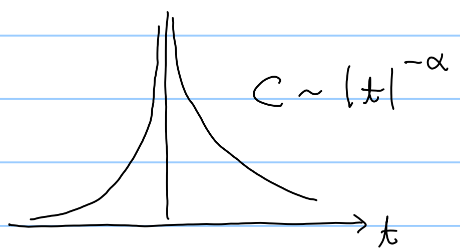
magnets at temperature T, external magnetic field H



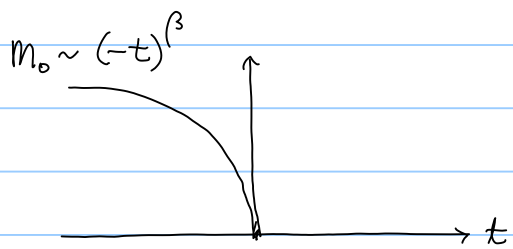


reduced temperature $t = (T-T_c)/T_c$

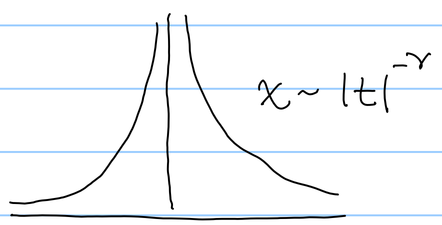
specific heat $C \equiv \dfrac{\partial U}{\partial T}$



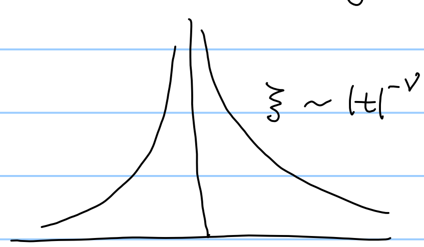
spontaneous magnetization



susceptibility $\chi \equiv \left.\dfrac{\partial m}{\partial H}\right|_{H=0}$



correlation length



correlation function $g(\vec{r}) = \langle m(\vec{r}_0)\, m(\vec{r}+\vec{r}_0)\rangle$

$$g \sim \begin{cases} m_0^2 + a\, e^{-r/\xi} , & t \neq 0 \\ r^{-(d-2+\eta)} , & t = 0 \end{cases}$$

critical phenomena :

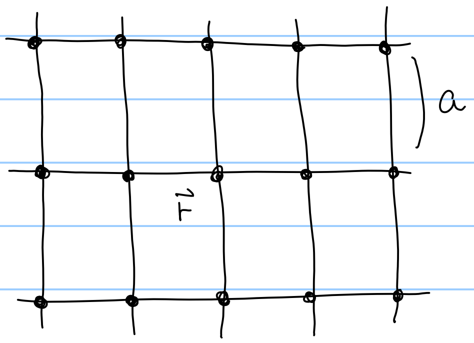
- ✓ power law singularity

  ← critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$

- ✓ universality

$\Leftarrow$ R. G.

II. Ising model



d-dimensional hyper-cubic lattice of volume $V = (La)^d$

$S_{\vec{r}} = \pm 1$: classical Ising spin at site $\vec{r}$

$(a \equiv 1)$ $\vec{r}$: in unit of a

- ferromagnetic Ising model Hamiltonian (Ernst Ising, 1924)

$$\mathcal{H} = -\frac{1}{2} \sum_{\vec{r},\vec{r}'} J(\vec{r},\vec{r}')\, S(\vec{r})\, S(\vec{r}') - \sum_{r} B(\vec{r})\, S(\vec{r})$$

- $J(\vec{r},\vec{r}') = J(\vec{r}-\vec{r}')$: ferromagnetic exchange coupling

in most case, n.n. interaction only

$$J(\vec{r},\vec{r}') = \begin{cases} J & \text{if } \vec{r} \text{ and } \vec{r}' \text{ are neighboring} \\ 0 & \text{otherwise} \end{cases}$$

- Lattice green function $G(\vec{r},\vec{r}') = J^{-1}(\vec{r},\vec{r}')$

$$J(r,r') = J(r-r') = \int \frac{d^d\vec{q}}{(2\pi)^d}\, J(\vec{q})\, e^{i\vec{q}\cdot(r-r')}$$

$$G(r) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{iq\cdot r}}{J(\vec{q})}$$

- $B(\vec{r})$: external magnetic field at site $\vec{r}$

∘ Statistical mechanics

- Equilibrium property at temperature T

"Cannonical Ensemble Theory"

$$P(s) \propto \exp\{-\beta H[s]\} \quad : \text{Boltzmann dist.}$$

$$\beta = 1/k_B T$$

Canonical ensemble average

$$\langle O \rangle = \sum_{\{s\}} O(s) e^{-\beta H[s]} \Big/ \sum_{\{s\}} e^{-\beta H[s]}$$

- Partition function (generating function)

$$Z(T,B) = \sum_{\{s\}} e^{-\beta H[S]}$$

from the partition function

energy density

$$e = \langle H/N \rangle = -\frac{\partial}{\partial \beta} \ln Z,$$

specific heat

$$C = \frac{\partial e}{\partial T}$$

magnetization

$$m(r) = \langle S(r) \rangle = \frac{1}{\beta} \frac{\partial}{\partial B(r)} \ln Z$$

Correlation function

$$g(r) = \langle S(r_0) S(r_0+r) \rangle - \langle S(r_0) \rangle \langle S(r_0+r) \rangle$$

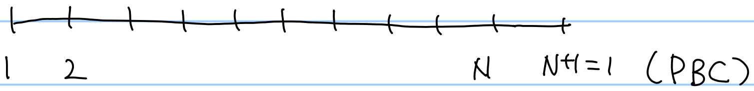
$$= \frac{1}{\beta^2} \frac{\partial^2}{\partial B(r_0) \partial B(r_0+r)} \ln Z$$

susceptibility $(B(r) \rightarrow B(r) + B)$

$$\chi = \frac{\partial}{\partial B} \left\langle \frac{1}{N} \sum_i S(r) \right\rangle$$

fluctuation - dissipation thm. $\quad \chi \sim \sum_r g(r)$

- 1D Ising model



$$\mathcal{H} = -J\sum_{i=1}^{N} S_i S_{i+1} - H\sum_i S_i$$

$$Z = \sum_{S_1}\cdots\sum_{S_N} e^{\beta J\sum_{i=1}^{N} S_i S_{i+1} + \beta H\sum_i S_i}$$

$$= \sum_{S_1}\cdots\sum_{S_N}\prod_{i=1}^{N} e^{\beta J S_i S_{i+1} + \frac{\beta H}{2}(S_i + S_{i+1})} \equiv T_{S_i, S_{i+1}}$$

transfer matrix (2×2) T

$$T_{S,S'} = e^{\beta J SS' + \frac{\beta H}{2}(S+S')}$$

$$= \begin{array}{c|cc} & + & - \\ \hline + & e^{\beta J + \beta H} & e^{-\beta J} \\ - & e^{-\beta J} & e^{\beta J - \beta H} \end{array}$$

$$\sigma_z = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= e^{\beta J}\cosh\beta H\,\mathbb{1} + e^{\beta J}(\sinh\beta H)\,\hat{\sigma}_z + e^{-\beta J}\,\hat{\sigma}_x$$

$$\equiv e^{-\hat{\mathcal{H}}}$$

$\hat{\mathcal{H}}$: quantum mechanical Hamiltonian for a single (0-Dim.) spin system.

$$\Rightarrow\quad Z = \mathrm{Tr}\, T^N = \mathrm{Tr}\, e^{-N\hat{\mathcal{H}}}$$

$$= \lambda_1^N + \lambda_2^N = e^{-NE_1} + e^{-NE_2}$$

diagonalization of T

$$(\lambda - e^{\beta J + \beta H})(\lambda - e^{\beta J - \beta H}) - e^{-2\beta J} = 0$$

$$\lambda^2 - 2e^{\beta J}\cosh\beta H \, \lambda + 2\sinh 2\beta J = 0$$

$$\lambda_{1,2} = e^{\beta J}\cosh\beta H \pm \sqrt{e^{2\beta J}\underbrace{\cosh^2\beta H}_{1+\sinh^2\beta H} - 2\sinh 2\beta J}$$

in the thermodynamic limit $(N \to \infty)$

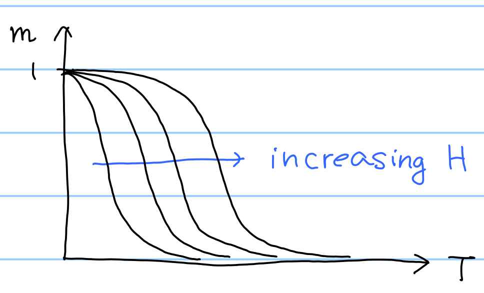
$$Z = \lambda_1^N + \lambda_2^N = \lambda_1^N\left(1 + (\lambda_2/\lambda_1)^N\right) \simeq \lambda_1^N$$

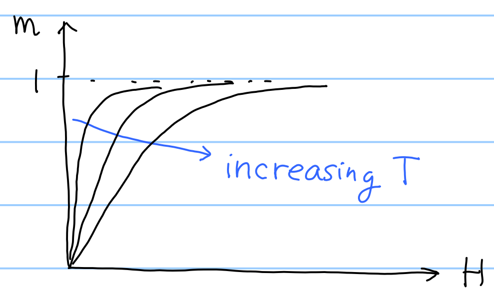
magnetization density

$$m = \frac{M}{N} = \frac{1}{N\beta}\frac{\partial}{\partial H}\ln Z = \frac{1}{\beta}\frac{\partial}{\partial H}\ln\lambda_1$$

$$= \frac{e^{\beta J}\sinh\beta H + e^{2\beta J}\sinh\beta H\cosh\beta H / \sqrt{}}{e^{\beta J}\cosh\beta H + \sqrt{}}$$

$$= \sinh\beta H\left(\sinh^2\beta H + e^{-4\beta J}\right)^{-1/2}$$





$m(H=0) = 0$ at all $T \neq 0$. : no phase transition at $T \neq 0$.

zero-field susceptibility

$$\chi = \left.\frac{\partial m}{\partial H}\right|_{H=0} = \frac{1}{T}e^{2\beta J} \to \infty \text{ at } T=0$$

: zero temperature phase transition

Correlation function $(H=0)$

$$g(r) = \langle S_i S_{i+r} \rangle = \frac{1}{Z} \sum_{S_1} \cdots \sum_{S_N} S_i S_{i+r}\, e^{\beta J \sum_i S_i S_{i+1}}$$

$$= \frac{1}{Z} \mathrm{Tr}\, T^{i-1} \sigma_z T^r \sigma_z T^{N-r-i+1}$$

$$= \frac{1}{Z} \mathrm{Tr}\, \sigma_z T^r \sigma_z T^{N-r}$$

$$= \frac{1}{\lambda_1^N + \cancel{\lambda_2^N}} \Big(\langle 1 | \sigma_z \underset{\wedge}{T^r} \sigma_z T^{N-r} | 1 \rangle + \cancel{\langle 2 | \cdots | 2 \rangle} \Big)$$

$$|1\rangle\langle 1| + |2\rangle\langle 2|$$

$$= |\langle 1 | \sigma_z | 1 \rangle|^2 + |\langle 1 | \sigma_z | 2 \rangle|^2 \left(\frac{\lambda_2}{\lambda_1} \right)^r$$

$$\lambda_{1,2}\,(H=0) = e^{\beta J} \pm e^{-\beta J}$$

$$\Rightarrow \frac{\lambda_2}{\lambda_1} = e^{-(E_2 - E_1)} = e^{-\Delta E} = \tanh \beta J$$

$$\Rightarrow g(r) \sim (\tanh \beta J)^r \sim e^{-r/\xi}$$

where correlation length

$$\xi = -1/\ln(\tanh \beta J) \sim \underset{T \to 0}{e^{2\beta J}} \to \infty \text{ as}$$

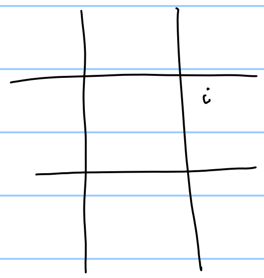
in summary,

- zero temperature phase transition in 1D Ising model
- stat mech $\longleftrightarrow$ field theory

d-dim. classical system	$((d-1)+1)$ dim. quantum system
	$\uparrow$ imaginary time direction
Configurations of degrees of freedom in d-dim. space	time trajectories of d.o.f. in $(d-1)$-dim. space

transfer mat. T	evolution op. $\hat{U} = e^{-(\delta\tau)\hat{H}/\hbar}$ in imaginary time
partition ftn $Z = \mathrm{Tr}\, T^N$	$\mathrm{Tr}\, \hat{U}^N$
free energy density	ground state energy density
correlation function	propagator
$(\text{correlation length})^{-1}$	mass gap ΔE

- 2-D Ising model



$$H = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i$$

exact solution by Lars Onsager (1944)
($H=0$)

$$T_c / J = \frac{2}{\ln(1+\sqrt{2})}$$

$$\alpha = 0 \quad (\text{log singularity})$$

$$\beta = \frac{1}{8}, \quad \gamma = \frac{7}{4}, \quad \nu = 1$$

cf) No analytic solution is available in 3D

(exercise) 1D quantum Hamiltonian for the 2D Ising model

$$T_{\text{2D Ising}} \sim e^{-\hat{H}_{1D}}$$