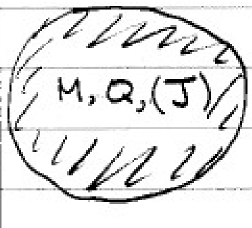
B.H. thermodynamics from

$\boxed{\text{Classical GR}}$ + Q.F.T.

Black hole




$\Rightarrow$

1. Entropy : $S = \dfrac{\text{Area}}{4G_N} \cdot \dfrac{c^3}{\hbar}$

horizon area

$\hbar$ ← quantum effect!

↖ Newton's const.

2. Hawking radiation
   - Thermal w/ $T(M, Q)$
   - $T$ decreas until $M = Q$ is reached.

$\boxed{\text{String Theory}}$

- is a candidate for Quantum Gravity.

  (GR is a low energy effective description of string theory)

- must be able to "explain" B.H. thermodynamics

- In string theory, all particles including gravitons are made of strings.

$\Rightarrow$ B.H. also should be made of strings!

* How to count entropy?

Give a macroscopic energy to the fundamental string

- For **free** strings, the degeneracy can be counted

$\Rightarrow$ $\boxed{S \sim M}$

- On the other hand, GR gives $\boxed{\text{Area} \sim M^{\frac{D-2}{D-3}}}$

  ($= M^2$ for $D=4$)

- The disrepancy presumably comes from large renormalization effect.

$\boxed{\text{Supersymmetry}}$ comes to rescue.

- In SUSY theories, renormalization effects are often much softer
- Degeneracy of BPS states are protected by SUSY

$\boxed{\text{BPS}}$ minimum energy for given charge

$\longleftrightarrow$ extremal black hole.

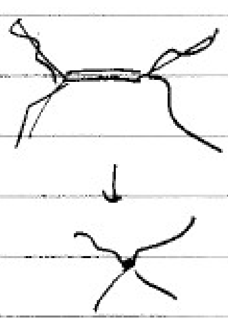
(More later)

⊙ BPS string states ($S \neq 0$)

fail to produce a macroscopic horizon

⇒ Disrepancy attributed to higher derivative corrections

$$I = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left[ R + \alpha' R^2 + \cdots \right]$$



$$\frac{p \cdot p}{p^2 + 1/\alpha'}$$

$\downarrow$ $(p^2 \ll 1/\alpha')$

$\alpha' \partial^2$

$[\alpha'] = M^{-2}$

$\frac{1}{2\pi\alpha'}$ = string tension

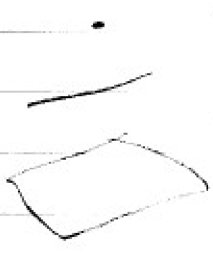
= string mass density

$(c = 1)$

⊙ String Theory contains more than just strings !

- Solitonic objects, called p-branes, have been "discovered".

  0-brane =
  1-brane =
  2-brane =
  ⋮



- A special class of branes, namely $\boxed{\text{D-branes}}$ will play a crucial role.

<u>Goal</u> of my lectures.

- Study the simplest example of extremal b.h. made of D-branes in string theory.
- Compute Area from GR
  - S by counting microstates living on the D-branes

and show that $\boxed{S = \dfrac{A}{4G_N}}$ !

<u>Keywords</u>

Supersymmetry, BPS states, soliton, D-brane
(EM duality, Dirac quantization condition)

# Supersymmetry

**Q.M.** States $\quad$ Hilbert space

$$|\Psi\rangle \in \mathcal{H}$$

**Q.F.T.** treats all multi-ptl states together

Ex) Free QED: $\underset{\sim\sim}{A_\mu(x)}$, $\underset{\longrightarrow}{\psi(x)}$

$$\left.\begin{array}{ll} 0: & |vac\rangle \\ 1: & a^+_{\vec{p},\epsilon}|vac\rangle \\ & b^+_{\vec{p},s}|vac\rangle \\ & \vdots \\ N: & a^+\cdots a^+ b^+ \cdots b^+|vac\rangle \end{array}\right\} \in \mathcal{H} \quad \text{Fock space.}$$

Spin - Statistics: $\left[\begin{array}{l}\text{Boson - Integer spin} \\ \text{Fermion - Half-integer spin}\end{array}\right.$

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$$

$$e^{2\pi i J_3}|B\rangle = +|B\rangle \qquad e^{2\pi i J_3}|F\rangle = -|F\rangle$$

$\mathcal{H}$ of an interacting theory may look completely different from that of free theory, but the splitting $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ remains valid.

( ! Caution: $\left[\begin{array}{l}\text{boson}\\ \text{fermion}\end{array}\right.$ vs. $\left[\begin{array}{l}\text{bosonic state}\\ \text{fermionic}\end{array}\right.$ )

## Symmetry

- QM / QFT incorporates symmetry in the language of group / algebra

- Symm. generators act on the Hilbert space as operators ( $P_\mu$, $M_{\mu\nu}$, etc. )

$$T : \mathcal{H} \to \mathcal{H}$$

$$T|\psi\rangle = |\psi'\rangle$$

- Generators form an algebra

$$[P_\lambda, P_\nu] = 0, \quad [M_{\mu\nu}, P_\lambda] = -i(\delta_{\nu\lambda} P_\mu - \delta_{\mu\lambda} P_\nu)$$

, etc.

- Some states are invariant under a subalgebra

Ex)

Localized, spherically symmetric state

$$P_i|\psi\rangle \neq 0$$

$$J_i|\psi\rangle = 0$$

- "ordinary" symms do not mix bosonic states w/ fermionic states

$$T : \mathcal{H}_B \to \mathcal{H}_B$$
$$\mathcal{H}_F \to \mathcal{H}_F$$

( Both $\left[\begin{array}{l}\text{space-time symm} \\ \text{Internal symm}\end{array}\right.$ )

## SUSY

◎ Supersymmetry

$$Q : \mathcal{H}_B \to \mathcal{H}_F$$
$$\mathcal{H}_F \to \mathcal{H}_B$$

- $Q$ changes <u>spin</u> and <u>statistics</u>

$Q$ must be a spinor $\left( D=4, \text{ minimal \# of components} = 4_R \right)$

and $\{ e^{2\pi i J_3}, Q \} = 0.$

★ Impossible to use spin $\geq 3/2$ generators

( Coleman - Mandula
Haag - Lopuszanski - Sohnius )

SUSY algebra

- $[P_\mu, Q_\alpha] = 0 \Rightarrow |\psi\rangle$ and $Q|\psi\rangle$ has the same energy!

- $[M_{\mu\nu}, Q_\alpha] = -\frac{i}{2}(\Gamma^{\mu\nu})_\alpha{}^\beta Q_\alpha$

- $\{Q_\alpha, Q_\beta\} = (\Gamma^\mu C)_{\alpha\beta} P_\mu$

$$E \simeq QQ^\dagger + Q^\dagger Q$$

$E \geq 0$ (unitarity), $E = 0$ iff $Q|\text{vac}\rangle = 0$

(some comments on spontaneous symm breaking)

Use of symm. — I. multiplet

- For any symm. <u>algebra</u>, physical states ( $\in \mathcal{H}$ ) form <u>representations</u>

  Ex) $[J_i, J_j] = i\epsilon_{ijk} J_k \Rightarrow \begin{cases} \vec{J}^2 |j,m\rangle = j(j+1)|j,m\rangle \\ J_3 |j,m\rangle = m|j,m\rangle \\ J_\pm |j,m\rangle \sim |j, m\pm 1\rangle \end{cases}$

- In SUSY theories, supercharges group bosonic states and fermionic states together to form a <u>multiplet</u>

  Ex) Free QED. $Q : a^\dagger_{\vec{p},\epsilon} |vac\rangle \longleftrightarrow b^\dagger_{\vec{p},s} |vac\rangle$

  Of course, even in a strongly interacting theories, states fall into multiplets.

Use of symm. — II. Interactions

- Symm. gives severe restrictions on interaction terms

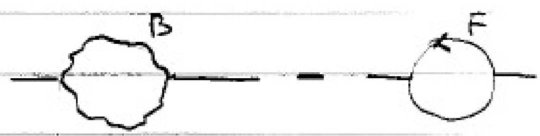
  Ex) $\mathcal{L} \sim A_\mu J^\mu$ $\qquad$ $\cancel{A_0 J^0 + 2A_x J^x - A_y J^y}$

- SUSY.

  $m^2\phi^2 \longleftrightarrow m\bar{\psi}\psi$ (equal mass)

  $\phi^4 \longleftrightarrow \phi\bar{\psi}\psi$

- Less renormalization




$= 0$

# SUSY Lagrangian

- Review : bosonic Lagrangian

$$S = \int d^4x \left[ \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \right]$$

Symm : $\phi \to e^{i\alpha} \phi \quad ( \delta\phi = i\epsilon\phi )$

Noether theorem : $J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$, $\partial_\mu J^\mu = 0$.

- SUSY

Majorana spinor

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar\Psi \Gamma^\mu \partial_\mu \Psi \right]$$

Symm: $\delta A_\mu = \bar\Psi \Gamma_\mu \epsilon \qquad ( = [\epsilon Q, A_\mu] )$

$\delta \Psi = \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon \qquad ( = [\epsilon Q, \Psi_\alpha] )$

Supercurrent

$J^\mu =$

- Classical solution preserving some part of a symmetry

. spherical $\quad \phi = \phi ( r = \sqrt{x^2+y^2+z^2} )$  $L_x, L_y, L_z$

. Axial $\quad \phi = \phi ( \rho = \sqrt{x^2+y^2}, z )$  $L_z \quad P_z$

- Supersymmetry ?

$\delta B = (F) \epsilon$

$\delta F = (B, \partial) \epsilon$

Classically, $F = 0 \Rightarrow \delta B = 0$ automatically.

Partially susy preserving solution $\Rightarrow \delta F = 0$ for some choice of $\epsilon$.

EM duality in $D=4$

- Maxwell eqs.

$$\nabla\cdot\vec{E} = \rho_e \qquad \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}_e$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0.$$

- When $\rho_e = 0 = \vec{J}_e$, these eqns are invariant under

$$D : \vec{E} \to \vec{B}, \quad \vec{B} \to -\vec{E}.$$

$D^2$ = charge conjugation.

- In relativistic form

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \quad \partial_\mu(*F^{\mu\nu}) = 0$$

$$*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

$$D : F_{\mu\nu} \to *F_{\mu\nu} \qquad (D^2 = -1 = **)$$

- Duality invariance of action.

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

Naive application of $D$ ($\vec{E}\to\vec{B}, \vec{B}\to -\vec{E}$) leads to

$$L = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \to \tilde{L} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) \quad ?!$$

$$L(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$L(F, \tilde{A}) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \partial^\mu \tilde{A}^\nu F^{\lambda\sigma}$$

$$\frac{\delta L}{\delta F} = 0 \Rightarrow -\frac{1}{2} F_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \partial^\lambda \tilde{A}^\sigma = 0.$$

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} (\partial^\lambda \tilde{A}^\sigma - \partial^\sigma \tilde{A}^\lambda)$$

$$= *\tilde{F}_{\mu\nu} \qquad \tilde{F}_{\mu\nu} = - * F_{\mu\nu}.$$

$$\Rightarrow L(\tilde{A}) = +\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$= -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}.$$

Dirac quantization in D=4

- EM duality of free Maxwell eq. broken by $j_e^\nu$.
  Could EM duality be restored somehow?
  - Add a magnetic source term! $\boxed{\partial_\mu * F^{\mu\nu} = j_m^\nu}$

- Field strength vs. vector potential.
  $\partial_\mu * F^{\mu\nu} \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\quad (A^\mu = (\phi, \vec{A}))$

  Ambiguity $A_\mu \to A_\mu - \partial_\mu \chi$ : gauge transformation.

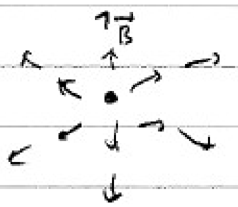
- Quantum mechanics
  $\vec{p} = -i\nabla \longrightarrow -i(\vec{\nabla} - ie\vec{A})$

  Schrodinger eq.

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}(\nabla - ie\vec{A})\psi + V\psi .$$

- Gauge transformation
  $\psi \to e^{-ie\chi}\psi$
  $\vec{A} \to \vec{A} - \nabla\chi \left( = \vec{A} - \frac{i}{e} e^{ie\chi} \vec{\nabla} e^{-ie\chi} \right)$

## Dirac quantization in D = 4

- 

$$\vec{B} = \frac{g}{4\pi} \cdot \frac{\hat{r}}{r^2}$$

- Quantum mechanics requires vector potential.
  - single vector potential leads to singularity
  - use several vector potentials, related to each other by gauge transf.

- A solution.

$$\vec{A}_N = \frac{g}{4\pi r} \frac{1-\cos\theta}{\sin\theta} \hat{e}_\phi$$

$$\vec{A}_S = -\frac{g}{4\pi r} \frac{1+\cos\theta}{\sin\theta} \hat{e}_\phi .$$

- Overlap region

$$\vec{A}_N - \vec{A}_S = -\nabla\chi \ , \quad \chi = -\frac{g}{2\pi}\phi .$$

$e^{-ie\chi}$ singlevalued along the equator

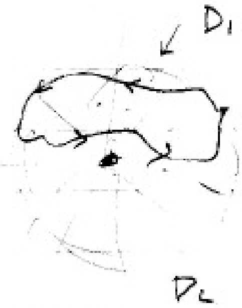
$$\Rightarrow \quad eg = 2\pi n \ , \quad n \in \mathbb{Z} .$$

# Dirac quantization in D=4

An alternative derivation

Path integral

$$K(x_i, x_f; t) = \int_{x_i}^{x_f} \mathcal{D}x \; e^{ie\int_0^t \vec{A}\cdot\frac{d\vec{x}}{dt}}$$




- $K(x_i = x_f) \sim e^{ie\oint \vec{A}}$

$$= e^{ie\int_{D_1}\vec{F}\cdot d\vec{s}} = e^{-ie\int_{D_2}\vec{F}\cdot d\vec{s}}$$

K is well-defined iff $\quad e\int_{S^2}\vec{F}\cdot d\vec{s} = eg = 2\pi n$.

't Hooft - Polyakov monopole

- $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D^\mu \phi^a D_\mu \phi^a - V(|\phi|)$

$$\begin{cases} F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc} A^b_\mu A^c_\nu \\ D_\mu \phi^a = \partial_\mu \Phi^a + e\epsilon^{abc} A^b_\mu \phi^c \end{cases} \qquad (a = 1, 2, 3)$$

$V(|\phi|)$ chosen so that $\langle \phi^a \phi^a \rangle = v^2$

This breaks $SU(2) \to U(1)$

- E.o.m.

$$\begin{cases} D_\mu F^{a\mu\nu} = e\epsilon^{abc} \phi^b D^\nu \phi^c \\ D^\mu D_\mu \phi^a = -\dfrac{\partial V}{\partial \phi^a} \end{cases}$$

- $H = \int d^3x \left[ \frac{1}{2} (\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a + \Pi^a \Pi^a + \vec{D}\phi^a \, \vec{D}\phi^a) + V \right]$

$\Pi^a = D_0 \Phi^a$, $E^{ai} = -F^{a0i}$, $B^{ai} = -\frac{1}{2} \epsilon^{ijk} F^a_{jk}$

SU(2) monopole

- Look for a static, spherically symmetric magnetic monopole sol'n.

$$H = \int d^3x \left[ \frac{1}{2}(\vec{B}^a \cdot \vec{B}^a + \vec{D}\phi^a \cdot \vec{D}\phi^a) + V \right]$$

- Monopole charge is classified by winding #. (Compare this w/ Dirac monopole)

$$V(\phi) = 0 \Rightarrow \phi^a\phi^a = v^2 \quad : \quad S^2\ (V=0)$$

Localized solution (finite engy) provides a map $\phi : S^2(r=\infty) \to S^2(V=0)$

Such a map is classified by an integer winding number. (w) → (This coincides w/ monopole # !)

- $w \neq 0 \Rightarrow$ monopole

· Suppose $w \neq 0$. If $A_i^a = 0$,

$$H = \int d^3x \frac{1}{2} \vec{\nabla}\phi^a \vec{\nabla}\phi^a + V \geq \int d^3x \frac{1}{2} \vec{\nabla}\phi^a \vec{\nabla}\phi^a$$

$$(\vec{\nabla}\phi^a)^2 = \left(\frac{\partial\phi^a}{\partial r}\right)^2 + \frac{1}{r^2}\left(\left(\frac{\partial\phi^a}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial\phi^a}{\partial\varphi}\right)^2\right)$$

$$\sim \frac{1}{r^2} \quad \text{as } r \to \infty.$$

$$H > \int \frac{r^2 dr}{r^2} \to \infty$$

SU(2) monopole.

⊙ Keep the energy finite by turning on $A_\mu^a$

$$D_\mu \phi^a = \partial_\mu \phi^a + e\epsilon^{abc} A_\mu^b \phi^c \sim 0 \quad \left(\tfrac{1}{r} \text{ term vanishes}\right)$$

$$\Rightarrow A_\mu^a \sim -\frac{1}{ev^2}\epsilon^{abc}\phi^b \partial_\mu \phi^c + \frac{1}{v}\phi^a A_\mu$$

$$\Rightarrow F^{a\mu\nu} = \frac{1}{v}\phi^a F^{\mu\nu}$$

$$F^{\mu\nu} = -\frac{1}{ev^3}\epsilon^{abc}\phi^a \partial^\mu \phi^b \partial^\nu \phi^c + \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$g = \int_{S^2(r=\infty)} \underbrace{\vec{B}}_{\left(\frac{\phi^a}{v}\vec{B}^a\right)} \cdot d\vec{S} = \frac{4\pi w}{e}$$

$$\therefore eg = 4\pi w = 2\pi(2w)$$

winding no. $w$ $\Rightarrow$ magnetic charge $2w$ in "Dirac unit"

Symm. sol'n and Bogomol'nyi bound.

◎ Exact solution ?

1) Set $V = 0$.

( $\phi^a\phi^a = v^2$ still makes sence )

2). Spherically symm. ansatz

$$\phi^a = v \cdot \frac{\hat{r}^a}{y} H(y) \qquad (y = ver)$$

$$A_i^a = -\epsilon^a{}_{ij} v \cdot \frac{\hat{r}^j}{y} (1 - K(y))$$

Boundary conditions

$$\boxed{\begin{array}{l} r \to 0: \quad K \to 1, \quad H \to 0 \\ r \to \infty: \quad K \to 0 \quad H/y \to 1. \end{array}}$$

⊙ Bogomol'nyi bound

; ~~Minimal energy~~ for a given $W$.

$$g = \int_{S^2_\infty} \vec{B} \cdot d\vec{S} = \frac{1}{v} \int \phi^a \vec{B}^a \cdot d\vec{S} = \frac{1}{v} \int \vec{B}^a \cdot (\vec{D}\phi^a) d^3 r$$

$$M_M = \int d^3 r \, \frac{1}{2} (\vec{B}^a \cdot \vec{B}^a + \vec{D}\phi^a \cdot \vec{D}\phi^a)$$

$$\geq \frac{1}{2} \int d^3 r \, (\vec{B}^a \pm \vec{D}\phi^a)^2 \mp \underbrace{\int \vec{B}^a \cdot \vec{D}\phi^a}_{vg}$$

$M_M \geq |vg|$ saturated iff

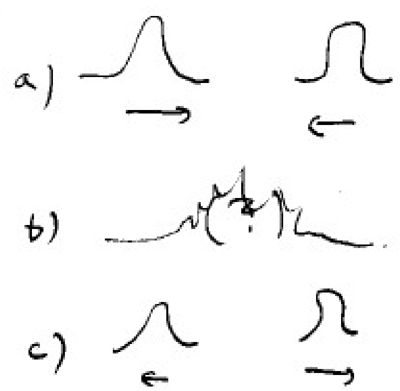
$$\vec{B}^a = \pm \vec{D}\phi^a.$$

⊙ Prasad - Sommerfield sol'n.

$$H(y) = y \coth y - 1, \quad K(y) = \frac{y}{\sinh y}$$

Magnetic monopole as quantum state.

- 't Hooft - Polyakov monopole is an example of "topological soliton"
- Soliton behaves much like an ordinary particle



For example, scattering of two solitons can be studied in classical field theory. The result is similar to

 of elementary quanta.

It turns out that at least in supersymmetric theories, it is possible to associate quantum states to solitons. In some cases, a dual description exists in which the solitons are replaced by elementary quanta and vice versa.

## BPS states

- Preserve some SUSY : $Q_\alpha|\psi\rangle = 0$ for some $Q_\alpha$.
- Minimal energy for a given charge
- Superalgebra modified to

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = \delta^I{}_J (\sigma^\mu)_{\alpha\dot\beta} P_\mu \qquad \{Q, Q\} = \begin{bmatrix} Z & P \\ P & Z \end{bmatrix}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}\epsilon^{IJ} Z$$

$$Q|\psi\rangle = 0 \Rightarrow M = |Z|$$

- Mass is ~~not~~ renormalized.
  (More precisely, exactly computable using ⟲ )

- Degeneracy protected
  (cannot mix w/ long multiplets)

Ex) $D=4$, $\mathcal{N}=2$ SUSY.

- Long multiplet : $8_B + 8_F$ states
- Short multiplet : $4_B + 4_F$ states

# BPS monopole — I.

- Let $|\psi\rangle$ be a BPS state. By definition,

  $Q_\alpha|\psi\rangle = 0$ for some $Q_\alpha$. Equivalently,

  $\epsilon^\alpha Q_\alpha|\psi\rangle \equiv \epsilon Q|\psi\rangle = 0$ for some $\epsilon$. ($\epsilon$: Grassmann-valued spinor)

- Then for any operator $\mathcal{O}$, $\langle\psi|[\epsilon Q, \mathcal{O}]|\psi\rangle = 0$

$$\begin{cases} \mathcal{O} \text{ bosonic} : \text{trivial} \\ \mathcal{O} \text{ fermionic} : \text{non-trivial} \end{cases}$$

- $[\epsilon Q, \mathcal{O}] = \delta_\epsilon \mathcal{O}$ (variation of $\mathcal{O}$ under susy transf. generated by $\epsilon Q$)

  - In the classical limit, $\delta_\epsilon\langle\mathcal{O}\rangle = \langle\delta_\epsilon\mathcal{O}\rangle$.
  - So, at tree level, a BPS state is a state satisfying $\delta_\epsilon\mathcal{O} = 0$ for all fermionic operators.
  - Also, in the classical limit, it is enough to check this for the elementary fermionic fields.

$D=4,\ \mathcal{N}=2$ SYM

- $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \cdots$

- SUSY transformation

$$\delta A_\mu =$$

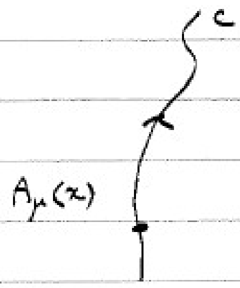
$$\delta \phi =$$

$$\boxed{\delta \lambda = \quad}$$

- ~~extremal~~ BPS solution $\Rightarrow$ SUSY !

EM duality in higher dim's.

· ptl (0-brane)




$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (F = dA)$$

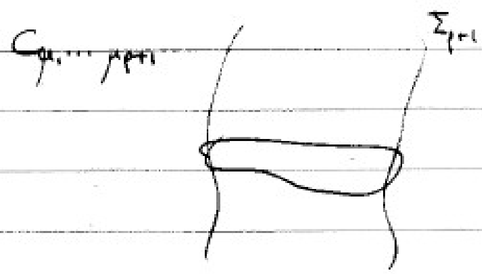
$$S \sim \int_{X^D} \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu} + e_0 \int_C A_\mu(x) \frac{dx^\mu(\tau)}{d\tau} d\tau$$

· String (1-brane)


$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \quad (H = dB)$$

$$S \sim \int_{X^D} \sqrt{-g}\, H_{\mu\nu\lambda} H^{\mu\nu\lambda} + e_1 \int_\Sigma B_{\mu\nu}\, dx^\mu dx^\nu$$

· p-brane




$$F_{\mu_1 \cdots \mu_{p+2}} = \partial_{\mu_1} C_{\mu_2 \cdots \mu_{p+2}} \quad (F_{p+2} = d\, C_{p+1})$$

$$S \sim \int_{X^D} \sqrt{-g}\, F^2 + e_p \int_{\Sigma_{p+1}} C_{\mu_1 \cdots \mu_{p+1}}\, dx^{\mu_1} \cdots dx^{\mu_{p+1}}$$

⊙ Coulomb potential

- Electrostatics ( $D=4$ , $p=0$ )

$$\nabla^2 A_0 = -e\,\delta^{(3)}(\vec{x})$$

$$\Rightarrow A_0 = \frac{e}{4\pi}\cdot\frac{1}{r}$$

- D-dim , p-brane

$$\nabla^2 C_{01\cdots p} = -e_p\,\delta^{(D-1-p)}(\vec{x})$$

$$\Rightarrow C_{01\cdots p} = \frac{e_p}{\omega_{D-p-2}}\cdot\frac{1}{r^{D-p-3}}$$

$$\left(\omega_n = \text{Area of } S^n = \frac{2\pi^{n+1/2}}{\Gamma(\frac{n+1}{2})}\right)$$

$$\int_{S^{D-p-2}} *F_{(p+2)} = e_p$$

⊙ Electromagnetic duality

$$p\text{-brane} \rightarrow C_{(p+1)} \rightarrow F_{(p+2)} = dC_{(p+1)}$$

$$\downarrow$$

$$\tilde{p} = D-p-4 \leftarrow \tilde{C}_{(D-p-3)} \leftarrow \hat{F}_{(D-p-2)} = *_D F_{(p+2)}$$

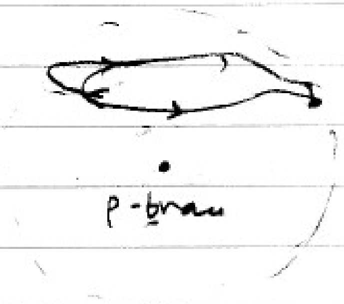
Ex) $D=5$ $\quad 0 \leftrightarrow 1$

$D=6$ $\quad 0 \leftrightarrow 2$ , $1 \leftrightarrow 1$

⋮

$D=10$ $\quad 0 \leftrightarrow 6$ , $1 \leftrightarrow 5$ , etc.

Dirac quantization in higher dim.




$$\Delta S = e_{\tilde{p}} \int_{\Sigma_{\tilde{p}+1}} \tilde{A}_{(\tilde{p}+1)} = e_{\tilde{p}} \int_{B_{\tilde{p}+2}} \tilde{F}_{(\tilde{p}+2)} =$$

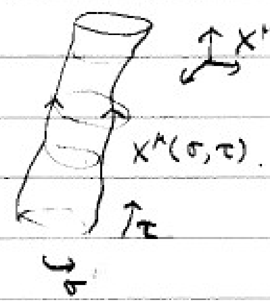
$\Delta S$ well-defined modulo $2\pi n$

$$\Rightarrow \quad e_{\tilde{p}} \int_{S^{\tilde{p}+2}} \tilde{F}_{(\tilde{p}+2)} = e_{\tilde{p}} \int_{S^{\tilde{p}+2}} * F_{(p+2)}$$

$$= e_{\tilde{p}}\, e_p = 2\pi n. \qquad !$$

# String quantization and effective gravity.

## ⊙ Classical relativistic string



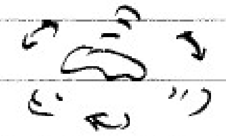

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \underbrace{\sqrt{\det\left( g_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \right)}}_{\text{"Area" of the world-sheet}}$$

$$[\alpha'] = (\text{length})^2 \qquad \sqrt{\alpha'} \equiv \underset{\parallel}{\overset{1/m_s}{}} l_s \quad (\text{string length})$$

$$T_{str} = \frac{1}{2\pi\alpha'} = \text{tension} = \text{mass density}.$$

$$(\, c = 1 \,!\,)$$

## ⊙ Quantum string.



- Discrete spectrum with mass (energy)
  (infinite tower of ptls) └ spin (angular momentum)

- Critical dim. ($D = 10$ for superstring)

- Spectrum always contains

$$g_{\mu\nu}, \; B_{\mu\nu}, \; \Phi \rightarrow \underline{\text{dilaton}} \; \left( \text{more on the next page} \right)$$

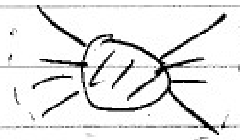
$g_{\mu\nu}$ ↓ GR!

$B_{\mu\nu}$ ↓ Analogue of $A_\mu$ for strings

- For interactions with $E \ll m_s$, wavelength $\gg l_s$
  effective gravity is a useful description

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g}\, e^{-2\phi} \left( R + 4(\nabla\phi)^2 - \frac{1}{12} H^2 + \cdots \right)$$

Interacting string




$$= \sum_{h=0}^{\infty} g_s^{2h-2+n} \int [\text{geometry}]$$


★ $g_s e^{\phi}$ is the effective string coupling at a given point in space-time!!

# IIB string in 10d

$128_B + 128_F$ states

$g_{\mu\nu}$ : $\frac{8\cdot 9}{2\cdot 1} - 1 = 44$

$B_{\mu\nu}$ : $\frac{8\cdot 7}{2\cdot 1} = 28$

$\phi$ : $= 1$

Common to all string theories

$\overline{\phantom{00}64}$

$C_{\mu\nu}$ : $\frac{8\cdot 7}{2\cdot 1} = 28$

$C^+_{\mu\nu\rho\sigma}$ : $\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}/2 = 35$

$C$ : $= 1$

$\overline{\phantom{00}64}$

Specific to IIB

$\psi_\mu$ : $7 \cdot 8 \times 2 = 112$

$\lambda$ : $8 \times 2 = 16$

$\overline{\phantom{0}128}$

---

String effective action

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \Big[ e^{-2\phi} \left( R + 4(\nabla\phi)^2 - \frac{1}{12} H^2 \right) - \frac{1}{2} F_1^2 - \frac{1}{2} F_3^2 - \cdots \Big]$$

- $G_{10} \sim g_s^2 \, l_s^8$
- Physical distance should be measured in Einstein metric : $g_E = e^{-\phi/2} g_{str}$.

# String soliton as Supergravity solution.

⊙ String analogue of Monopole sol'n.

- To be precise, full string eqn. should be considered. (exact 2d CFT background)
- SUGRA sol'n gives a good low energy description of the soliton.

⊙ Killing spinor eqn

Recall $\delta F = (B, \partial)\epsilon = 0$ for some $\epsilon$.

In SUGRA, the main diff is that

$$\begin{cases} \delta\psi_\mu = D_\mu \epsilon + (F\ldots \Gamma^{\cdots})\epsilon = 0 \\ \delta\lambda = (F\ldots\Gamma^{\cdots})\epsilon = 0 \end{cases}$$

Often, we can take $\epsilon = f \times \epsilon^0$ ↳ "const" spinor

↓ scalar function.

- Unbroken SUSY : $\boxed{\Gamma^{\cdots\cdots}\epsilon = \pm\epsilon}$

Qu) What kind of solitons are present in (say, IIB) string theory?

String solitons : summary of SUGRA sol'ns.

① Fundamental string

$$\begin{cases} ds^2 = H_{F1}^{-1}(-dt^2 + dx_9^2) + dx_1^2 + \cdots + dx_8^2 \\ B_{09} = H_{F1}^{-1} \\ e^{\phi} = H_{F1}^{-1/2} \end{cases}$$

$$\left( H_{F1} = 1 + c_{F1} \frac{G_{10} T_{F1}}{r^6} \times Q_{F1} \right)$$

$$r^2 = x_1^2 + \cdots + x_8^2$$

⊙ $\Gamma^0 \Gamma^9 \epsilon = -\epsilon^*$

② Solitonic 5-brane.

$$\begin{cases} ds^2 = -dt^2 + dx_5^2 + \cdots + dx_9^2 + H_{S5}(dx_1^2 + \cdots + dx_4^2) \\ H_{ijk} = (dB)_{ijk} = \epsilon_{ijkl} \partial_l H_{S5} \quad (i = 5 \sim 8) \\ (H = \text{"}*_4\text{"} dH) \\ e^{\phi} = H_{S5}^{+1/2} \end{cases}$$

$$\left( H_{S5} = 1 + c_{S5} \frac{G_{10} T_{S5}}{r^2} \times Q_{S5} \right)$$

$$r^2 = x_1^2 + \cdots + x_4^2$$

⊙. $\Gamma^{5678} \epsilon = + \epsilon^*$

③ D-p-brane

$$ds^2 = H_p^{-1/2}(-dt^2 + dx_1^2 + \cdots + dx_p^2) + H_p^{+1/2}(dx_{p+1}^2 + \cdots + dx_9^2)$$

$$C_{0 \cdots p} = H_p^{-1}$$

$$\left( H_p = 1 + c_p \cdot \frac{G_{10} T_p Q_p}{r^{7-p}} \right)$$

$$e^{\phi} = H_p^{\frac{3-p}{4}}$$

$$r^2 = x_{p+1}^2 + \cdots + x_9^2$$

D - brane

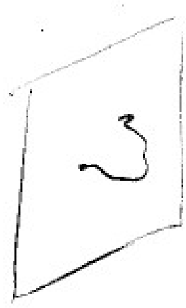
- $B_{\mu\nu}$ couples to F1 just as $A_\mu$ couples to a charged particle. It is nice to see that $B_{\mu\nu}$ pops out <u>automatically</u> in the spectrum.
- S5 brane is the string theoretic analog of magnetic monopole.

$$T_{S5} \sim \frac{1}{g_s^2 \ell_s^6} \quad \left( \because \; I \sim \frac{1}{g^2} \int d^{10}x \sqrt{-g} \, [R + \cdots] \right)$$

- What about $C_{(p+1)}$ ?

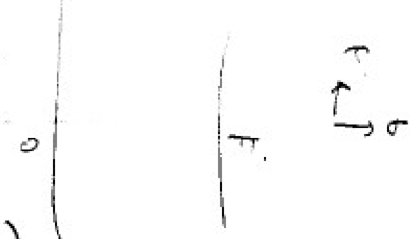
What are the charged BPS states ?

★ Solution : D - brane = end-point of open strings.



$$I = \frac{1}{4\pi\alpha'} \int_0^\pi d^2\sigma \left[ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right]$$

$X^\mu(\sigma, \tau)$



- Open string :

$$\begin{cases} \text{e.o.m.} : & (\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0 . \quad (0 < \sigma < \pi) \\ \text{b.c.} : & \underline{\delta X_\mu \partial_\sigma X^\mu = 0} \quad (\sigma = 0, \pi) \end{cases}$$

$$\longrightarrow \begin{cases} \partial_\sigma X^\mu = 0 & (\text{freely moving}) \\ \delta X^\mu = 0 & (\text{string end-point fixed}) \end{cases}$$

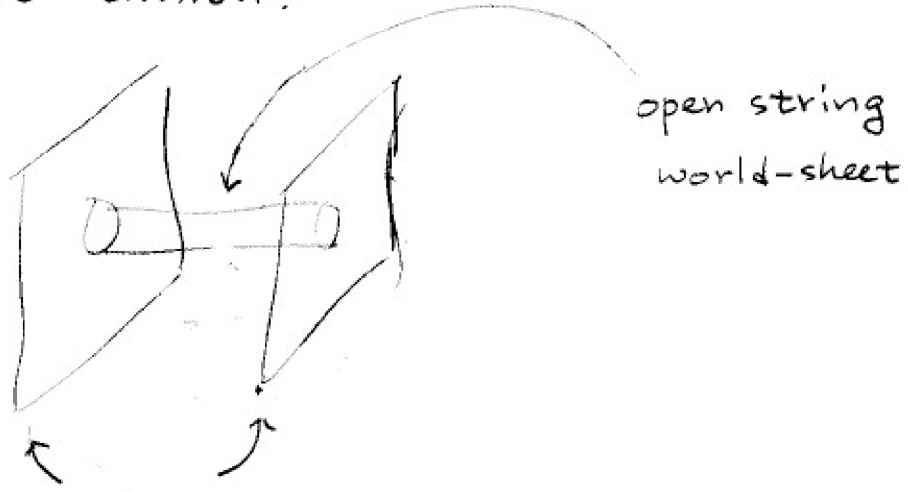
* - D-p-brane corresponds to

$$\partial_\sigma X^\mu = 0 \qquad (\mu = 0 \cdots p)$$

$$X^\mu = \text{const.} \quad (\mu = p+1 \cdots D-1)$$

# D-brane.

. D-brane tension.




⊙ Compute : Energy / (p-volume)

① Closed string picture
Coulomb-like potential between D-p-branes.

$$\frac{G_D (T_p)^2}{r^{D-p-3}} \left(1 + e^{-\#} + \cdots\right)$$

Newton's const.

↳ massive ptl exchange.

② Open string picture
Casimir energy due to open string states.

$$-\mathrm{Tr} \int d^{p+1}k \, \log\left(k^2 + m_i^2(r)\right) \quad \left( m_i^2(r) = \left(\frac{r}{2\pi\alpha'}\right)^2 + m_i^2(0) \right)$$

$$= \mathrm{Tr} \int d^{p+1}k \int \frac{ds}{s} e^{-s(k^2+m_i^2(r))}$$

(non-trivial steps)

$$\sim \frac{l_s^{D-2}}{r^{D-p-3}} \left(1 + e^{-\#} + \cdots\right)$$

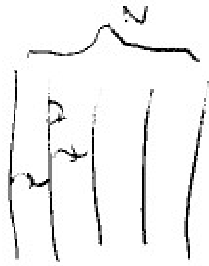
Recall : $G_D \sim g_s^2 \, l_s^{D-2} \Rightarrow \boxed{T_p \sim \frac{1}{g_s \, l_s^{p+1}}}$

D-brane - world volume theory.

◎ Quantize open string : $A_\mu$ + fermion

( cf. $g_{\mu\nu}$, $B_{\mu\nu}$, $\phi$ for closed strings )

◎ Multiple branes

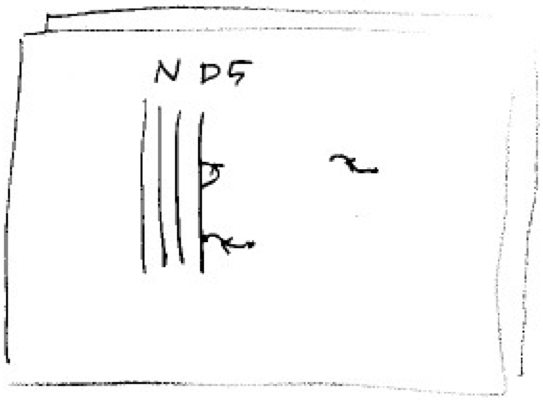



N branes

$(A_\mu)^i{}_j$ $(i, j = 1, \cdots, N)$.

$\Rightarrow$ $U(N)$ gauge theory.

◎ Composite brane config




Neglect 9-9 strings

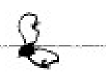
$\Rightarrow$ 5d $U(N)$ gauge theory plus matter multiplet in $N \oplus \bar{N}$ rep. with $U(K)$ flavor symmetry.

# D1-D5-P black-hole



◎ Strategy

$g_s \ll 1$ ⟶ $g_s \lesssim 1$

D-brane ⟶ 

Microscopic
world-volume theory
from open strings


Macroscopic
black-hole

In the extremal (BPS) case,
the microscopic counting at weak coupling
can be extrapolated to "strong" coupling.

# D1-D5-p black hole - I. SUGRA sol'n. 

⊙ Recipe for D-brane black holes.

1) Bring 10d down to lower d by compactification.
(Usually, $d=4$, we will do $d=5$)

2) Wrap some branes and combine them to cook up a black hole.

⊙ Single brane won't do: why?

We want:

1) Macroscopic Horizon
2) Finite dilaton $e^{\phi}$.
3) Small but finite size of internal manifold.

(2) and 3) throughout the space from horizon to asymptotic infinity)

★ Single brane is not "strong" enough to produce 1), and never satisfy 2) and 3).

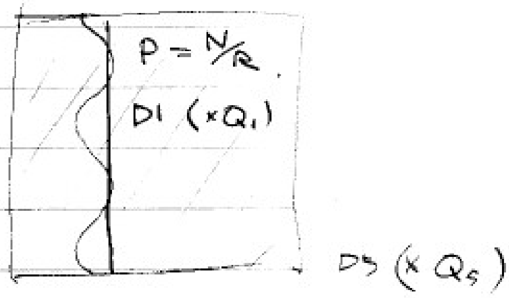
⇒ Combine several species of branes to keep the balance.

Recall
$$\begin{cases} ds^2 = H_p^{-1/2}(-dt^2+dx_1^2+\cdots+dx_p^2) + H^{1/2}(dx_{p+1}^2+\cdots+dx_9^2) \\ e^{\phi} = H_p^{\frac{3-p}{4}} \end{cases}$$

## D1-D5-P model : Set-up.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D1 | × | × | | | | | | | | × |
| D5 | × | | | | | × | × | × | × | × |
| P | × | | | | | | | | | × |

$\underbrace{0\ 1\ 2\ 3\ 4}_{\text{space-time } \mathbb{R}^{1,4}}$ $\underbrace{5\ 6\ 7\ 8\ 9}_{\text{compact } T^5}$




$T^5$ : $x^9$ ↑, → $x^{5,6,7,8}$

D1 - D5 - P solution in D = 10

- $ds^2 = H_1^{-1/2} H_5^{-1/2} \left( -dt^2 + dx_9^2 + K(dt - dx_9)^2 \right) + H_1^{1/2} H_5^{1/2} \left( dx_1^2 + \cdots + dx_4^2 \right) + H_1^{1/2} H_5^{-1/2} \left( dx_5^2 + \cdots + dx_8^2 \right)$

$e^{\phi} = H_1^{1/2} H_5^{-1/2}$

$C_{09} = H_1^{-1} - 1$

$F_{ijk} = \epsilon_{ijkl} \partial_l H_5 \qquad (i = 1 \sim 4, \text{ etc.})$

$H_1 = 1 + \frac{h_1}{r^2}, \; H_5 = 1 + \frac{h_5}{r^2}, \quad K = 1 + \frac{k}{r^2}$

- SUSY

- D1 : $\epsilon_R = \Gamma^0 \Gamma^9 \epsilon_L$
- D5 : $\epsilon_R = \Gamma^0 \Gamma^{56789} \epsilon_L$
- P : $\epsilon_{R,L} = \Gamma^0 \Gamma^9 \epsilon_{R,L}$

$\Bigg\}$ • mutually compatible
• preserve $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ SUSY.

D1-D5-P solution in $D = 10-5$.

- $$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g}\left[e^{-2\phi} R + \cdots \right]$$

integrate over $x^{5\sim 9}$.

$$\to \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g}\left[e^{-2\phi}\, V\, R + \cdots \right]$$

volume of the internal manifold at point $x$.

- $$ds_{10}^2 = (H_1 H_5)^{-1/2}\left(-dt^2 + dx_9^2 + K\,(dt - dx_9)^2\right) + (H_1 H_5)^{1/2}\, dx_{1\sim 4}^2 + H_1^{1/2} H_5^{-1/2}\, dx_{5\sim 9}^2$$

$$= (H_1 H_5)^{-1/2}\left(-\frac{1}{K+1}\,dt^2 + (1+K)\left(dx_9 - \frac{K}{K+1}\,dt\right)^2\right)$$

$$+ \cdots$$

$$= -(H_1 H_5)^{-1/2}(1+K)^{-1}\, dt^2 + (H_1 H_5)^{1/2}\left(dx_1^2 + \cdots + dx_4^2\right)$$

$$+ \underbrace{(1+K)}_{(H_1 H_5)^{-1/2}}\left(dx_9 + \omega\right)^2 + H_1^{1/2} H_5^{-1/2}\, dx_{5-9}^2$$

- $$ds_5^2 = -\left(H_1 H_5 H_p^2\right)^{-1/2} dt^2 + (H_1 H_5)^{1/2}\left(dx_1^2 + \cdots + dx_4^2\right)$$

$$e^{-2\phi}\cdot V = \left(H_1^{-1} \cancel{H_5}\right)\left[(H_1 H_5)^{-1/4} H_p^{1/2} \cdot \left(H_1 \cancel{H_5^{-1}}\right)\right]$$

$$= (H_1 H_5)^{-1/4} H_p^{1/2}$$

- Rescale to obtain the 5d Einstein metric.

$$\boxed{\begin{aligned} ds_{5E}^2 &= \left(e^{-2\phi} V\right)^{\frac{2}{3}} ds_5^2 \\ &= -(H_1 H_5 H_p)^{-2/3}\, dt^2 + (H_1 H_5 H_p)^{1/3}\, d\vec{x}^{\,2} \end{aligned}}$$

## Properties of the SUGRA solution

1) Macroscopic horizon at $r = 0$ ( more below )

2) finite dilaton $e^{\phi} \to \left(\frac{h_1}{h_5}\right)^{1/2}$ as $r \to 0$

3) finite size of internal $T^5$.

$$ds_{T^5}^2 = \frac{H_p}{(H_1 H_5)^{1/2}} (dx_9 + \omega)^2 + H_1^{1/2} H_5^{-1/2} dx_{5-8}^2$$

balance !      valance !

### Horizon area

$$ds_{5E}^2 = -(\quad)^{-2/3} dt^2 + (H_1 H_5 H_p)^{1/3} d\vec{x}^2$$

$$\left[\left(1+\frac{h_1}{r^2}\right)\left(1+\frac{h_5}{r^2}\right)\left(1+\frac{k}{r^2}\right)\right]^{1/3} \left[dr^2 + r^2 d\Omega_3^2\right]$$

$$(r \to 0) \quad \to (h_1 h_5 k)^{1/3} d\Omega_3^2 + \cdots$$

$$\boxed{A \sim (h_1 h_5 k)^{1/2}}$$

The constants

- $h_1 \sim G_5 M_1$ , $h_5 = G_5 M_5$ , $k = G_5 M_p$

$$M_1 \sim \frac{R}{g_s l_s^2} Q_1 , \quad M_5 = \frac{RL^4}{g_s l_s^6} Q_5 , \quad M_p = \frac{1}{R} N .$$

- $$G_5 \sim \frac{G_{10}}{\mathrm{Vol}(T^5)} \quad \left( \because \ \frac{1}{16\pi G_{10}} \int d^{10}x \to \frac{1}{16\pi G_5} \int d^5 x \right)$$

$$\sim g_s^2 l_s^8 / L^4 R .$$

- $$h_1 h_5 k \sim (G_5 M_1)(G_5 M_5)(G_5 M_p)$$

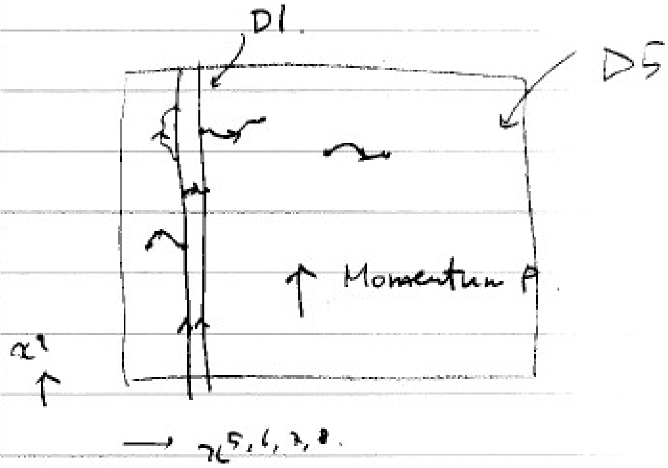
$$= G_5^2 \ (G_5 M_1 M_5 M_p)$$

$$\sim G_5^2 \ Q_1 Q_5 N .$$

$$S = \frac{A}{4G_5} \sim \frac{\sqrt{h_1 h_5 k}}{G_5} \sim \sqrt{Q_1 Q_5 N} .$$

Fix the numerical factors $\Rightarrow$ $S = 2\pi \sqrt{Q_1 Q_5 N}$

Microscopics of D1-D5-P b.h.




- Three types of excitations
  (11)-string, (55)-string, (15, 51) string.
- Entropy comes from the number of ways to distribute the momentum among the string modes. (BPS : massless, right-movers only)
- Most of the contribution come from (15, 51) strgs.

because
- Excite (11), (55) strgs
  $\rightarrow$ (15, 51) strings obtain mass and get frozen
  $\sim Q_1 + Q_5$ degrees of freedom
- Excite (15), (51) strgs
  $\rightarrow$ (11), (55) strings get frozen
  $\sim Q_1 Q_5$ d.o.f.

Microscopics : Quantization of (5, 51) open strings

- 

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| D1 | | | | | | | | | × |
| D5 | | | | | × | × | × | × | × |
| P | | | | | | | | | × |

$\underbrace{1\ 2\ 3\ 4}_{SO(4)_E}$ $\underbrace{5\ 6\ 7\ 8}_{SO(4)_I}$

Boson : Scalar 2 spinor $\rightarrow 4_{IR}$ states

Fermion : 2 spinor scalar $\rightarrow 4_{IR}$ states

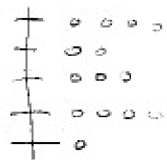
- Counting the entropy.

Only right movers : Energy = Momentum.

$S = \log$ ( All possible way to distribute N units of momenta among $\underbrace{4Q_1Q_5}_{\text{massless}}$ bosons and $\underbrace{4Q_1Q_5}_{\text{massless}}$ fermions )

- Introduce.

$$Z = \sum_{A.P.S.} e^{-\beta E} = \sum_N d(N)\, q^N \qquad (q \equiv e^{-\beta})$$

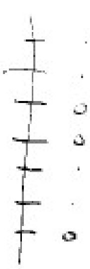


- $Z$ (boson)

$$Z_B = (1+q+q^2+\dots)(1+q^2+q^4+\dots)\cdots$$
$$= \prod_{n=1}^{\infty}(1-q^n)^{-1}$$

- $Z$ (fermion)

$$Z_F = (1+q)(1+q^2)(1+q^3)\cdots$$
$$= \prod_{n=1}^{\infty}(1+q^n)$$



Microscopics

- $$Z_{tot} = (Z_B Z_F)^{4Q_1Q_5} = \left( \prod_{n=1}^{\infty} (1-q^n)^{-1} (1+q^n) \right)^{4Q_1Q_5}$$

- $$\equiv \sum_{N=0}^{\infty} d(N, Q_1Q_5)\, q^N$$

- Two steps to compute $d(N, Q_1Q_5)$
  for $N, Q_1, Q_5 \gg 1$.

  1) Modular transf.

  $$Z(\beta) = (\quad)\, Z\left(\tfrac{1}{\beta}\right)$$

  2) Saddle point approx.

<u>End result</u>

$$S = \log d(Q_1, Q_5, N) \approx 2\pi\sqrt{Q_1 Q_5 N}.$$

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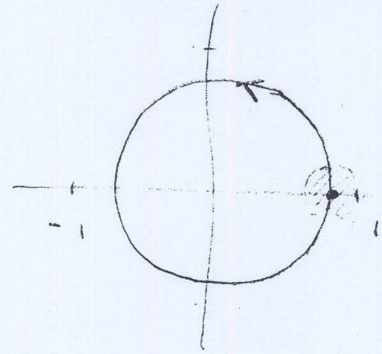
$$Z = \sum d_N q^N.$$

$$\Longleftrightarrow \quad d_N = \frac{1}{2\pi i} \oint \frac{dq}{q} \cdot \frac{Z}{q^N}$$




$$q = e^{2\pi i \tau}$$

$$\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\sqrt{\frac{\theta_4}{\eta}} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}})$$

$$\sqrt{\frac{\theta_2}{\eta}} = \sqrt{2}\, q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n).$$

$$\eta(-\tfrac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau)$$

$$\theta_2(-\tfrac{1}{\tau}) = (-i\tau)^{1/2} \theta_4(\tau)$$

$$\theta_4(-\tfrac{1}{\tau}) = (-i\tau)^{1/2} \theta_2(\tau)$$

$$Z_B Z_F = \prod_{n=1}^{\infty} (1 - q^n)^{-1} (1 + q^n)$$

$$\tilde{q} = e^{2\pi i (-\frac{1}{\tau})}$$

$$= \frac{1}{\sqrt{2}} \eta^{-1} \left( \frac{\theta_2}{\eta} \right)^{1/2}$$

$$= \frac{1}{\sqrt{2}} (-i\tau)^{1/2} \eta(-\tfrac{1}{\tau})^{-1} \left[ \frac{\theta_4(-\frac{1}{\tau})}{\eta(-\frac{1}{\tau})} \right]^{1/2}.$$

$$= \frac{1}{\sqrt{2}} (-i\tau)^{1/2} e^{-\frac{2\pi i}{\tau}(-\frac{1}{24} - \frac{1}{48})} \prod_{n=1}^{\infty} (1 - \tilde{q}^n)^{-1} (1 - \tilde{q}^{n-\frac{1}{2}})$$

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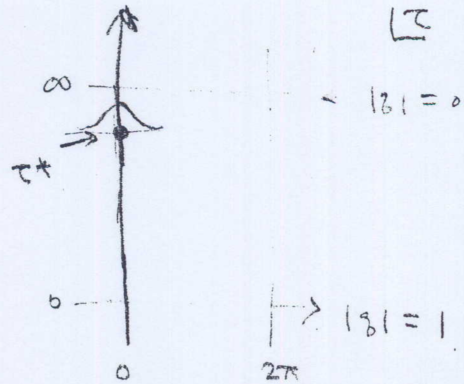
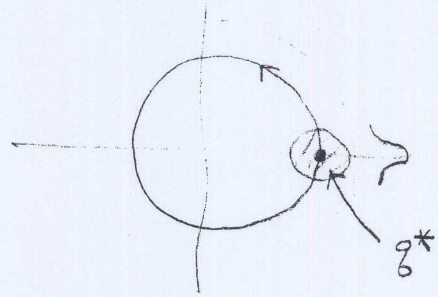
$$d_N \sim \oint e^{2\pi i\left(-N\tau + \frac{4Q_1Q_5}{16}\cdot\frac{1}{\tau}\right)}\, d\tau .$$




$$f(\tau) \equiv 2\pi i \left(-N\tau + \frac{4Q_1Q_5}{16}\cdot\frac{1}{\tau}\right)$$

saddle pt. : $f'(\tau) = 0$

$$\Rightarrow \tau^* = \frac{i}{2}\sqrt{\frac{Q_1Q_5}{N}}.$$




$$f(\tau_* + \epsilon) = f(\tau_*) + \frac{1}{2} f''(\tau_*)\,\epsilon^2 + \cdots$$

$$= 2\pi\sqrt{Q_1Q_5N} - 8\pi \frac{N^{3/2}}{(Q_1Q_5)^{1/2}}\,\epsilon^2 + \cdots$$

$$S = \log d_N = 2\pi\sqrt{Q_1Q_5N}\left[1 + O\left(\frac{N}{Q_1Q_5}\right)\right]$$

$$+ \text{(logarithmic terms)}$$

$$\simeq 2\pi\sqrt{Q_1Q_5N} \quad !$$