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Lecture note on Clifford algebra

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ABSTRACT: This lecture note surveys the gamma matrices in general dimensions with arbitrary signatures, the study of which is essential to understand the supersymmetry in the corresponding spacetime. The contents supplement the lecture presented by the author at *Modave Summer School in Mathematical Physics, Belgium*, june, 2005.

KEYWORDS: gamma matrix, supersymmetry, octonion.

Contents

1. Preliminary

Where do we see Clifford algebra?

- Dirac equation, for sure.
- Supersymmetry algebra.
- Non-anti-commutative superspace.
- Division algebra, $\mathbf{R}, C, H, O$.
- ADHM construction for instantons, $F = \pm * F$.

The gamma matrices in the Euclidean two-dimensions provide the *fermionic oscillators*,

$$f^2 = 0\,, \qquad \bar{f}^{\,2} = 0\,, \qquad \{f, \bar{f}\} = 1\,, \tag{1.1}$$

where $f = \frac{1}{2}(\gamma^1 + i\gamma^2)$, $\bar{f} = \frac{1}{2}(\gamma^1 - i\gamma^2)$. Consequently, the irreducible representation is given uniquely by 2×2 matrices acting on two dimensional spinors, $|+\rangle$ and $|-\rangle$,

$$f = |-\rangle\langle+| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \qquad \bar{f} = |+\rangle\langle-| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,. \tag{1.2}$$

Higher dimensional gamma matrices are then constructed by the direct products of them.

2. Gamma Matrix

We start with the following *Theorem* on linear algebra.

Theorem
Any matrix, M, satisfying $M^2 = \lambda^2 \neq 0$, $\lambda \in \mathbf{C}$ is diagonalizable, and furthermore if there is another invertible matrix, N, which anti-commutes with M, $\{N, M\} = 0$, then M is $2n \times 2n$ matrix of the form

$$M = S \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} S^{-1}\,. \tag{2.1}$$

In particular, $\mathrm{tr}M = 0$. See Sec. A for our proof.

2.1 In Even Dimensions

In even $d = t + s$ dimensions, with metric[1]

$$\eta^{\mu\nu} = \mathrm{diag}(\underbrace{+ + \cdots +}_{t}\underbrace{- - \cdots -}_{s})\,, \tag{2.2}$$

[1]Note that throughout the lecture note we adopt the field theorists' convention rather than string theorists such that the time directions have the positive signature. The conversion is straightforward.

gamma matrices, γ^μ, satisfy the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,. \tag{2.3}$$

With[2]

$$\gamma^{\mu_1\mu_2\cdots\mu_m} = \gamma^{[\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_m]}\,, \tag{2.4}$$

we define Γ^M, $M = 1, 2, \cdots 2^d$ by assigning numbers to independent $\gamma^{\mu_1\mu_2\cdots\mu_m}$, e.g. imposing $\mu_1 < \mu_2 < \cdots < \mu_m$,

$$\Gamma^M = (1, \gamma^\mu, \gamma^{\mu\nu}, \cdots, \gamma^{\mu_1\mu_2\cdots\mu_m}, \cdots, \gamma^{12\cdots d})\,. \tag{2.5}$$

Then $\{\Gamma^M\}/Z_2$ forms a group

$$\Gamma^M\Gamma^N = \Omega^{MN}\Gamma^L\,, \qquad \Omega^{MN} = \pm 1\,, \tag{2.6}$$

where L is a fuction of M, N and $\Omega_{MN} = \pm 1$ does not depend on the specific choice of representation of the gamma matrices.
Theorem (2.1) implies

$$\frac{1}{2n}\mathrm{tr}(\Gamma^M\Gamma^N) = \Omega_{MN}\delta^{MN}\,, \tag{2.7}$$

which shows the linear independence of $\{\Gamma^M\}$ so that any gamma matrix should not be smaller than $2^{d/2} \times 2^{d/2}$.

In two-dimensions, one can take the Pauli sigma matrices, σ^1, σ^2 as gamma matrices with a possible factor, i, depending on the signature. In general, one can construct $d+2$ dimensional gamma matrices from d dimensional gamma matrices by taking tensor products as

$$(\gamma^\mu \otimes \sigma^1,\ 1 \otimes \sigma^2,\ 1 \otimes \sigma^3) \qquad : \text{up to a factor } i\,. \tag{2.8}$$

Thus, the smallest size of irreducible representations is $2^{d/2} \times 2^{d/2}$ and $\{\Gamma^M\}$ forms a basis of $2^{d/2} \times 2^{d/2}$ matrices.

By induction on the dimensions, from eq.(2.8), we may require gamma matrices to satisfy the hermiticity condition

$$\gamma^{\mu\dagger} = \gamma_\mu = \begin{cases} +\gamma^\mu & \text{for time-like } \mu \\ \\ -\gamma^\mu & \text{for space-like } \mu \end{cases}\,. \tag{2.9}$$

With this choice of gamma matrices we define $\gamma^{(d+1)}$ as

$$\gamma^{(d+1)} = \sqrt{(-1)^{\frac{t-s}{2}}}\,\gamma^1\gamma^2\cdots\gamma^d\,, \tag{2.10}$$

[2] "[]" means the standard anti-symmetrization with "strength one".

satisfying

$$\gamma^{(d+1)} = (\gamma^{(d+1)})^{-1} = \gamma^{(d+1)\dagger}\,, \qquad\qquad \{\gamma^\mu, \gamma^{(d+1)}\} = 0\,. \tag{2.11}$$

For two sets of irreducible gamma matrices, γ^μ, γ'^μ which are $2n \times 2n$, $2n' \times 2n'$ respectively, we consider a matrix

$$S = \sum_M \Gamma'^M T (\Gamma^M)^{-1}\,, \tag{2.12}$$

where T, is an arbitrary $2n' \times 2n$ matrix.
This matrix satisfies for any N from eq.(2.6)

$$\Gamma'^N S = S\Gamma^N\,. \tag{2.13}$$

By Schur's Lemmas, it should be either $S = 0$ or $n = n', \det S \neq 0$. Furthermore, S is unique up to constant, although T is arbitrary. This implies the uniqueness of the irreducible $2^{d/2} \times 2^{d/2}$ gamma matrices in even d dimensions, up to the similarity transformations. These similarity transformations are also unique up to constant. Consequently there exist similarity transformations which relate γ^μ to $\gamma^{\mu\dagger}$, $\gamma^{\mu *}$, $\gamma^{\mu T}$ since the latter form also representations of the Clifford algebra. By combining $\gamma^{(d+1)}$ with the similarity transformations, from eq.(2.11), we may acquire the opposite sign, $-\gamma^{\mu\dagger}$, $-\gamma^{\mu *}$, $-\gamma^{\mu T}$ as well. Explicitly we define[3]

$$A = \sqrt{(-1)^{\frac{t(t-1)}{2}}}\,\gamma^1\gamma^2\cdots\gamma^t\,, \tag{2.14}$$

satisfying

$$A = A^{-1} = A^\dagger\,, \tag{2.15}$$

$$\gamma^{\mu\dagger} = (-1)^{t+1} A\gamma^\mu A^{-1}\,. \tag{2.16}$$

If we write

$$\pm\gamma^{\mu *} = B_\pm \gamma^\mu B_\pm^{-1}\,, \tag{2.17}$$

then from

$$\gamma^\mu = (\gamma^{\mu *})^* = B_\pm^* B_\pm \gamma^\mu (B_\pm^* B_\pm)^{-1}\,, \tag{2.18}$$

one can normalize $B_\pm$ to satisfy [2, 3]

$$B_\pm^* B_\pm = \varepsilon_\pm 1\,, \qquad \varepsilon_\pm = (-1)^{\frac{1}{8}(s-t)(s-t\pm 2)}\,, \tag{2.19}$$

$$B_\pm^\dagger B_\pm = 1\,, \tag{2.20}$$

$$B_\pm^T = \varepsilon_\pm B_\pm\,, \tag{2.21}$$

[3] Alternatively, one can construct $C_\pm$ explicitly out of the gamma matrices in a certain representation [1].

where the unitarity follows from

$$\gamma^{\mu}=\gamma_{\mu}^{\dagger}=(\pm B_{\pm}^{-1}\gamma_{\mu}^{*}B_{\pm})^{\dagger}=\pm B_{\pm}^{\dagger}\gamma^{\mu *}(B_{\pm}^{\dagger})^{-1}=B_{\pm}^{\dagger}B_{\pm}\gamma^{\mu}(B_{\pm}^{\dagger}B_{\pm})^{-1}\,, \tag{2.22}$$

and the positive definiteness of $B_{\pm}^{\dagger}B_{\pm}$. The calculation of $\varepsilon_{\pm}$ is essentially counting the dimensions of symmetric and anti-symmetric matrices [2, 3][4].

What is worth to note is the case $\varepsilon_{\pm}=+1$. As we see later in (4.4), (4.5), **if $\varepsilon_{+}=+1$, the gamma matrices can be chosen to real, i.e. $B_{+}=1$, while if $\varepsilon_{-}=+1$, the gamma matrices can be chosen to pure imaginary, i.e. $B_{-}=1$.** Especially when the gamma matrices are real we say they are in the Majorana representation.

The charge conjugation matrix, $C_{\pm}$, given by

$$C_{\pm}=B_{\pm}^{T}A\,, \tag{2.23}$$

satisfies[5] from the properties of A and $B_{\pm}$

$$C_{\pm}\gamma^{\mu}C_{\pm}^{-1}=\zeta\gamma^{\mu T}\,,\qquad\qquad \zeta=\pm(-1)^{t+1}\,, \tag{2.24}$$

$$C_{\pm}^{\dagger}C_{\pm}=1\,, \tag{2.25}$$

$$C_{\pm}^{T}=(-1)^{\frac{1}{8}d(d-\zeta 2)}\,C_{\pm}=\varepsilon_{\pm}(\pm 1)^{t}(-1)^{\frac{1}{2}t(t-1)}\,C_{\pm}\,, \tag{2.26}$$

$$\zeta^{t}(-1)^{\frac{1}{2}t(t-1)}A^{T}=B_{\pm}AB_{\pm}^{-1}=C_{\pm}AC_{\pm}^{-1}\,. \tag{2.27}$$

$\varepsilon_{\pm}$ is related to ζ as

$$\varepsilon_{\pm}=\zeta^{t}(-1)^{\frac{1}{2}t(t-1)+\frac{1}{8}d(d-\zeta 2)}\,. \tag{2.28}$$

Eqs.(2.24, 2.26) imply

$$\begin{aligned}(C_{\pm}\gamma^{\mu_1\mu_2\cdots\mu_n})^{T}&=\zeta^{n}(-1)^{\frac{1}{8}d(d-\zeta 2)+\frac{1}{2}n(n-1)}\,C_{\pm}\gamma^{\mu_1\mu_2\cdots\mu_n}\\&=\varepsilon_{\pm}(\pm 1)^{t+n}(-1)^{n+\frac{1}{2}(t+n)(t+n-1)}\,C_{\pm}\gamma^{\mu_1\mu_2\cdots\mu_n}\,.\end{aligned} \tag{2.29}$$

$\gamma^{(d+1)}$ satisfies

$$\begin{aligned}\gamma^{(d+1)\dagger}&=(-1)^{t}A_{\pm}\gamma^{(d+1)}A_{\pm}^{-1}=\gamma^{(d+1)}\,,\\\gamma^{(d+1)*}&=(-1)^{\frac{t-s}{2}}B_{\pm}\gamma^{(d+1)}B_{\pm}^{-1}\,,\\\gamma^{(d+1)T}&=(-1)^{\frac{t+s}{2}}C_{\pm}\gamma^{(d+1)}C_{\pm}^{-1}\,,\end{aligned} \tag{2.30}$$

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[4] From (2.24) we have $(C_{\pm}\gamma^{\mu_1\mu_2\cdots\mu_n})^{T}=\chi_{n\pm}\,C_{\pm}\gamma^{\mu_1\mu_2\cdots\mu_n}$, $\chi_{n\pm}:=\varepsilon_{\pm}(\pm 1)^{t+n}(-1)^{n+\frac{1}{2}(t+n)(t+n-1)}$ (2.29). Thus, one can obtain the dimension of the symmetric $2^{d/2}\times 2^{d/2}$ matrices as

$$2^{d/2-1}\left(2^{d/2}+1\right)=\sum_{n=0}^{d}\tfrac{1}{2}\left(1+\chi_{n\pm}\right)\frac{d!}{n!(d-n)!}\,.$$

From this one can obtain the value of $\varepsilon_{\pm}$ (2.19).

[5] Essentially all the properties of the charge conjugation matrix, $C_{\pm}$ depends only on d and ζ. However it is useful here to have expression in terms of the signature to dicuss the Majorana supersymmetry later.

where $\{A_+, A_-\} = \{A, \gamma^{(d+1)}A\}$.

In stead of eq.(2.8) one can construct $d+2$ dimensional gamma matrices from d dimensional gamma matrices by taking tensor products as

$$(\gamma^\mu \otimes \sigma^1,\ \gamma^{(d+1)} \otimes \sigma^1,\ 1 \otimes \sigma^2) \qquad : \text{up to a factor } i\,. \tag{2.31}$$

Therefore the gamma matrices in even dimensions can be chosen to have the "off-block diagonal" form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^{(d+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.32}$$

where the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ matrices, $\sigma^\mu, \tilde{\sigma}^\mu$ satisfy

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = 2\eta^{\mu\nu}\,, \tag{2.33}$$

$$\sigma^{\mu\dagger} = \tilde{\sigma}_\mu\,. \tag{2.34}$$

In this choice of gamma matrices, from eq.(2.30), $A_\pm, B_\pm, C_\pm$ are either "block diagonal" or "off-block diagonal" depending on whether t, $\frac{t-s}{2}$, $\frac{t+s}{2}$ are even or odd respectively.
In particular, in the case of odd t, we write from eqs.(2.14, 2.15) A as

$$A = \begin{pmatrix} 0 & \mathrm{a} \\ \tilde{\mathrm{a}} & 0 \end{pmatrix}, \qquad \mathrm{a} = \sqrt{(-1)^{\frac{t(t-1)}{2}}}\,\sigma^1\tilde{\sigma}^2 \cdots \sigma^t = \tilde{\mathrm{a}}^\dagger = \tilde{\mathrm{a}}^{-1}\,, \tag{2.35}$$

and in the case of odd $\frac{t+s}{2}$ we write from eq.(2.26) $C_\pm$ as

$$C_\pm = \begin{pmatrix} 0 & \mathrm{c} \\ \pm\tilde{\mathrm{c}} & 0 \end{pmatrix}, \qquad \mathrm{c} = \varepsilon_+(-1)^{\frac{t(t-1)}{2}}\tilde{\mathrm{c}}^T = (\mathrm{c}^\dagger)^{-1}\,, \tag{2.36}$$

where a, ã, c, c̃ satisfy from eqs.(2.16, 2.24)

$$\begin{aligned} \sigma^{\mu\dagger} &= \tilde{\mathrm{a}}\sigma^\mu\tilde{\mathrm{a}}\,, & \tilde{\sigma}^{\mu\dagger} &= \mathrm{a}\tilde{\sigma}^\mu\mathrm{a}\,, \\ \sigma^{\mu T} &= (-1)^{t+1}\tilde{\mathrm{c}}\sigma^\mu\mathrm{c}^{-1}\,, & \tilde{\sigma}^{\mu T} &= (-1)^{t+1}\mathrm{c}\tilde{\sigma}^\mu\tilde{\mathrm{c}}^{-1}\,. \end{aligned} \tag{2.37}$$

If both of t and $\frac{t+s}{2}$ are odd then from eq.(2.27)

$$\mathrm{a}^T = (-1)^{\frac{t-1}{2}}\,\tilde{\mathrm{c}}\,\mathrm{a}\,\mathrm{c}^{-1}\,, \qquad \tilde{\mathrm{a}}^T = (-1)^{\frac{t-1}{2}}\,\mathrm{c}\,\tilde{\mathrm{a}}\,\tilde{\mathrm{c}}^{-1}\,. \tag{2.38}$$

2.2 In Odd Dimensions

The gamma matrices in odd $d+1 = t+s$ dimensions are constructed by combining a set of even d dimensional gamma matrices with either $\pm\gamma^{(d+1)}$ or $\pm i\gamma^{(d+1)}$ depending on the signature of even d dimensions. This way of construction is general, since $\gamma^{(d+1)}$ serves the role of γ^{d+1}

$$\begin{gathered} -\gamma^\mu = \gamma^{d+1}\gamma^\mu(\gamma^{d+1})^{-1}\,, \qquad \text{for } \mu = 1, 2, \cdots, d\,, \\ (\gamma^{d+1})^2 = \pm 1\,, \end{gathered} \tag{2.39}$$

and such a matrix is unique in irreducible representations up to sign.

However, contrary to the even dimensional Clifford algebra, in odd dimensions two different choices of the signs in γ^{d+1} bring two irreducible representations for the Clifford algebra, which can not be mapped to each other[6] by similarity transformations

$$\gamma^\mu = (\gamma^1, \gamma^2, \cdots, \gamma^{d+1}) \quad \text{and} \quad \gamma'^\mu = (\gamma^1, \gamma^2, \cdots, \gamma^d, -\gamma^{d+1}) \,. \tag{2.40}$$

If there were a similarity transformation between these two, it should have been identity up to constant because of the uniqueness of the similarity transformation in even dimensions. Clearly this would be a contradiction due to the presence of the two opposite signs in γ^{d+1}.

In general one can put[7]

$$\gamma^{d+1} = \begin{cases} \pm\gamma^{12\cdots d} & \text{for } t-s \equiv 1 \bmod 4\,, \\ \\ \pm i\gamma^{12\cdots d} & \text{for } t-s \equiv 3 \bmod 4\,. \end{cases} \tag{2.41}$$

$2^{d/2} \times 2^{d/2}$ gamma matrices in odd $d+1$ dimensions, $\gamma^\mu, \mu = 1, 2, \cdots, d+1$, induce the following basis of $2^{d/2} \times 2^{d/2}$ matrices, $\tilde{\Gamma}^M$

$$\tilde{\Gamma}^M = (1, \gamma^\mu, \gamma^{\mu\nu}, \cdots, \gamma^{\mu_1\mu_2\cdots\mu_{d/2}}) \,, \qquad M = 1, 2, \cdots 2^d \,. \tag{2.42}$$

From eq.(2.41)

$$\tilde{\Gamma}^M\tilde{\Gamma}^N = \tilde{\Omega}_{MN}\tilde{\Gamma}^L \,,$$

$$\tilde{\Omega}_{MN} = \begin{cases} \pm 1 & \text{for } t-s \equiv 1 \bmod 4\,, \\ \\ \pm 1, \pm i & \text{For } t-s \equiv 3 \bmod 4\,. \end{cases} \tag{2.43}$$

Here, contrary to the even dimensional case, $\tilde{\Omega}_{MN}$ depends on each particular choice of the representations due to the arbitrary sign factor in γ^{d+1}. This is why eq.(2.13) does not hold in odd dimensions. Therefore it is not peculiar that not all of $\pm\gamma^{\mu\dagger}, \pm\gamma^{\mu *}, \pm\gamma^{\mu T}$ are related to γ^μ by similarity transformations. In fact, if it were true, say for $\pm\gamma^{\mu *}$, then the similarity transformation should have been $B_\pm$ (2.17) by the uniqueness of the similarity transformations in even dimensions, but this would be a contradiction to eq.(2.30), where the sign does not alternate under the change of $B_+ \leftrightarrow B_-$. Thus, in odd dimensions, only the half of $\pm\gamma^{\mu\dagger}, \pm\gamma^{\mu *}, \pm\gamma^{\mu T}$ are related to γ^μ by similarity transformations and hence

[6] Nevertheless, this can be cured by the following transformation. Under $x^\mu = (x^1, x^2, \cdots, x^{d+1}) \rightarrow x'^\mu = (x^1, x^2, \cdots, -x^{d+1})$, we transform the Dirac field $\psi(x)$ as $\psi(x) \rightarrow \psi'(x') = \psi(x)$, to get $\bar{\psi}(x)\gamma\cdot\partial\psi(x) \rightarrow \bar{\psi}'(x')\gamma'\cdot\partial'\psi'(x') = \bar{\psi}(x)\gamma\cdot\partial\psi(x)$. Hence those two representations are equivalent describing the same physical system.

[7] Our results (2.41-2.50) do not depend on the choice of the signature in d dimensions, i.e. they hold for either increasing the time dimensions, $d = (t-1)+s$ or the space dimensions, $d = t + (s-1)$.

from eq.(2.30) there exist three similarity transformations, A, B, C such that

$$(-1)^{t+1}\gamma^{\mu\dagger} = A\gamma^{\mu}A^{-1}\,, \tag{2.44}$$

$$(-1)^{\frac{t-s-1}{2}}\gamma^{\mu *} = B\gamma^{\mu}B^{-1}\,, \tag{2.45}$$

$$(-1)^{\frac{t+s-1}{2}}\gamma^{\mu T} = C\gamma^{\mu}C^{-1}\,. \tag{2.46}$$

A, B, C are all unitary and satisfy

$$A = A^{-1} = A^{\dagger}\,, \qquad\qquad C = B^{T}A\,, \tag{2.47}$$

$$B^{*}B = \varepsilon\,1 = (-1)^{\frac{1}{8}(t-s+1)(t-s-1)}\,1\,, \tag{2.48}$$

$$B^{T} = \varepsilon B\,, \qquad\qquad C^{T} = \varepsilon(-1)^{\frac{ts}{2}}\,C = (-1)^{\frac{1}{8}(t+s+1)(t+s-1)}C\,, \tag{2.49}$$

$$(-1)^{\frac{ts}{2}}A^{T} = BAB^{-1} = CAC^{-1}\,. \tag{2.50}$$

In particular, A is given by eq.(2.14).

2.3 Lorentz Transformations

Lorentz transformations, L can be represented by the following action on gamma matrices in a standard way

$$\mathcal{L}^{-1}\gamma^{\mu}\mathcal{L} = L^{\mu}{}_{\nu}\gamma^{\nu}\,, \tag{2.51}$$

where L and $\mathcal{L}$ are given by

$$\begin{gathered} L = e^{w_{\mu\nu}M^{\mu\nu}}\,, \qquad\qquad \mathcal{L} = e^{\frac{1}{2}w_{\mu\nu}\gamma^{\mu\nu}}\,, \\ (M^{\mu\nu})^{\lambda}{}_{\rho} = \eta^{\mu\lambda}\delta^{\nu}{}_{\rho} - \eta^{\nu\lambda}\delta^{\mu}{}_{\rho}\,. \end{gathered} \tag{2.52}$$

For even d, if a $2^{d/2}\times 2^{d/2}$ matrix, $M^{\mu_1\mu_2\cdots\mu_n}$, is totally anti-symmetric over the n spacetime indices

$$M^{\mu_1\mu_2\cdots\mu_n} = M^{[\mu_1\mu_2\cdots\mu_n]}\,, \tag{2.53}$$

and transforms covariantly under Lorentz transformations in d or $d+1$ dimensions as

$$\mathcal{L}^{-1}M^{\mu_1\mu_2\cdots\mu_n}\mathcal{L} = \prod_{i=1}^{n} L^{\mu_i}{}_{\nu_i}\,M^{\nu_1\nu_2\cdots\nu_n}\,, \tag{2.54}$$

then for $0 \leq n \leq \max(d/2, 2)$, the general forms of $M^{\mu_1\mu_2\cdots\mu_n}$ are

$$M^{\mu_1\mu_2\cdots\mu_n} = \begin{cases} (1 + c\gamma^{(d+1)})\gamma^{\mu_1\mu_2\cdots\mu_n} & \text{In even } d \text{ dimensions}\,, \\ \\ \gamma^{\mu_1\mu_2\cdots\mu_n} & \text{In odd } d+1 \text{ dimensions}\,, \end{cases} \tag{2.55}$$

where c is a constant.

To show this, one may first expand $M^{\mu_1\mu_2\cdots\mu_n}$ in terms of $\gamma_{\nu_1\nu_2\cdots\nu_m}$, $\gamma^{(d+1)}\gamma_{\nu_1\nu_2\cdots\nu_m}$ or $\gamma_{\nu_1\nu_2\cdots\nu_m}$ depending on the dimensions, d or $d+1$, with $0 \leq m \leq d/2$. Then eq.(2.54) implies that the coefficients of them, say $T^{\mu_1\mu_2\cdots\mu_{m+n}}$, are Lorentz invariant tensors satisfying

$$\prod_{i=1}^{m+n} L^{\mu_i}{}_{\nu_i}\, T^{\nu_1\nu_2\cdots\nu_{m+n}} = T^{\mu_1\mu_2\cdots\mu_{m+n}} \tag{2.56}$$

Finally one can recall the well known fact [4] that the general forms of Lorentz invariant tensors are multi-products of the metric, $\eta^{\mu\nu}$, and the totally antisymmetric tensor, $\epsilon^{\mu_1\mu_2\cdots}$, which verifies eq.(2.55).

2.4 Crucial Identities for Super Yang-Mills

The following identities are crucial to show the existence of the non-Abelian super Yang-Mills in THREE, FOUR, SIX and TEN dimensions.

(i) The following identity holds only in THREE or FOUR dimensions with arbitrary signature

$$0 = (\gamma^\mu C^{-1})_{\alpha\beta}(\gamma_\mu C^{-1})_{\gamma\delta} + \text{cyclic permutations of } \alpha,\beta,\gamma \tag{2.57}$$

To verify the identity in even dimensions we contract $(\gamma^\mu C^{-1})_{\alpha\beta}(\gamma_\mu)_{\gamma\delta}$ with $(C\gamma^{\nu_1\nu_2\cdots\nu_n})_{\beta\alpha}$ and take cyclic permutations of α,β,γ to get

$$0 = 2^{d/2}\delta^n_1 + (d-2n)(\zeta + \zeta^n(-1)^{\frac{1}{2}n(n-1)})(-1)^{n+\frac{1}{8}d(d-\zeta 2)} \tag{2.58}$$

This equation must be satisfied for all $0 \leq n \leq d$, which is valid only in $d = 4, \zeta = -1$. Similar analysis can be done for the $d+1$ odd dimensions by adding $(\gamma^{(d+1)}C^{-1})_{\alpha\beta}(\gamma^{(d+1)}C^{-1})_{\gamma\delta}$ term into eq.(2.57). We get

$$0 = 2^{d/2}(\delta^n_1 + \delta^n_d) + (d-2n+1)(\zeta + \zeta^n(-1)^{\frac{1}{2}n(n-1)})(-1)^{n+\frac{1}{8}d(d-\zeta 2)}, \qquad \zeta = (-1)^{d/2} \tag{2.59}$$

Only in $d = 2$ and hence three dimensions, this equation is satisfied for all $0 \leq n \leq d$.

(ii) The following identity holds only in TWO, FOUR or SIX dimensions with arbitrary signature

$$0 = (\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} + (\sigma^\mu)_{\gamma\beta}(\sigma_\mu)_{\alpha\delta} \tag{2.60}$$

To verify this identity we take d dimensional sigma matrices from $f = d-2$ dimensional gamma matrices as in eq.(2.31)

$$\sigma^\mu = (\gamma^\mu, \gamma^{(f+1)}, i) \tag{2.61}$$

to get

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = (\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} + (\gamma^{(f+1)})_{\alpha\beta}(\gamma^{(f+1)})_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2.62}$$

Again this expression is valid for any signature, (t,s). Now we contract this equation with $(\gamma^{\nu_1\nu_2\cdots\nu_n}C_+^{-1})_{\beta\delta}$. From eqs.(2.24, 2.30) in the case of odd t we get

$$\left((-1)^n(f-2n)+(-1)^{\frac{f}{2}+n}-1\right)(\gamma^{\nu_1\nu_2\cdots\nu_n}C_+^{-1})_{\alpha\gamma} \tag{2.63}$$

To satisfy eq.(2.60) this expression must be anti-symmetric over $\alpha\leftrightarrow\gamma$ for any $0\leq n\leq f$. Thus from eq.(2.29) we must require $0=(-1)^n(f-2n)+(-1)^{\frac{f}{2}+n}-1$ for all n satisfying $(-1)^{\frac{1}{8}f(f-2)+\frac{1}{2}n(n-1)}=1$. This condition is satisfied only in $f=0,2,4$ and hence $d=2,4,6$ ($f=6$ case is excluded by choosing $n=6$ and $f\geq 8$ cases are excluded by choosing either $n=0$ or $n=3$).

(iii) The following identity holds only in TWO or TEN dimensions with arbitrary signature

$$0=(\sigma^\mu \mathrm{c}^{-1})_{\alpha\beta}(\sigma_\mu \mathrm{c}^{-1})_{\gamma\delta}+\text{cyclic permutations of }\alpha,\beta,\gamma \tag{2.64}$$

3. Spinors

3.1 Weyl Spinor

In any even d dimensions, Weyl spinor, ψ, satisfies

$$\gamma^{(d+1)}\psi = \psi \tag{3.1}$$

and so $\bar{\psi} = \psi^\dagger A$ satisfies from eq.(2.30)

$$\bar{\psi}\gamma^{(d+1)} = (-1)^t\bar{\psi} \qquad \gamma^{(d+1)}C_\pm^{-1}\bar{\psi}^T = (-1)^{\frac{t-s}{2}}C_\pm^{-1}\bar{\psi}^T \tag{3.2}$$

3.2 Majorana Spinor

By definition Majorana spinor satisfies

$$\bar{\psi} = \psi^T C_\pm \qquad \text{or} \qquad \bar{\psi} = \psi^T C \tag{3.3}$$

depending on the dimensions, even or odd. This is possible only if $\varepsilon_\pm, \varepsilon = 1$ and so from eqs.(2.19, 2.48)

$$\begin{aligned} \eta = +1 \ &: \qquad t - s = 0, 1, 2 \bmod 8 \\ \eta = -1 \ &: \qquad t - s = 0, 6, 7 \bmod 8 \end{aligned} \tag{3.4}$$

where η is the sign factor, ± 1, occuring in eq.(2.17) or eq.(2.45)[8].

3.3 Majorana-Weyl Spinor

Majorana-Weyl spinor satisfies both of the two conditions above

$$\gamma^{(d+1)}\psi = \psi \qquad \bar{\psi} = \psi^T C_\pm \tag{3.5}$$

Majorana-Weyl Spinor exists only if

$$\begin{aligned} \eta = +1 \ &: \qquad t - s = 0 \bmod 8 \\ \eta = -1 \ &: \qquad t - s = 0 \bmod 8 \end{aligned} \tag{3.6}$$

[8] In [2], $\eta = -1$ case is called Majorana and $\eta = +1$ case is called pseudo-Majorana.

4. Majorana Representation and SO(8)

Fact 1:
Consider a finite dimensional vector space, $\mathcal{V}$ with the unitary and symmetric matrix, $\mathcal{B} = \mathcal{B}^T$, $\mathcal{B}\mathcal{B}^\dagger = 1$. For every $|v\rangle \in \mathcal{V}$ if $\mathcal{B}|v\rangle^* \in \mathcal{V}$ then there exists an orthonormal "semi-real " basis, $\mathcal{V} = \{|l\rangle,\, l = 1, 2, \cdots\}$ such that $\mathcal{B}|l\rangle^* = |l\rangle$.

Proof
Start with an arbitrary orthonormal bais, $\{|v_l\rangle,\, l = 1, 2, \cdots\}$ and let $|1\rangle \propto |v_1\rangle + \mathcal{B}|v_1\rangle^*$. After the normalization, $\langle 1|1\rangle = 1$, we can take a new orthonormal basis, $\{|1\rangle, |2'\rangle, |3'\rangle, \cdots\}$. Now we assume that $\{|1\rangle, |2\rangle, \cdots |k-1\rangle, |k'\rangle, |(k+1)'\rangle, \cdots\}$ is an orhonormal basis such that $\mathcal{B}|j\rangle^* = |j\rangle$ for $1 \leq j \leq k-1$. To construct the k th such a vector, $|k\rangle$ we set $|k\rangle \propto |k'\rangle + \mathcal{B}|k'\rangle^*$ with the normalization. We check this is orthogonal to $|j\rangle$, $1 \leq j \leq k-1$

$$\langle j|\big(\,|k'\rangle + \mathcal{B}|k'\rangle^*\,\big) = 0 + \langle k|j\rangle = 0\,. \tag{4.1}$$

In this way one can construct the desired basis.

In the spacetime which admits Majorana spinor from Eq.(3.4)

$$\begin{aligned} \eta = +1 \;\; &: \qquad t - s = 0, 1, 2 \text{ mod } 8 \\ \eta = -1 \;\; &: \qquad t - s = 0, 6, 7 \text{ mod } 8\,, \end{aligned} \tag{4.2}$$

more explicitly in the even dimensions having $\varepsilon_+ = 1$ (or $\varepsilon_- = 1$) where B_+ (or B_-) is symmetric and also in the odd dimensions of $\varepsilon = 1$ where B is symmetric, from the **fact 1** above we can choose an "semi-real " orthonormal basis such that $B_\eta|l\rangle^* = |l\rangle$ In the basis, we write the gamma matrices

$$\gamma^\mu = \sum R^\mu_{lm}|l\rangle\langle m|\,. \tag{4.3}$$

From $\eta\,\gamma^{\mu *} = B_\eta \gamma^\mu B_\eta^{-1}$ and the property of the semi-real basis, $B_\eta|l\rangle^* = |l\rangle$ we get

$$(R^\mu_{lm})^* = \eta R^\mu_{lm}\,. \tag{4.4}$$

Since R^μ is also a representation of the gamma matrix

$$R^\mu R^\nu + R^\nu R^\mu = 2\eta^{\mu\nu}\,, \tag{4.5}$$

adopting the true real basis, we conclude that **there exists a Majorana represention where the gamma matrices are real, $\eta = +$ or pure imaginary, $\eta = -$ in any spacetime admitting Majorana spinors.**

Furthermore from Eq.(2.30), in the even dimension of $t - s \equiv 0$ mod 8, $\varepsilon_\pm = 1$ and $\gamma^{(d+1)*} = B\gamma^{(d+1)}B^{-1}$ (here we omit the subscript index $\pm$ or η for simplicity.). The action, $|v\rangle \rightarrow B^\dagger|v\rangle^*$ preserves the chirality, and from the **fact 1** above we can choose an orthonormal semi-real basis for the chiral and anti-chiral spinor spaces, $\mathcal{V} = \mathcal{V}_+ + \mathcal{V}_-$, $\mathcal{V}_\pm = \{|l_\pm\rangle\}$ such that

$$\langle l_\pm|m_\pm\rangle = \delta_{lm}\,, \qquad \langle l_\pm|m_\mp\rangle = 0\,, \qquad \gamma^{(d+1)}|l_\pm\rangle = \pm|l_\pm\rangle\,, \qquad B^\dagger|l_\pm\rangle^* = |l_\pm\rangle\,. \tag{4.6}$$

With the semi-real basis

$$\gamma^{(d+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{4.7}$$

and the gamma matrices are in the Majorana representation

$$\gamma^\mu = \begin{pmatrix} 0 & r^\mu \\ r^T_\mu & 0 \end{pmatrix}\,, \qquad r^\mu \in \mathrm{O}(2^{d/2-1})\,, \qquad r^\mu r^{\nu T} + r^\nu r^{\mu T} = 2\delta^{\mu\nu}\,. \tag{4.8}$$

From Eq.(6.8) any two sets of semi-real basis, say $\{|l_\pm\rangle\}$ and $\{|\tilde{l}_\pm\rangle\}$ are connected by an $\mathrm{O}((2^{d/2-1}))$ transformation

$$|\tilde{l}_\pm\rangle = \sum_m \Lambda_{\pm ml}|m_\pm\rangle\,, \qquad \sum_m \Lambda_{\pm lm}\Lambda_{\pm nm} = \delta_{ln}\,. \tag{4.9}$$

If we define

$$\Lambda_\pm = \sum_{l,m} \Lambda_{\pm lm}|l_\pm\rangle\langle m_\pm|\,, \tag{4.10}$$

then $|\tilde{l}_\pm\rangle = \Lambda_\pm|l_\pm\rangle$ and from the definition of the semi-real basis

$$\Lambda_\pm = B^\dagger\Lambda^*_\pm B = \Lambda_\pm P_\pm = P_\pm\Lambda_\pm\,, \qquad \Lambda_\pm\Lambda^\dagger_\pm = P_\pm\,. \tag{4.11}$$

We write

$$\Lambda_\pm = e^{M_\pm}\,, \qquad M_\pm \equiv \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(\Lambda_\pm - P_\pm)^n = \ln\,\Lambda_\pm\,. \tag{4.12}$$

Thus for $\Lambda_\pm$ such that the infinity sum converges we have

$$M_\pm = -M^\dagger_\pm = B^\dagger M^*_\pm B = M_\pm P_\pm = P_\pm M_\pm\,. \tag{4.13}$$

This gives a strong constraint when we express $M_\pm$ by the gamma matrix products. For the Eucledean eight dimensions only the SO(8) generators for the spinors survive in the expansion!

$$M_\pm = \tfrac{1}{2}w_{ab}\gamma^{ab}P_\pm\,. \tag{4.14}$$

Namely we find an isomorphism between the two SO(8)'s, one for the semi-real vectors and the other for the spinors in the conventional sense. Alternatively this can be seen from

$$\gamma^{ab} = \begin{pmatrix} r^{[a}r^{b]\,T} & 0 \\ 0 & r^{[a\,T}r^{b]} \end{pmatrix}\,, \tag{4.15}$$

where the each block diagonal is a generator of SO(D) while the dimension of the chiral space is $2^{d/2-1}$. Only in $d = 8$ both coincide leading to the "so(8) triolity" among $\mathrm{so}_v(8)$, $\mathrm{so}_c(8)$ and $\mathrm{so}_{\bar{c}}(8)$.

Fact 2: Relation to octonions.

In Euclidean eight dimensions, the 16×16 gamma matrices can be taken of the off-block diagonal form,

$$\gamma_a = \begin{pmatrix} 0 & r_a \\ r_a^T & 0 \end{pmatrix}, \qquad r_a r_b^T + r_b r_a^T = 2\delta_{ab}\,, \tag{4.16}$$

where the 8×8 real matrices, r_a, $1 \leq a \leq 8$, give the multiplication of the octonions, o_a,

$$o_a o_b = (r_a)_b{}^c o_c\,. \tag{4.17}$$

Fact 3:

Consider an arbitrary real self-dual or anti-self-dual four form in $D = 8$

$$T^{\pm}_{abcd} = \pm\tfrac{1}{4!}\,\epsilon_{abcdefgh}T^{\pm efgh}\,. \tag{4.18}$$

Using the SO(8) rotations one can transform the four form into the canonical form where the non-vanishing components are $T^{\pm}_{1234}$, $T^{\pm}_{1256}$, $T^{\pm}_{1278}$, $T^{\pm}_{1357}$, $T^{\pm}_{1368}$, $T^{\pm}_{1458}$, $T^{\pm}_{1467}$ and their dual counter parts only.

Proof

We start with the seven linearly independent traceless Hermitian matrices

$$\begin{aligned} &E_{\pm 1} = \gamma^{2341}P_{\pm}\,, \quad E_{\pm 2} = \gamma^{2561}P_{\pm}\,, \quad E_{\pm 3} = \gamma^{2781}P_{\pm}\,, \quad E_{\pm 4} = \gamma^{1357}P_{\pm}\,, \\ &E_{\pm 5} = \gamma^{3681}P_{\pm}\,, \quad E_{\pm 6} = \gamma^{4581}P_{\pm}\,, \quad E_{\pm 7} = \gamma^{4671}P_{\pm}\,. \end{aligned} \tag{4.19}$$

As they commute with each other, there exists a basis $\mathcal{V}_{\pm} = \{|l_{\pm}\rangle\}$ diagonalizing the seven quantities

$$E_{\pm r} = \sum_l \lambda_{rl}|l_{\pm}\rangle\langle l_{\pm}|\,, \qquad (\lambda_{rl})^2 = 1\,. \tag{4.20}$$

Further, since $C|l_{\pm}\rangle^*$ is also an eigenvector of the same eigenvalues, from the **fact 1** we can impose the semi-reality condition without loss of generality, $C|l_{\pm}\rangle^* = |l_{\pm}\rangle$.

Now for the self-dual four form we let

$$T^{\pm} = \tfrac{1}{4}T^{\pm}_{abcd}\gamma^{abcd}\,. \tag{4.21}$$

Since $T^{\pm}$ is Hermitian and $C(T^{\pm})^*C^{\dagger} = T^{\pm}$, one can diagonalize $T^{\pm}$ with a semi-real basis

$$T^{\pm} = \sum_l \lambda_l|\tilde{l}_{\pm}\rangle\langle\tilde{l}_{\pm}|\,, \qquad C|\tilde{l}_{\pm}\rangle^* = |\tilde{l}_{\pm}\rangle\,. \tag{4.22}$$

For the two semi-real basis above we define a transformation matrix

$$O_{\pm} = |l_{\pm}\rangle\langle\tilde{l}_{\pm}|\,. \tag{4.23}$$

Then, since $T^{\pm}$ is traceless, $O_{\pm}T^{\pm}O_{\pm}^{\dagger}$ can be written in terms of $E_{\pm i}$'s. Finally the fact $O_{\pm}$ gives a spinorial SO(8) rotation completes our proof.

Some useful formulae are

$$\begin{aligned}\pm P_{\pm} &= E_{\pm 1}E_{\pm 2}E_{\pm 3} = E_{\pm 1}E_{\pm 4}E_{\pm 5} = E_{\pm 1}E_{\pm 6}E_{\pm 7} = E_{\pm 2}E_{\pm 4}E_{\pm 6} \\ &= E_{\pm 2}E_{\pm 5}E_{\pm 7} = E_{\pm 3}E_{\pm 4}E_{\pm 7} = E_{\pm 3}E_{\pm 5}E_{\pm 6}\,.\end{aligned} \tag{4.24}$$

For an arbitrary self-dual or anti-self-dual four form tensor in $D = 8$, from

$$\begin{aligned}T^{\pm}_{acde}T^{\pm bcde} &= (\tfrac{1}{4!})^2\epsilon_{acdefghi}\epsilon^{bcdejklm}T^{\pm fghi}T^{\pm}_{jklm} \\ &= \tfrac{1}{4}\,\delta_a{}^b T^{\pm}_{cdef}T^{\pm cdef} - T^{\pm}_{acde}T^{\pm bcde}\,,\end{aligned} \tag{4.25}$$

we obtain an identity

$$T^{\pm}_{acde}T^{\pm bcde} = \tfrac{1}{8}\,\delta_a{}^b\,T^{\pm}_{cdef}T^{\pm cdef}\,. \tag{4.26}$$

5. Superalgebra

5.1 Graded Lie Algebra

Supersymmetry algebra is a $\hat{Z}_2$ graded Lie algebra, $\mathbf{g} = \{T_a\}$, which is an algebra with commutation and anti-commutation relations [5, 6]

$$[T_a, T_b\} = C^c_{ab} T_c \tag{5.1}$$

where C^c_{ab} is the structure constant and

$$[T_a, T_b\} = T_a T_b - (-1)^{\#a\#b} T_b T_a \tag{5.2}$$

with $\#a$, the $\hat{Z}_2$ grading of T_a,

$$\#a = \begin{cases} 0 \text{ for bosonic } a \\ 1 \text{ for fermionic } a \end{cases} \tag{5.3}$$

The generalized Jacobi identity is

$$[T_a, [T_b, T_c\}\} - (-1)^{\#a\#b}[T_b, [T_a, T_c\}\} = [[T_a, T_b\}, T_c\} \tag{5.4}$$

which implies

$$(-1)^{\#a\#c} C^d_{ab} C^e_{dc} + (-1)^{\#b\#a} C^d_{bc} C^e_{da} + (-1)^{\#c\#b} C^d_{ca} C^e_{db} = 0 \tag{5.5}$$

For a graded Lie algebra we consider

$$g(z) = \exp(z^a T_a) \tag{5.6}$$

where z^a is a superspace coordinate component which has the same bosonic or fermionic property as T_a and hence $z^a T_a$ is bosonic.
In the general case of non-commuting objects, say A and B, the Baker-Campbell-Haussdorff formula gives

$$e^A e^B = \exp\left(\sum_{n=0}^{\infty} C_n(A, B)\right) \tag{5.7}$$

where $C_n(A, B)$ involves n commutators. The first three of these are

$$\begin{aligned} C_0(A, B) &= A + B \\ C_1(A, B) &= \tfrac{1}{2}[A, B] \\ C_2(A, B) &= \tfrac{1}{12}[[A, B], B] + \tfrac{1}{12}[A, [A, B]] \end{aligned} \tag{5.8}$$

Since for the graded algebra

$$[z^a T_a, z^b T_b] = z^b z^a [T_a, T_b\} = z^b z^a C^c_{ab} T_c \tag{5.9}$$

the Baker-Campbell-Haussdorff formula (5.7) implies that $g(z)$ forms a group, the graded Lie group. Hence we may define a function on superspace, $f^a(w,z)$, by

$$g(w)g(z) = g(f(w,z)) \tag{5.10}$$

Since $g(0) = e$, the identity, we have $f(0,z) = z$, $f(w,0) = w$ and further we assume that $f(w,z)$ has a Taylor expansion in the neighbourhood of $w = z = 0$.
Associativity of the group multiplication requires $f(w,z)$ to satisfy

$$f(f(u,w),z) = f(u,f(w,z)) \tag{5.11}$$

5.2 Left & Right Invariant Derivatives

For a graded Lie group, left and right invariant derivatives, L_a, R_a are defined by

$$L_a g(z) = g(z)T_a \tag{5.12}$$

$$R_a g(z) = -T_a g(z) \tag{5.13}$$

Explicitly we have

$$L_a = L_a{}^b(z)\partial_b \qquad L_a{}^b(z) = \left.\frac{\partial f^b(z,u)}{\partial u^a}\right|_{u=0} \tag{5.14}$$

$$R_a = R_a{}^b(z)\partial_b \qquad R_a{}^b(z) = -\left.\frac{\partial f^b(u,z)}{\partial u^a}\right|_{u=0} \tag{5.15}$$

where $\partial_b = \frac{\partial}{\partial z^b}$.
It is easy to see that L_a is invariant under left action, $g(z) \to hg(z)$, and R_a is invariant under right action, $g(z) \to g(z)h$.
From eqs.(5.12, 5.13) we get

$$[L_a, L_b\} = C^c_{ab} L_c \tag{5.16}$$

$$[R_a, R_b\} = C^c_{ab} R_c \tag{5.17}$$

and from eqs.(5.12, 5.13) we can also easily show

$$[L_a, R_b\} = 0 \tag{5.18}$$

Thus, $L_a(z)$, $R_a(z)$ form representations of the graded Lie algebra separately. For the supersymmetry algebra, the left invariant derivatives become covariant derivatives, while the right invariant derivatives become the generators of the supersymmetry algebra acting on superfields.

5.3 Superspace & Supermatrices

In general a superspace may be denoted by $\mathbf{R}^{p|q}$, where p, q are the number of real commuting (bosonic) and anti-commuting (fermionic) variables respectively. A supermatrix which takes $\mathbf{R}^{p|q} \to \mathbf{R}^{p|q}$ may be represented by a $(p+q)\times(p+q)$ matrix, M, of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.19}$$

where a, d are $p\times p$, $q\times q$ matrices of Grassmanian even or bosonic variables and b, c are $p\times q$, $q\times p$ matrices of Grassmanian odd or fermionic variables respectively.
The inverse of M can be expressed as

$$M^{-1} = \begin{pmatrix} (a-bd^{-1}c)^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ -d^{-1}c(a-bd^{-1}c)^{-1} & (d-ca^{-1}b)^{-1} \end{pmatrix} \tag{5.20}$$

where we may write

$$(a-bd^{-1}c)^{-1} = a^{-1} + \sum_{n=1}^{\infty}(a^{-1}bd^{-1}c)^n a^{-1} \tag{5.21}$$

Note that due to the fermionic property of b, c, the power series terminates at $n \le pq+1$.
The supertrace and the superdeterminant of M are defined as

$$\operatorname{str} M = \operatorname{tr} a - \operatorname{tr} d \tag{5.22}$$

$$\operatorname{sdet} M = \det(a-bd^{-1}c)/\det d = \det a/\det(d-ca^{-1}b) \tag{5.23}$$

The last equality comes from

$$\det(1-a^{-1}bd^{-1}c) = \det{}^{-1}(1-d^{-1}ca^{-1}b) \tag{5.24}$$

which may be shown using

$$\det(1-a) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{tr} a^n\right) \tag{5.25}$$

and observing

$$\operatorname{tr}(a^{-1}bd^{-1}c)^n = -\operatorname{tr}(d^{-1}ca^{-1}b)^n \tag{5.26}$$

From eq.(5.23) we note that $\operatorname{sdet} M \neq 0$ implies the existence of M^{-1}. Thus the set of supermatrices for $\operatorname{sdet} M \neq 0$ forms the supergroup, $\mathrm{Gl}(p|q)$. If $\operatorname{sdet} M = 1$ then $M \in \mathrm{Sl}(p|q)$.
The supertrace and the superdeterminant have the properties

$$\operatorname{str}(M_1M_2) = \operatorname{str}(M_2M_1) \tag{5.27}$$

$$\operatorname{sdet}(M_1M_2) = \operatorname{sdet} M_1 \operatorname{sdet} M_2 \tag{5.28}$$

We may define the transpose of the supermatrix, M, either as

$$M^t = \begin{pmatrix} a^t & c^t \\ -b^t & d^t \end{pmatrix} \tag{5.29}$$

or as

$$M^{t'} = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} \tag{5.30}$$

where a^t, b^t, c^t, d^t are the ordinary transposes of a, b, c, d respectively.
We note that

$$(M_1 M_2)^t = M_2^t M_1^t \qquad (M_1 M_2)^{t'} = M_2^{t'} M_1^{t'} \tag{5.31}$$

$$(M^t)^{t'} = (M^{t'})^t = M \tag{5.32}$$

6. Super Yang-Mills

6.1 $(3+1)D$ $\mathcal{N}=1$ super Yang-Mills

In four-dimensional Minkowskian spacetime of the metric, $\eta=\mathrm{diag}(-+++)$, the 4×4 gamma matrices satisfy with $\mu=0,1,2,3$,

$$\begin{array}{lll}\Gamma^{\mu\dagger}=\Gamma_\mu=-A\Gamma^\mu A^\dagger\,, & A=\Gamma^t=-A^\dagger\,, & \\ \Gamma^{\mu *}=+B\Gamma^\mu B^\dagger\,, & B^T=B\,, & B^\dagger=B^{-1}\,, \\ \Gamma^{\mu T}=-C\Gamma^\mu C^\dagger\,, & C=-C^T=B\Gamma^t\,, & C^\dagger=C^{-1}\,.\end{array} \tag{6.1}$$

The Majorana spinor, ψ satisfies then

$$\bar{\psi}=\psi^\dagger\Gamma^t=\psi^T C \qquad \Longleftrightarrow \qquad \psi^*=B\psi\,. \tag{6.2}$$

The four-dimensional super Yang-Mills Lagrangian reads

$$\mathcal{L}_{4D}=\mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-i\tfrac{1}{2}\bar{\psi}\Gamma^\mu D_\mu\psi\right)\,. \tag{6.3}$$

The supersymmetry transformations are

$$\delta A_\mu=i\bar{\varepsilon}\Gamma_\mu\psi=-i\bar{\psi}\Gamma_\mu\varepsilon\,, \qquad \delta\psi=-\tfrac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon\,. \tag{6.4}$$

6.2 $(5+1)D$ $(1,0)$ super Yang-Mills

In six-dimensional Minkowskian spacetime of the metric, $\eta=\mathrm{diag}(-+++++)$, the 8×8 gamma matrices satisfy with $M=0,1,2,3,4,5$,

$$\begin{array}{lll}\Gamma^{M\dagger}=\Gamma_M=A\Gamma^M A^\dagger\,, & A:=\Gamma^{12345}=A^\dagger=A^{-1}\,, & \\ \Gamma^{MT}=C\Gamma^M C^\dagger\,, & C^T=-C\,, & C^\dagger=C^{-1}\,, \\ \Gamma^{M*}=B\Gamma^M B^\dagger\,, & B=CA=-B^T\,, & B^\dagger=B^{-1}\,.\end{array} \tag{6.5}$$

The gamma "seven" is given by $\Gamma^{(7)}=\Gamma^{012345}$ to satisfy $\Gamma^{(7)}=\Gamma^{(7)\dagger}=\Gamma^{(7)-1}$ and

$$\Gamma^{LMN}=\tfrac{1}{6}\epsilon^{LMNPQR}\,\Gamma_{PQR}\Gamma^{(7)}\,, \tag{6.6}$$

where $\epsilon^{012345}=+1$.

The su(2) Majorana-Weyl spinor, ψ_i, $i=1,2$, satisfies then

$$\begin{array}{lll}\Gamma^{(7)}\psi_i=+\psi_i\,, \quad \bar{\psi}^i\Gamma^{(7)}=-\bar{\psi}^i & : & \text{chiral}\,, \\ \bar{\psi}^i=(\psi_i)^\dagger A=\epsilon^{ij}(\psi_j)^T C & : & \text{su(2) Majorana}\,,\end{array} \tag{6.7}$$

where ϵ^{ij} is the usual 2×2 skew-symmetric unimodular matrix. It is worth to note that $\bar{\psi}^i\Gamma^{M_1M_2\cdots M_{2n}}\rho_i = 0$ and

$$\mathrm{tr}(i\bar{\psi}^i\Gamma^{M_1M_2\cdots M_{2n+1}}\rho_i) = \left[\mathrm{tr}(i\bar{\psi}^i\Gamma^{M_1M_2\cdots M_{2n+1}}\rho_i)\right]^\dagger = -(-1)^n\mathrm{tr}(i\bar{\rho}^i\Gamma^{M_1M_2\cdots M_{2n+1}}\psi_i)\,, \tag{6.8}$$

where ψ_i, ρ_i are two arbitrary Lie algebra valued su(2) Majorana-Weyl spinors.

The six-dimensional super Yang-Mills Lagrangian reads

$$\mathcal{L}_{6D} = \mathrm{tr}\left(-\tfrac{1}{4}F_{LM}F^{LM} - i\tfrac{1}{2}\bar{\psi}^i\Gamma^L D_L\psi_i\right)\,, \tag{6.9}$$

where all the fields are in the adjoint representation of the gauge group such that, with the Hermitian Lie algebra valued gauge field, A_M,

$$D_L\psi_i = \partial_L\psi_i - i[A_L,\psi_i]\,, \qquad F_{LM} = \partial_L A_M - \partial_M A_L - i[A_L, A_M]\,. \tag{6.10}$$

From (6.8) the action is real valued.

The supersymmetry transformations are given by with a su(2) Majorana-Weyl supersymmetry parameter, ε_i,

$$\delta A_M = +i\bar{\varepsilon}^i\Gamma_M\psi_i = -i\bar{\psi}^i\Gamma_M\varepsilon_i\,, \qquad \delta\psi_i = -\tfrac{1}{2}F_{MN}\Gamma^{MN}\varepsilon_i\,, \tag{6.11}$$

so that, in particular, $\delta\bar{\psi}^i = +\frac{1}{2}F_{MN}\bar{\varepsilon}^i\Gamma^{MN}$. The crucial Fierz identity for the supersymmetry invariance is with the chiral projection matrix, $P := \frac{1}{2}(1+\Gamma^{(7)})$,

$$\left(\Gamma^L P\right)_{\alpha\beta}(\Gamma_L P)_{\gamma\delta} + \left(\Gamma^L P\right)_{\gamma\beta}(\Gamma_L P)_{\alpha\delta} = 0\,, \tag{6.12}$$

which ensures the vanishing of the terms cubic in ψ_i,

$$\mathrm{tr}\left(\bar{\psi}^i\Gamma^L[\delta A_L,\,\psi_i]\right) = \mathrm{tr}\left(\bar{\psi}^i\Gamma^L[\,i\bar{\varepsilon}^j\Gamma_L\psi_j\,,\,\psi_i]\right) = 0\,. \tag{6.13}$$

The equations of motion are

$$D_L F^{LM} + \bar{\psi}^i\Gamma^M\psi_i = 0\,, \qquad \Gamma^M D_M\psi_i = 0\,. \tag{6.14}$$

6.3 $6D$ super Yang-Mills in the spacetime of arbitrary signature

With

$$\left(\Gamma^M\right)^T = \pm\mathcal{C}_\pm\Gamma^M\mathcal{C}_\pm^{-1}\,, \qquad \mathcal{C}_\pm^T = \mp\mathcal{C}_\pm\,, \tag{6.15}$$

we have

$$\left(\mathcal{C}_\pm\Gamma^M\right)^T = -\mathcal{C}_\pm\Gamma^M\,. \tag{6.16}$$

We introduce a pair of Weyl spinors of the same chirality,

$$(\psi_1,\psi_2)\ , \qquad \Gamma^{(7)}\psi_i = s\psi_i\,, \qquad s^2 = 1\,, \tag{6.17}$$

and define the charge conjugate spinor by

$$\bar{\psi}_c^i := \epsilon^{-1ij}\psi_j^T\mathcal{C}_\pm\,. \tag{6.18}$$

The super Yang-Mills Lagrangian reads

$$\mathcal{L}_{6D} = \mathrm{tr}\left(\tfrac{1}{4}F_{MN}F^{MN} + \tfrac{1}{2}\bar{\psi}_c^i\Gamma^M D_M\psi_i\right)\ , \tag{6.19}$$

and the supersymmetry transformations are given by

$$\begin{aligned}\delta A_M &= \bar{\varepsilon}_c^i\Gamma_M\psi_i = -\bar{\psi}_c^i\Gamma_M\varepsilon_i\,,\\ \delta\psi_i &= -\tfrac{1}{2}F_{MN}\Gamma^{MN}\varepsilon_i\,,\end{aligned} \tag{6.20}$$

so that, in particular, $\delta\bar{\psi}_c^i = +\frac{1}{2}F_{MN}\bar{\varepsilon}_c^i\Gamma^{MN}$. The Lagrangian transforms as, from (2.60),

$$\delta\mathcal{L}_{6D} = \partial_M\mathrm{tr}\left(F^{MN}\delta A_N - \tfrac{1}{2}\bar{\psi}_c^i\Gamma^M\delta\psi_i\right)\ . \tag{6.21}$$

Only if $B_\pm^*B_\pm = -1$, as in the Minkowskian signature, one can impose the pseudo-Majorana condition,

$$\bar{\psi}_c^i = \bar{\psi}_D^i := (\psi_i)^\dagger A\,. \tag{6.22}$$

6.4 $(9+1)D$ SYM, its reduction, and $4D$ superconformal symmetry

- **Conventions for $(9+1)D$ gamma matrices**

Spacetime signature : $\eta = \mathrm{diag}(-++\cdots+)$, mostly plus signature.
32×32 Gamma matrices:
i) Hermitian conjugate,

$$(\Gamma^M)^\dagger = \Gamma_M = -\Gamma^0\Gamma^M\Gamma_0 = \mathcal{A}\Gamma^M\mathcal{A}^\dagger\,,$$
$$\mathcal{A} = \Gamma^{12\cdots 9} = \mathcal{A}^\dagger = \mathcal{A}^{-1}\,, \tag{6.23}$$
$$(\mathcal{A}\Gamma^{M_1M_2\cdots M_n})^\dagger = (-1)^{\frac{1}{2}n(n-1)}\mathcal{A}\Gamma^{M_1M_2\cdots M_n}\,,$$

ii) Complex conjugate,

$$(\Gamma^M)^* = \pm\mathcal{B}_\pm\Gamma^M\mathcal{B}_\pm^\dagger\,,$$
$$\mathcal{B}_\pm = \mathcal{B}_\pm^T = (\mathcal{B}_\pm^\dagger)^{-1}\,, \tag{6.24}$$

iii) Transpose,

$$(\Gamma^M)^T = \pm\mathcal{C}_\pm\Gamma^M\mathcal{C}_\pm^\dagger\,,$$
$$\mathcal{C}_\pm = \mathcal{B}_\pm^T\mathcal{A} = \pm\mathcal{C}_\pm^T = (\mathcal{C}_\pm^\dagger)^{-1}\,, \tag{6.25}$$
$$(\mathcal{C}_+\Gamma^{M_1M_2\cdots M_n})^T = (-1)^{\frac{1}{2}n(n-1)}\mathcal{C}_+\Gamma^{M_1M_2\cdots M_n}\,.$$

Let the spinorial indices be located as

$$(\Gamma^M)^\alpha{}_\beta\,,\qquad (\mathcal{A})^\alpha{}_\beta\,,\qquad (\mathcal{B}_\pm)_{\alpha\beta} = (\mathcal{B}_\pm)_{\beta\alpha}\,,\qquad (\mathcal{C}_\pm)_{\alpha\beta} = \pm(\mathcal{C}_\pm)_{\beta\alpha}\,. \tag{6.26}$$

Define

$$\Gamma^{(10)} = \Gamma^{012\cdots 9} = (\Gamma^{(10)})^\dagger = (\Gamma^{(10)})^{-1} = -\mathcal{C}_+^\dagger(\Gamma^{(10)})^T\mathcal{C}_+\,. \tag{6.27}$$

The crucial identity for the super Yang-Mills action is

$$(\mathcal{C}_+\Gamma^M\Gamma_\pm)_{(\alpha\beta}(\mathcal{C}_+\Gamma_M\Gamma_\pm)_{\gamma)\delta} = 0 \tag{6.28}$$

where $\Gamma_\pm = \frac{1}{2}(1\pm\Gamma^{(10)})$ is either the chiral or the anti-chiral projector, and α,β,γ are symmetrized. Note also the symmetric property, $(\mathcal{C}_+\Gamma^M\Gamma_\pm)^T = \mathcal{C}_+\Gamma^M\Gamma_\pm$.

For spinors we set

$$\bar\psi = \psi^\dagger\mathcal{A}\,. \tag{6.29}$$

Majorana-Weyl Spinor, ψ, satisfies

$$\Gamma^{(10)}\psi = +\psi \qquad : \quad \text{Weyl condition}\,, \tag{6.30}$$

$$\psi^* = \mathcal{B}_+\psi \qquad : \quad \text{Majorana condition}\,, \tag{6.31}$$

or equivalently,

$$\bar{\psi}\Gamma^{(10)} = -\bar{\psi} \quad : \text{ opposite chirality}\,, \qquad \bar{\psi} = \psi^T \mathcal{C}_+\,. \tag{6.32}$$

Hence for the fermionic Majorana-Weyl spinors,

$$\bar{\psi}_1 \Gamma^{M_1 M_2 \cdots M_{2n}} \psi_2 = 0\,, \tag{6.33}$$

and[9]

$$\bar{\psi}_1 \Gamma^{M_1 M_2 \cdots M_{2n+1}} \psi_2 = (-1)^{n+1} \bar{\psi}_2 \Gamma^{M_1 M_2 \cdots M_{2n+1}} \psi_1 = -(\bar{\psi}_1 \Gamma^{M_1 M_2 \cdots M_{2n+1}} \psi_2)^\dagger \quad : \text{ imaginary}\,. \tag{6.34}$$

We can further set

$$\Gamma^M = \begin{pmatrix} 0 & \tilde{\gamma}^M \\ \gamma^M & 0 \end{pmatrix}\,, \qquad \gamma^M \tilde{\gamma}^N + \gamma^N \tilde{\gamma}^M = 2\eta^{MN}\,, \qquad \eta = \mathrm{diag}(- + + + \cdots +)\,. \tag{6.35}$$

Namely, $(\gamma^M, \tilde{\gamma}^N)$ are the real 16×16 matrices appearing in the off block-diagonal parts of the 32×32 gamma matrices,

satisfying[10]

$$\begin{aligned} &(\gamma^M)^* = \gamma^M\,, && (\gamma^M)^T = \tilde{\gamma}^0 \gamma^M \tilde{\gamma}^0 = \tilde{\gamma}_M\,, \\ &\tilde{\gamma}^0 \gamma^1 \tilde{\gamma}^2 \cdots \gamma^9 = +1\,, && \gamma^0 \tilde{\gamma}^1 \gamma^2 \cdots \tilde{\gamma}^9 = -1\,. \end{aligned} \tag{6.36}$$

- **Lagrangian.**
  Let the gauge group be su(N) or u(N).
  Lie algebra valued fields,

$$A_M = A_M^p T_p\,, \qquad \Psi = \Psi^p T_p\,, \qquad (T_p)^\dagger = T_p\,. \tag{6.37}$$

Field strength and the covariant derivative are

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]\,, \qquad D_M \Psi = \partial_M \Psi - i[A_M, \Psi]\,. \tag{6.38}$$

Bianchi identity reads

$$D_L F_{MN} + D_M F_{NL} + D_N F_{LM} = 0\,. \tag{6.39}$$

The gauge symmetry is given by, for $g^\dagger = g^{-1}$,

$$A_M \rightarrow g A_M g^{-1} + i g \partial_M g^{-1}\,, \qquad F_{MN} \rightarrow g F_{MN} g^{-1}\,, \qquad \Psi \rightarrow g \Psi g^{-1}\,. \tag{6.40}$$

[9] When the spinor is Lie algebra valued, Eq.(6.34) does not hold in general.

[10] From $\tilde{\gamma}^M = (\gamma_M)^{-1}$ it also follows that $\tilde{\gamma}^M \gamma^N + \tilde{\gamma}^N \gamma^M = 2\eta^{MN}$. One may further impose the symmetric property, $(\gamma^M)^T = \gamma^M$, but it is not necessary in our paper.

The Lagrangian of $10D$ super Yang-Mills theory reads

$$\begin{aligned}\mathcal{L} &= \mathrm{tr}\left[-\tfrac{1}{4}F_{MN}F^{MN} - i\tfrac{1}{2}\bar{\Psi}\Gamma^{M}D_{M}\Psi\right] \\ &= \mathrm{tr}\left[-\tfrac{1}{4}F_{MN}F^{MN} - i\tfrac{1}{2}\bar{\psi}\gamma^{M}D_{M}\psi\right] ,\end{aligned} \tag{6.41}$$

where $\Psi \equiv (\psi\ 0)^{T}$ and ψ^{α} is a sixteen component spinor and $\bar{\psi} := \psi^{T}\tilde{\gamma}^{0}$.

Under arbitrary infinitesimal transformations, δA_{M}, $\delta\Psi$,

$$\delta\mathcal{L} = \mathrm{tr}\left[\left(D_{L}F^{LM} + \bar{\Psi}\Gamma^{M}\Psi\right)\delta A_{M} - i\bar{\Psi}\Gamma^{M}D_{M}\delta\Psi\right] + \partial_{N}\mathrm{tr}\left[F^{MN}\delta A_{M} - i\tfrac{1}{2}\delta\bar{\Psi}\Gamma^{N}\Psi\right] . \tag{6.42}$$

- **Summary of supersymmetry in $D \leq 10$.**
The ordinary supersymmetry and kinetic supersymmetry are given by

$$\delta A_{M} = i\bar{\Psi}\Gamma_{M}\xi_{+} = -i\bar{\xi}_{+}\Gamma_{M}\Psi , \qquad \delta\Psi = \tfrac{1}{2}F_{MN}\Gamma^{MN}\xi_{+} + \xi'_{+}1_{N\times N} , \tag{6.43}$$

so that

$$\delta\bar{\Psi} = -\tfrac{1}{2}\bar{\xi}_{+}F_{MN}\Gamma^{MN} + \bar{\xi}'_{+}1_{N\times N} , \tag{6.44}$$

where ξ_{+} and ξ'_{+} are constant Majornana-Weyl spinors corresponding to the ordinary and kinetic supersymmetry parameters. $+$ denotes the chirality. The above is the symmetry of the $(9{+}1)D$ and also any dimensionally reduced super Yang-Mills action.

In four-dimensions of either Minkowskian or Euclidean signature, the supersymmetry gets enhanced to the superconformal symmetry as

$$\delta A_{M} = i\bar{\Psi}\Gamma_{M}\mathcal{E}(x) = -i\bar{\mathcal{E}}(x)\Gamma_{M}\Psi , \qquad \delta\Psi = \tfrac{1}{2}F_{MN}\Gamma^{MN}\mathcal{E}(x) - 2\Phi_{a}\Gamma^{a}\xi_{-} + \xi'_{+}1_{N\times N} , \tag{6.45}$$

where m is for the four-dimensions and a is for the rest. ξ_{-} is a constant Majornana-Weyl spinor of the opposite chirality corresponding to the special superconformal symmetry parameter, and

$$\mathcal{E}(x) = x^{m}\Gamma_{m}\xi_{-} + \xi_{+} . \tag{6.46}$$

In any case, the conserved supercurrent is of the universal form,

$$J^{M} = -i\mathrm{tr}\left(\bar{\Psi}\Gamma^{M}\delta\Psi\right) = +i\mathrm{tr}\left(\delta\bar{\Psi}\Gamma^{M}\Psi\right) . \tag{6.47}$$

In Appendix C, we present the derivation.

- **Superconformal symmetry in $4D$ of arbitrary signature.**
  The 32 supersymmetries in $4D$ super Yang-Mills which consist of ordinary supersymmetry and special superconformal symmetry read

$$
\begin{aligned}
\delta A_M &= i\bar{\Psi}\Gamma_M(1+x^m\Gamma_m)\xi = -i\bar{\xi}(1+x^m\Gamma_m)\Gamma_M\Psi\,,\\
\delta\Psi &= \tfrac{1}{2}(1+\Gamma^{(10)})\left[\tfrac{1}{2}F_{MN}\Gamma^{MN}(1+x^m\Gamma_m)-2\Phi_a\Gamma^a\right]\xi\,,\\
\delta\bar{\Psi} &= \bar{\xi}\left[-\tfrac{1}{2}(1+x^m\Gamma_m)F_{MN}\Gamma^{MN}-2\Phi_a\Gamma^a\right]\tfrac{1}{2}(1-\Gamma^{(10)})\,,
\end{aligned}
\tag{6.48}
$$

  where ξ is a 32 component Majorana spinor,

$$
\xi^* = \mathcal{B}_+\xi\,. \tag{6.49}
$$

  The chiral decomposition of the spinor gives the ordinary supersymmetry and special superconformal symmetry,[11]

$$
\xi = \xi_+ + \xi_-\,, \qquad \xi_\pm = \tfrac{1}{2}(1\pm\Gamma^{(10)})\xi\,. \tag{6.50}
$$

  The 32 component Majorana supercurrent is of the form,

$$
\begin{aligned}
J^M &= +i\bar{\mathcal{Q}}^M\xi = -i\bar{\xi}\mathcal{Q}^M\,,\\
\mathcal{Q}^M &= \mathrm{tr}\left[\left(\tfrac{1}{2}(1+x^m\Gamma_m)F_{KL}\Gamma^{KL}+2\Phi_a\Gamma^a\right)\Gamma^M\Psi\right]\,,\\
\bar{\mathcal{Q}}^M &= \mathrm{tr}\left[\bar{\Psi}\Gamma^M\left(-\tfrac{1}{2}F_{KL}\Gamma^{KL}(1+x^m\Gamma_m)+2\Phi_a\Gamma^a\right)\right] = (\mathcal{Q}^M)^\dagger\mathcal{A} = (\mathcal{Q}^M)^T\mathcal{C}_+\,.
\end{aligned}
\tag{6.51}
$$

  The supercharge is given by

$$
\mathcal{Q} = \int d^3x\;\mathcal{Q}^0\,. \tag{6.52}
$$

[11] Note also $\mathcal{E}(x) = \frac{1}{2}(1+\Gamma^{(10)})(1+x^m\Gamma_m)\xi$.

A. Proof of the Theorem

Theorem 1
Any $N \times N$ matrix, M, satisfying $M^2 = \lambda^2 1_{N\times N}$, $\lambda \neq 0$, is diagonalizable.

Proof
Suppose for some K, $1 \leq K \leq N$, we have found a basis,

$$\{e_a,\ v_r\ :\ 1 \leq a \leq K,\ 1 \leq r \leq N-K\} \tag{A.1}$$

such that

$$\begin{aligned} Me_a &= \lambda_a e_a\,, && \text{for } 1 \leq a \leq K\,, \\ Mv_r &= P^s{}_r v_s + h^a{}_r e_a\,, && \text{for } K+1 \leq r,s \leq N\,. \end{aligned} \tag{A.2}$$

From $M^2 = \lambda^2 1_{N\times N}$,

$$\begin{aligned} &\lambda_a^2 = \lambda^2\,, \\ &\lambda^2 v_r = (P^2)^s{}_r v_s + \left[(hP)^a{}_r + \lambda_a h^a{}_r\right] e_a\,, \end{aligned} \tag{A.3}$$

and hence,

$$\begin{aligned} &P^2 = \lambda^2 1_{(N-K)\times(N-K)}\,, \\ &(hP)^a{}_r + \lambda_a h^a{}_r = 0\,. \end{aligned} \tag{A.4}$$

The assumption holds for $K = 1$ surely. In order to construct e_{K+1} we first consider an eigenvector of the $(N-K)\times(N-K)$ matrix, P,

$$P^r{}_s c^s = \lambda_{K+1} c^r\,, \qquad \lambda_{K+1}^2 = \lambda^2\,, \tag{A.5}$$

and set

$$\begin{aligned} &v = c^r v_r\,, \qquad h^a = h^a{}_r c^r\,, \\ &Mv = \lambda_{K+1} v + h^a e_a\,. \end{aligned} \tag{A.6}$$

Consequently

$$(\lambda_{K+1} + \lambda_a) h^a = 0 \qquad : \quad \text{not } a \text{ sum}\,, \tag{A.7}$$

so that

$$h^a = 0 \qquad \text{if} \qquad \lambda_{K+1} + \lambda_a \neq 0\,. \tag{A.8}$$

We construct e_{K+1}, with K unknown coefficients, d^a, as

$$e_{K+1} = v + d^a e_a\,. \tag{A.9}$$

From

$$Me_{K+1} = \lambda_{K+1} e_{K+1} + \left[h^a + (\lambda_a - \lambda_{K+1}) d^a\right] e_a\,, \tag{A.10}$$

we determine

$$d^a = \begin{cases} \dfrac{h^a}{\lambda_{K+1} - \lambda_a} & \text{if} \quad \lambda_{K+1} \neq \lambda_a \,, \\ \\ \text{any number} & \text{if} \quad \lambda_{K+1} = \lambda_a \,. \end{cases} \tag{A.11}$$

From (A.8) and $\lambda_{K+1}^2 = \lambda_a^2 = \lambda^2 \neq 0$, we have

$$Me_{K+1} = \lambda_{K+1} e_{K+1} \,. \tag{A.12}$$

This completes our proof.

If we set a $N \times N$ invertible matrix, S, by

$$(S)^b{}_a = (e_a)^b \,, \qquad Me_a = \lambda_a e_a \,, \qquad 1 \leq a, b \leq N \,, \tag{A.13}$$

then

$$S^{-1} M S = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N) \,. \tag{A.14}$$

B. Gamma matrices in 4,6,10,12 dimensions

Our conventions are such that

$$
\begin{array}{lcll}
\hat{\gamma}^m & : & m = 0,1,2,3 & \text{for } \ 1+3D\,, \\
\gamma^\mu & : & \mu = 1,2,\cdots,6 & \text{for } \ 2+4D\,, \\
\gamma^a & : & a = 7,8,\cdots,12 & \text{for } \ 0+6D\,, \\
\Gamma^M & : & M = 0,1,2,3,7,\cdots,12 & \text{for } \ 1+9D\,, \\
\mathbf{\Gamma^M} & : & \mathbf{M} = 1,2,\cdots,12 & \text{for } \ 2+10D\,.
\end{array}
\tag{B.1}
$$

B.1 Four dimensions

In Minkowskian four dimension of the metric, $\hat{\eta} = \text{diag}(-+++)$, the gamma matrices satisfy

$$
\hat{\gamma}^m\hat{\gamma}^n + \hat{\gamma}^n\hat{\gamma}^m = 2\hat{\eta}^{mn}\,, \qquad (\hat{\gamma}^m)^\dagger = \hat{\gamma}_m\,, \tag{B.2}
$$

where $m,n = 0,1,2,3$. The chiral matrix reads

$$
\hat{\gamma}^{(5)} = -i\hat{\gamma}^{0123} = (\hat{\gamma}^{(5)})^{-1} = (\hat{\gamma}^{(5)})^\dagger\,. \tag{B.3}
$$

The three pairs of unitary matrices, $\hat{A}_\pm, \hat{B}_\pm, \hat{C}_\pm$, relate the hermitain conjugate, complex conjugate, and the transpose of the gamma matrices,

$$
\begin{array}{ll}
\pm(\hat{\gamma}^m)^\dagger = \hat{A}_\pm\hat{\gamma}^m\hat{A}_\pm^\dagger\,, & \hat{A}_\pm^\dagger\hat{A}_\pm = 1\,, \\
\pm(\hat{\gamma}^m)^* = \hat{B}_\pm\hat{\gamma}^m\hat{B}_\pm^\dagger\,, & \hat{B}_\pm^\dagger\hat{B}_\pm = 1\,, \\
\pm(\hat{\gamma}^m)^T = \hat{C}_\pm\hat{\gamma}^m\hat{C}_\pm^\dagger\,, & \hat{C}_\pm^\dagger\hat{C}_\pm = 1\,.
\end{array}
\tag{B.4}
$$

Especially in Minkowskian four dimensions, they can be chosen further to satisfy

$$
\begin{array}{lll}
\hat{A}_+ = -i\gamma^{123}\,, & \hat{A}_- = -\hat{\gamma}^0\,, & \hat{A}_- = \hat{A}_+\hat{\gamma}^{(5)}\,, \\
\hat{B}_\pm^*\hat{B}_\pm = \pm 1\,, & \hat{B}_\pm^T = \pm\hat{B}_\pm\,, & \hat{B}_- = \hat{B}_+\hat{\gamma}^{(5)}\,, \\
\hat{C}_\pm = \hat{B}_+^T\hat{A}_\pm = \hat{B}_\pm^T\hat{A}_+\,, & \hat{C}_\pm^T = -\hat{C}_\pm\,, & \hat{C}_- = \hat{C}_+\hat{\gamma}^{(5)}\,.
\end{array}
\tag{B.5}
$$

B.2 Four to six dimensions

Using the four dimensional gamma matrices above, one can construct the six dimensional gamma matrices in the off-block diagonal form,

$$
\gamma^\mu = \begin{pmatrix} 0 & \rho^\mu \\ \bar{\rho}^\mu & 0 \end{pmatrix}\,, \qquad \mu = 1,2,\cdots,6\,, \qquad \rho^\mu\bar{\rho}^\nu + \rho^\nu\bar{\rho}^\mu = 2\eta^{\mu\nu}\,. \tag{B.6}
$$

With the relevant choice of the metric,

$$\eta = \mathrm{diag}(--++++)\,, \tag{B.7}$$

we require $\bar{\rho}^{\mu} = (\rho_{\mu})^{\dagger}$ and set

$$\begin{aligned}
&\gamma^{1} = U\,(-i\tau_{2}\otimes 1)\,U^{\dagger}\,, && \gamma^{m+2} = U\,(\tau_{1}\otimes\hat{\gamma}^{m})\,U^{\dagger}\,,\\
&\gamma^{6} = U\left(\tau_{1}\otimes\hat{\gamma}^{(5)}\right)U^{\dagger}\,, && U = \begin{pmatrix}\hat{C}_{+} & 0\\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{B.8}$$

Explicitly with (B.3), (B.5)

$$\begin{aligned}
&\rho^{1} = -\hat{C}_{+}\,, && \rho^{m+2} = \hat{C}_{+}\hat{\gamma}^{m}\,, && \rho^{6} = \hat{C}_{-}\,,\\
&\bar{\rho}^{1} = +\hat{C}_{+}^{-1}\,, && \bar{\rho}^{m+2} = \hat{\gamma}^{m}\hat{C}_{+}^{-1}\,, && \bar{\rho}^{6} = \hat{C}_{-}^{-1}\,.
\end{aligned} \tag{B.9}$$

Note

$$\gamma^{(7)} = i\gamma^{1}\gamma^{2}\cdots\gamma^{6} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{B.10}$$

and especially the anti-symmetric property of the 4×4 matrices,

$$(\rho_{\mu})_{\alpha\beta} = -(\rho_{\mu})_{\beta\alpha}\,, \qquad (\bar{\rho}^{\mu})^{\alpha\beta} = -\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\rho^{\mu})_{\gamma\delta}\,. \tag{B.11}$$

The spinorial indices, $\alpha,\beta = 1,2,3,4$, denote the fundamental representation of su(2,2). It follows that $\{\rho^{\mu}\}$ and $\{\bar{\rho}^{\mu}\}$ separately form bases for the anti-symmetric 4×4 matrices with the completeness relation,

$$\mathrm{tr}(\rho^{\mu}\bar{\rho}_{\nu}) = 4\delta^{\mu}{}_{\nu}\,, \qquad (\rho^{\mu})_{\alpha\beta}(\bar{\rho}_{\mu})^{\gamma\delta} = 2(\delta_{\alpha}{}^{\delta}\delta_{\beta}{}^{\gamma} - \delta_{\beta}{}^{\delta}\delta_{\alpha}{}^{\gamma})\,. \tag{B.12}$$

On the other hand, Eq.(B.8) implies that[12]

$$\rho^{[\mu}\bar{\rho}^{\nu}\rho^{\lambda]} = +i\tfrac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau\kappa}\rho_{[\sigma}\bar{\rho}_{\tau}\rho_{\kappa]}\,, \qquad \bar{\rho}^{[\mu}\rho^{\nu}\bar{\rho}^{\lambda]} = -i\tfrac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau\kappa}\bar{\rho}_{[\sigma}\rho_{\tau}\bar{\rho}_{\kappa]}\,, \tag{B.13}$$

so each of the sets $\rho^{[\mu}\bar{\rho}^{\nu}\rho^{\lambda]} \equiv \rho^{\mu\nu\lambda}$ or $\bar{\rho}^{[\mu}\rho^{\nu}\bar{\rho}^{\lambda]} \equiv \bar{\rho}^{\mu\nu\lambda}$ has only 10 independent components and forms a basis for symmetric 4×4 matrices,

$$\begin{aligned}
&\mathrm{tr}(\rho^{\mu\nu\lambda}\bar{\rho}_{\sigma\tau\kappa}) = -i4\,\epsilon^{\mu\nu\lambda}{}_{\sigma\tau\kappa} - 24\delta^{[\mu}_{\sigma}\delta^{\nu}{}_{\tau}\delta^{\lambda]}_{\kappa}\,,\\
&(\rho^{\mu\nu\lambda})_{\alpha\beta}(\bar{\rho}_{\mu\nu\lambda})^{\gamma\delta} = -24(\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} + \delta_{\beta}{}^{\gamma}\delta_{\alpha}{}^{\delta})\,.
\end{aligned} \tag{B.14}$$

Finally, $\{\rho^{\mu\nu} \equiv \frac{1}{2}(\rho^{\mu}\bar{\rho}^{\nu} - \rho^{\nu}\bar{\rho}^{\mu})\}$ or $\{\bar{\rho}^{\mu\nu} \equiv \frac{1}{2}(\bar{\rho}^{\mu}\rho^{\nu} - \bar{\rho}^{\nu}\rho^{\mu})\}$ forms an orthonormal basis for the general 4×4 traceless matrices,

$$\mathrm{tr}(\rho^{\mu\nu}\rho_{\lambda\kappa}) = 4(\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\lambda} - \delta^{\nu}{}_{\kappa}\delta^{\mu}{}_{\lambda})\,, \quad -\tfrac{1}{8}(\rho^{\mu\nu})_{\alpha}{}^{\beta}(\rho_{\mu\nu})_{\gamma}{}^{\delta} + \tfrac{1}{4}\delta_{\alpha}{}^{\beta}\delta_{\gamma}{}^{\delta} = \delta_{\alpha}{}^{\delta}\delta_{\gamma}{}^{\beta}\,, \tag{B.15}$$

satisfying

$$(\bar{\rho}^{\mu\nu})^{\alpha}{}_{\beta} = -(\rho^{\mu\nu})_{\beta}{}^{\alpha}\,. \tag{B.16}$$

[12]We put $\epsilon^{123456} = 1$ and "[ ]" denotes the standard anti-symmetrization with "strength one".

B.3 Six dimensions

The result above can be straightforwardly generalized to other signatures in six dimensions. In Euclidean six dimensions, gamma matrices satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}\,, \tag{B.17}$$

where we set a, b run from 7 to 12, instead of 1 to 6, as the latter have been reserved for so$(2,4)$. With the choice,

$$\gamma^{(7)} = i\gamma^7\gamma^8\cdots\gamma^{12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{B.18}$$

the six dimensional gamma matrices are in the block diagonal form,

$$\gamma^a = \begin{pmatrix} 0 & \rho^a \\ \bar{\rho}^a & 0 \end{pmatrix}\,, \tag{B.19}$$

satisfying the hermiticity conditions,

$$\bar{\rho}^a = (\rho^a)^\dagger\,. \tag{B.20}$$

We can further set all the 4×4 matrices, ρ^a, $\bar{\rho}^a$ to be anti-symmetric [**?**]

$$(\rho^a)_{\dot{\alpha}\dot{\beta}} = -(\rho^a)_{\dot{\beta}\dot{\alpha}}\,, \qquad (\bar{\rho}^a)^{\dot{\alpha}\dot{\beta}} = -\tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(\rho^a)_{\dot{\gamma}\dot{\delta}}\,, \tag{B.21}$$

which makes the relation, su$(4) \equiv$ so(6), manifest. That is, the indices, $\dot{\alpha}, \dot{\beta} = 1, 2, 3, 4$, denote the fundamental representation of su(4).

Note that precisely the same equations as (B.12)-(B.16) hold for the so(6) gamma matrices, $\{\rho^a, \bar{\rho}^b\}$ after replacing μ, ν, α, β by a, b, $\dot{\alpha}, \dot{\beta}$, etc.

B.4 Ten dimensions again

Using the four and six dimensional gamma matrices above, we write the ten dimensional gamma matrices,

$$\begin{aligned} \Gamma^m &= \hat{\gamma}^m \otimes \gamma^{(7)} \qquad \text{for} \quad m = 0, 1, 2, 3 \\ \Gamma^a &= 1 \otimes \gamma^a \qquad\quad \text{for} \quad a = 7, 8, 9, 10, 11, 12\,. \end{aligned} \tag{B.22}$$

In the above choice of gamma matrices, we have from (6.27), (B.3), (B.18)

$$\Gamma^{(10)} = \hat{\gamma}^{(5)} \otimes \gamma^{(7)}\,, \tag{B.23}$$

and

$$\begin{aligned} &\mathcal{A} = \hat{A}_+ \otimes 1\,, && \mathcal{B}_\pm = \mathcal{C}_\pm\mathcal{A}\,, \\ &\mathcal{B}_+ = \hat{B}_- \otimes \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\,, && \mathcal{B}_- = \hat{B}_+ \otimes \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}\,, \\ &\mathcal{C}_+ = \hat{C}_- \otimes \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\,, && \mathcal{C}_- = \hat{C}_+ \otimes \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}\,. \end{aligned} \tag{B.24}$$

Majorana spinor is now of the form,

$$\Psi = \mathcal{B}_+^{-1}\Psi^* = \begin{pmatrix} \psi_{+\dot{\alpha}}^{\alpha} \\ \\ \psi_{-}^{\alpha\dot{\alpha}} \end{pmatrix}, \qquad (\psi_+^{\dagger})_{\alpha}{}^{\dot{\alpha}} = (\hat{B}_-)_{\alpha\beta}\psi_-^{\beta\dot{\alpha}}, \tag{B.25}$$

where α is the so$(1,3)$ spinor index and $\pm$ denote the so(6) chirality.

Further to have 10 dimensional Majorana-Weyl spinor, imposing the chirality condition, $\Gamma^{(10)}\Psi = \Psi$, we also have

$$\hat{\gamma}^{(5)}\psi_{\pm} = \pm\psi_{\pm}\,. \tag{B.26}$$

For the later convenience, we define $\psi_{\alpha\dot{\alpha}}$, $\bar{\psi}^{\alpha\dot{\alpha}}$ by

$$\psi_{\alpha\dot{\alpha}} = i(\hat{C}_+)_{\alpha\beta}\psi_{+\dot{\alpha}}^{\beta}\,, \qquad \bar{\psi}^{\alpha\dot{\alpha}} = \psi_-^{\alpha\dot{\alpha}}\,. \tag{B.27}$$

The Majorana condition is equivalent to

$$\bar{\psi}^{\alpha\dot{\alpha}} = A^{\alpha}{}_{\beta}(\psi^{\dagger})^{\beta\dot{\alpha}}\,, \qquad A = i\hat{A}_- = A^{\dagger} = A^{-1}\,. \tag{B.28}$$

B.5 Twelve dimensions

In order to make the SO$(2,4)\times$SO(6) isometry of $AdS_5\times S^5$ geometry manifest, it is convenient to employ the twelve dimensional gamma matrices of spacetime signature, $(--++++++++++)$, and write them in terms of two sets of six dimensional gamma matrices, $\{\gamma^{\mu}\}$, $\{\gamma^{a}\}$, which we reviewed above,

$$\begin{aligned} \mathbf{\Gamma}^{\mu} &= \gamma^{\mu}\otimes\gamma^{(7)} \quad \text{for} \quad \mu = 1,2,3,4,5,6 \\ \mathbf{\Gamma}^{a} &= 1\otimes\gamma^{a} \quad \text{for} \quad a = 7,8,9,10,11,12\,. \end{aligned} \tag{B.29}$$

In the above choice of gamma matrices, the twelve dimensional charge conjugation matrices, $\mathbf{C}_{\pm}$, are given by

$$\pm(\mathbf{\Gamma}^{\mathbf{M}})^T = \mathbf{C}_{\pm}\mathbf{\Gamma}^{\mathbf{M}}\mathbf{C}_{\pm}^{-1}\,, \quad \mathbf{M} = 1,2,\cdots,12, \qquad \mathbf{C}_{\pm} = \begin{pmatrix} 0 & 1 \\ \pm1 & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & 1 \\ \mp1 & 0 \end{pmatrix}, \tag{B.30}$$

while the complex conjugate matrices, $\mathbf{A}_{\pm}$, read

$$\mathbf{A}_{\pm} = \begin{pmatrix} A^t & 0 \\ 0 & \mp A \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & \pm1 \end{pmatrix}, \qquad A = -i\bar{\rho}_{12} = -i\hat{\gamma}^0 = i\hat{A}_- = A^{\dagger} = A^{-1}\,, \tag{B.31}$$

satisfying

$$\pm(\mathbf{\Gamma}^{\mathbf{M}})^{\dagger} = \mathbf{A}_{\pm}\mathbf{\Gamma}^{\mathbf{M}}\mathbf{A}_{\pm}^{-1}\,. \tag{B.32}$$

In particular, for $\mu = 1,2,\cdots,6$, we have

$$(\rho^{\mu})^{\dagger} = -A\bar{\rho}^{\mu}A^t = \bar{\rho}_{\mu}\,, \qquad (\bar{\rho}^{\mu})^{\dagger} = -A^t\rho^{\mu}A = \rho_{\mu}\,. \tag{B.33}$$

Now if we define the twelve dimensional chirality operator as

$$\mathbf{\Gamma^{(13)}} \equiv \gamma^{(7)} \otimes \gamma^{(7)} \,, \tag{B.34}$$

then

$$\{\mathbf{\Gamma}^{(13)}, \mathbf{\Gamma^{M}}\} = 0 \,, \qquad \mathbf{C}_- = \mathbf{C}_+ \mathbf{\Gamma^{(13)}} \,, \qquad \mathbf{A}_- = \mathbf{A}_+ \mathbf{\Gamma^{(13)}} \,. \tag{B.35}$$

In 2+10 dimensions it is possible to impose the Majorana-Weyl condition on spinors to have sixteen independent complex components which coincides with the number of supercharges in the $AdS_5 \times S^5$ superalgebra, su$(2,2|4)$. Up to the redefinition of the spinor by a phase factor, there are essentially two choices for the Majorana-Weyl condition depending on the chirality,

$$\mathbf{\Psi} = \pm \mathbf{\Gamma}^{(13)} \mathbf{\Psi} \,, \quad \text{and} \quad \bar{\mathbf{\Psi}} = \mathbf{\Psi}^\dagger \mathbf{A}_+ = \mathbf{\Psi^T} \mathbf{C}_+ \,. \tag{B.36}$$

Our choice will be the plus sign so that the $2+10$ dimensional Weyl spinor carries the same chiral indices for su$(2,2)$ and su(4), i.e. $\mathbf{\Psi} = (\psi_{\alpha\dot{\alpha}} \,,\ \bar{\psi}^{\alpha\dot{\alpha}})^T$, while the Majorana condition relates them as $\bar{\psi}^{\alpha\dot{\alpha}} = A^{\alpha}{}_{\beta} (\psi^\dagger)^{\beta\dot{\alpha}}$ which is identical to (B.28). Hence, the Majorana-Weyl spinor in $2+10$ dimenisons can be identified as the Majorana spinor in $1+9$ dimensions.

C. Looking for the general odd symmetry

With a Majorana-Weyl spinor, $\mathcal{E}$, Δ_Ψ, which may depend on x^M, we focus on the following transformations,

$$\delta A_M = i\bar{\Psi}\Gamma_M\mathcal{E} = -i\bar{\mathcal{E}}\Gamma_M\Psi\,, \qquad \delta\Psi = \tfrac{1}{2}F_{MN}\Gamma^{MN}\mathcal{E} + \Delta_\Psi\,, \tag{C.1}$$

so that

$$\delta\bar{\Psi} = -\tfrac{1}{2}\bar{\mathcal{E}}F_{MN}\Gamma^{MN} + \overline{\Delta_\Psi}\,. \tag{C.2}$$

Note that Δ_Ψ is Lie algebra valued, while $\mathcal{E}$ is not.

From

$$\Psi^{p\alpha}\Psi^{q\beta}\Psi^{r\gamma}\mathrm{tr}(T_pT_qT_r) = \Psi^{p\gamma}\Psi^{q\alpha}\Psi^{r\beta}\mathrm{tr}(T_pT_qT_r) = \Psi^{p\beta}\Psi^{q\gamma}\Psi^{r\alpha}\mathrm{tr}(T_pT_qT_r)\,, \tag{C.3}$$

and the identity (6.28), we note that the second term in (6.42) vanishes

$$\mathrm{tr}\left(\bar{\Psi}\Gamma^M\Psi\,\bar{\Psi}\Gamma_M\mathcal{E}\right) = 0\,. \tag{C.4}$$

We also get, using the Bianchi identity (6.39),

$$\begin{aligned}\bar{\Psi}\Gamma^M D_M\delta\Psi &= \tfrac{1}{2}D_LF_{MN}\bar{\Psi}(\Gamma^{LMN} + 2\eta^{LM}\Gamma^N)\mathcal{E} + \tfrac{1}{2}\bar{\Psi}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E}F_{MN} + \bar{\Psi}\Gamma^LD_L\Delta_\Psi\\ &= -iD_MF^{MN}\delta A_N + \tfrac{1}{2}\bar{\Psi}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E}F_{MN} + \bar{\Psi}\Gamma^LD_L\Delta_\Psi\,.\end{aligned} \tag{C.5}$$

Thus, semi-finally, we obtain

$$\delta\mathcal{L} = -i\,\mathrm{tr}\left[\tfrac{1}{2}F_{MN}\bar{\Psi}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E} + \bar{\Psi}\Gamma^LD_L\Delta_\Psi\right] + \partial_N\mathrm{tr}\left[F^{MN}\delta A_M + i\tfrac{1}{2}\bar{\Psi}\Gamma^N\delta\Psi\right]\,. \tag{C.6}$$

We first note that constant $\mathcal{E}$, and constant Δ_Ψ which is central in the Lie algebra lead to the ordinary and kinetic supersymmetries

$$\mathcal{E},\ \Delta_\Psi\ \ :\ \text{constant} \qquad \text{and} \qquad \Delta_\Psi \propto 1_{N\times N}\,. \tag{C.7}$$

Henceforth, keeping *the dimensional reduction either to Minkowskian d-dimensions,* $0 \le m \le d-1$, $d \le a \le 9$, *or Euclidean d-dimensions,* $1 \le m \le d$, $a = 0$, $d+1 \le a \le 9$, we set $A_a = \Phi_a$, "$\partial_a \equiv 0$", and look for some possibilities of more general symmetries.

Since

$$F_{MN}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E} = \left(F_{mn}\Gamma^l\Gamma^{mn} + 2D_m\Phi_b\Gamma^l\Gamma^{mb} + D_a\Phi_b\Gamma^l\Gamma^{ab}\right)\partial_l\mathcal{E}\,, \tag{C.8}$$

we first require

$$\Gamma^l\Gamma^{mn}\partial_l\mathcal{E} = 0\,, \tag{C.9}$$

or equivalently

$$\Gamma^{mn}\Gamma^{l}\partial_{l}\mathcal{E} = 2\Gamma^{m}\partial^{n}\mathcal{E} - 2\Gamma^{n}\partial^{m}\mathcal{E}\,. \tag{C.10}$$

It follows after multiplying Γ_{nm} without m, n summing,

$$\Gamma^{l}\partial_{l}\mathcal{E} = 2\Gamma^{m}\partial_{m}\mathcal{E} + 2\Gamma^{n}\partial_{n}\mathcal{E} \quad : \quad \text{no sum for } m \neq n\,. \tag{C.11}$$

Eqs.(C.9), (C.10), (C.11) are trivial when $d = 0, 1$. For $d \geq 2$, summing over $m \neq n$ in (C.11) we get

$$(d-1)(d-4)\Gamma^{l}\partial_{l}\mathcal{E} = 0\,. \tag{C.12}$$

Hence, for $d = 2, 3$, $d \geq 5$,

$$\Gamma^{m}\partial_{m}\mathcal{E} = -\Gamma^{n}\partial_{n}\mathcal{E} \quad : \quad \text{no sum and } m \neq n\,. \tag{C.13}$$

- For $d = 3$, $d \geq 5$, we easily conclude $\partial_{m}\mathcal{E} = 0$, i.e. constant parameter, $\mathcal{E}$.
- When $d = 2$, we get

  $$\partial_{m}\mathcal{E} = -\Gamma_{mn}\partial^{n}\mathcal{E} \quad : \quad \text{for } d = 2\,, \tag{C.14}$$

  so that

  $$\partial^{m}\partial_{m}\mathcal{E} = 0\,. \tag{C.15}$$

  Let $\sigma \neq \tau$ be the two different spacetime indices in $d = 2$ case. Eq.(C.9) is simply equivalent to

  $$(\partial_{\sigma} + \Gamma_{\sigma}{}^{\tau}\partial_{\tau})\mathcal{E} = 0\,. \tag{C.16}$$

  This can be solved easily in the diagonal basis of $\Gamma_{\sigma}{}^{\tau}$. In the Minkowskian two-dimensions, as $\Gamma_{0}{}^{1}$ is hermitian, the solution is given by the left and right modes, $\sigma \pm \tau$. On the other hand, in the Euclidean two-dimensions, $\Gamma_{1}{}^{2}$ is anti-hermitian and the solution involves holomorphic functions, $\sigma \pm i\tau$.
- For $d = 4$ we have for any m,

  $$\partial_{m}\mathcal{E} = \Gamma_{m}\xi_{-}\,, \qquad \xi_{-} = \tfrac{1}{4}\Gamma^{l}\partial_{l}\mathcal{E}\,. \tag{C.17}$$

  From $\partial_{[m}\partial_{n]}\mathcal{E} = 0$ we get an essenitally same relation as (C.13),

  $$\Gamma^{m}\partial_{m}\xi_{-} = -\Gamma^{n}\partial_{n}\xi_{-} \quad : \quad \text{no sum and } m \neq n\,. \tag{C.18}$$

  Hence, ξ_{-} is a constant spinor, and

  $$\mathcal{E} = x^{m}\Gamma_{m}\xi_{-} + \xi_{+}\,, \tag{C.19}$$

  where ξ_{+}, ξ_{-} are constant Majorana-Weyl spinors of the opposite chiralities, corresponding to the ordinary supersymmetry and special superconformal symmetry, respectively.

Provided the above solutions for (C.9), we are ready for the full analysis.

1. When $d=0$: IKKT matrix model.
   Eq.(C.8) becomes trivial, and we natually require

$$\Gamma^a[\Phi_a, \Delta_\Psi] = 0\,. \tag{C.20}$$

   We need to find the algebraic solution for Δ_Ψ in terms of the Lie algebra valued fields, Φ_a, $d \leq a \leq 9$. Clearly, the kinetic supersymmetry transformation, i.e. $\Delta_\Psi \propto 1_{N\times N}$, satisfies the above equation. In fact, we can show that this is the most general solution.

   Proof
   We consider the special case, $\Phi_a = 0$, $d \leq a \leq 7$. Eq.(C.20) gives

$$[\Phi_8, \Gamma^8\Delta_\Psi] + [\Phi_9, \Gamma^9\Delta_\Psi] = 0\,. \tag{C.21}$$

   Multiplying Φ_8 and taking the u(N) trace we get

$$\mathrm{tr}\left([\Phi_8, \Phi_9]\Delta_\Psi\right) = 0\,. \tag{C.22}$$

   Since the commutator, $[\Phi_8, \Phi_9]$, can be arbitrary except $1_{N\times N}$, we conclude that $\Delta_\Psi \propto 1_{N\times N}$. This completes our proof.

   Therefore, when $d=0$, $\mathcal{E}$ and Δ_Ψ are simply constant Majorana-Weyl spinors corresponding to the ordinary and the kinetic supersymmetries.

2. When $d=1$: BFSS matrix model.
   Eq.(C.9) is trivial, and with the coordinate, τ for $d=1$, From Eq.(C.6) we require

$$\begin{aligned}0 &= \tfrac{1}{2}F_{MN}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E} + \Gamma^L D_L\Delta_\Psi \\ &= \Gamma^\tau D_\tau(\Delta_\Psi + \Phi_a\Gamma^{\tau a}\partial_\tau\mathcal{E}) + \Gamma^b D_b(\Delta_\Psi - \tfrac{1}{2}\Phi_a\Gamma^{\tau a}\partial_\tau\mathcal{E}) - \Phi_a\Gamma^a\partial^\tau\partial_\tau\mathcal{E}\,.\end{aligned} \tag{C.23}$$

   The only possible algebraic solutions are (C.7) corresponding to the ordinary and the kinetic supersymmetries.

3. When $d=2$.
   From (C.6) we require, using (C.14), (C.15),

$$\begin{aligned}0 &= \tfrac{1}{2}F_{MN}\Gamma^L\Gamma^{MN}\partial_L\mathcal{E} + \Gamma^L D_L\Delta_\Psi \\ &= \Gamma^m D_m\Delta_\Psi + \Gamma^a D_a\Delta_\Psi + 2(D^\tau\Phi_a - D^\sigma\Phi_a\Gamma_\sigma{}^\tau)\Gamma^a\partial_\tau\mathcal{E}\,.\end{aligned} \tag{C.24}$$

   We conclude again that the only possible algebraic solutions are (C.7) corresponding to the ordinary and the kinetic supersymmetries.

4. When $d=3$, $d\geq 5$.
   Since $\mathcal{E}$ is constant, the only possible algebraic solutions are (C.7) corresponding to the ordinary and the kinetic supersymmetries.

5. When $d = 4$.
   From (C.6) we require, using (C.19),

$$\begin{aligned} 0 &= \tfrac{1}{2} F_{MN} \Gamma^L \Gamma^{MN} \partial_L \mathcal{E} + \Gamma^L D_L \Delta_\Psi \\ &= \Gamma^L D_L (\Delta_\Psi + 2\Phi_a \Gamma^a \xi_-) \,. \end{aligned} \tag{C.25}$$

   Thus the algebraic solution reads

$$\Delta_\Psi + 2\Phi_a \Gamma^a \xi_- \propto 1_{N\times N} \,. \tag{C.26}$$

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