QCD PRACTICE

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Contents

Chapter 1

QED Review

1.1 Maxwell's equations

1. Let us define covariant(x_μ) and contravariant (x^μ) vectors as

$$x^\mu \equiv (t, +\boldsymbol{x}) \, , \quad x_\mu \equiv (t, -\boldsymbol{x}) \, . \tag{1.1}$$

2. Show that

$$\partial_\mu \;\equiv\; \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, +\boldsymbol{\nabla}\right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{1.2a}$$

3. Show that $\partial_\mu(\partial^\mu)$ transforms like $x_\mu(x^\mu)$.

4. Show

$$\partial^2 = \frac{\partial}{\partial t}^2 - \boldsymbol{\nabla}^2. \tag{1.3}$$

5. Show

$$\partial \cdot J = \frac{\partial}{\partial t} J^0 + \boldsymbol{\nabla} \cdot \boldsymbol{J}. \tag{1.4}$$

6. Using the potential $A^\mu = (\phi, \boldsymbol{A})$, express the electromagnetic fields

$$\boldsymbol{E} \;=\; -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi, \tag{1.5a}$$

$$\boldsymbol{B} \;=\; \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1.5b}$$

7. Show that

$$E^i \;=\; -\frac{\partial}{\partial x^0} A^i - \frac{\partial}{\partial x^i} A^0 = -\partial^0 A^i + \partial^i A^0, \tag{1.6a}$$

$$B^i \;=\; \epsilon^{ijk} \frac{\partial}{\partial x^j} A^k = -\epsilon^{ijk} \partial^j A^k. \tag{1.6b}$$

8. Defining the **field-strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{1.7}$$

show that

$$\begin{aligned} E^i &= \partial^i A^0 - \partial^0 A^i = -F^{0i}, \\ B^i &= -\frac{1}{2}\epsilon^{ijk} F^{jk}. \end{aligned}$$

9. Express the field strength tensor in a matrix form as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{1.8a}$$

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{1.8b}$$

10. Show that

$$F_{\mu\nu} = F^{\mu\nu}(\boldsymbol{E} \to -\boldsymbol{E}). \tag{1.9}$$

11. Let us define the **dual field-strength tensor** $\mathcal{F}^{\mu\nu}$ as

$$\mathcal{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{1.10}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is a totally antisymmetric tensor and $\epsilon_{0123} = -\epsilon^{0123} = +1$.

12. Show that

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} = F^{\mu\nu}(\boldsymbol{E} \to \boldsymbol{B}, \boldsymbol{B} \to -\boldsymbol{E}). \tag{1.11}$$

13. Show that

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \partial_i V^i, \quad \left(\frac{\partial}{\partial t}\boldsymbol{V}\right)^i = \partial^0 V^i \tag{1.12a}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{V})^i = \epsilon^{ijk}\partial_j V^k = -\epsilon^{ijk}\partial^j V^k \tag{1.12b}$$

$$\epsilon^{ijk}\partial^i\partial^j V^k = 0 \tag{1.12c}$$

$$\epsilon^{ijk}\partial^i F^{jk} = 0 \tag{1.12d}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{E})^i = \frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk}, \quad \frac{\partial B^i}{\partial t} = -\frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk} \tag{1.12e}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{B})^i = \partial_j F^{ji}, \quad -\frac{\partial E^i}{\partial t} = \partial_0 F^{0i} \tag{1.12f}$$

14. Show that

$$\nabla \cdot \boldsymbol{E} = \rho \quad \rightarrow \quad \partial_0 F^{00} + \partial_i F^{i0} = J^0 \tag{1.13a}$$

$$\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J} \quad \rightarrow \quad \partial_0 F^{0i} + \partial_j F^{ji} = J^i \tag{1.13b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \rightarrow \quad \frac{1}{2}\epsilon^{ijk}\partial^i F^{jk} \equiv 0 \; : \; \partial_i \mathcal{F}^{i0} = 0 \tag{1.13c}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \quad \rightarrow \quad \frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk} - \frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk} = 0 \tag{1.13d}$$

$$: \partial_0 \mathcal{F}^{0i} + \partial_j \mathcal{F}^{ji} = 0 \tag{1.13e}$$

15. You have shown that Maxwell's equations reduce into the form

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{1.14}$$

1.2 Gauge

16. Show that $\partial^\mu \partial^\nu \chi - \partial^\nu \partial^\mu \chi = 0$, where χ is a Lorentz scalar function.

17. Show that $F^{\mu\nu}$ is invariant under the gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi, \tag{1.15}$$

and the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are also gauge invariant.

18. Show that the Maxwell's equations are gauge invariant.

19. Let us use the Lorentz gauge $\partial \cdot A = 0$. Show that the Maxwell's equations reduce into

$$\partial^2 A^\mu = J^\mu. \tag{1.16}$$

20. We can make another gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \Lambda. \tag{1.17}$$

Show that Λ satisfies the Lorentz condition $\partial^2 \Lambda = 0$.

21. Show that

$$A^\mu = \epsilon^\mu(\boldsymbol{p}) e^{-ip\cdot x}. \tag{1.18}$$

is a solution to the free photon($J^\mu = 0$) equation with

$$p^2 = 0, \quad \epsilon \cdot p = 0. \tag{1.19}$$

22. Choosing $\Lambda = iae^{-ip\cdot x}$, where a is a scalar, show that the gauge invariance condition ensures that the transformation

$$\epsilon^\mu \rightarrow \epsilon'^\mu = \epsilon^\mu + ap^\mu \tag{1.20}$$

does not change physical results.

23. If we choose a such that $\epsilon'^0 = 0$, show that

$$\boldsymbol{\epsilon} \cdot \boldsymbol{p} = 0 \tag{1.21}$$

and it is equivalent to the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

24. Show that the transverse condition $\epsilon \cdot p = 0$ and Coulomb gauge condition $\boldsymbol{\epsilon} \cdot \boldsymbol{p} = 0$ restricts the photon to have only two degrees of freedom

$$\epsilon^\mu = (0, 1, 0, 0), \quad (0, 0, 1, 0) \tag{1.22}$$

if $p^\mu = (E, 0, 0, E)$.

25. We have shown that there are only two independent parameters describing the polarization vector ϵ^μ of the photon. There are only two polarization(spin) states for a massless spin-1 particle.

26. Show that spin-1 wavefunction is expressed in terms of spherical harmonics for $|\ell = 1, \ell_z\rangle$ states as

$$\begin{aligned} Y_1^{\pm}(\theta, \phi) &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} = \sqrt{\frac{3}{4\pi}} \hat{\boldsymbol{r}} \cdot \boldsymbol{\epsilon}(\pm), & (1.23a) \\ Y_1^0(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \hat{\boldsymbol{r}} \cdot \boldsymbol{\epsilon}(0). & (1.23b) \end{aligned}$$

where $\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$.

27. We can express the spherical harmonics in Cartesian coordinate system, which is convenient for vector transformation. The polarization vectors $\boldsymbol{\epsilon}(\lambda)$ are defined by

$$\begin{aligned} \boldsymbol{\epsilon}(\pm) &= \mp\frac{1}{\sqrt{2}} (\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) = \mp\frac{1}{\sqrt{2}}(1, \pm i, 0) & (1.24a) \\ \boldsymbol{\epsilon}(0) &= \hat{\boldsymbol{z}} = (0, 0, 1) & (1.24b) \end{aligned}$$

28. Show that the polarization vectors are expressed using four-vector nontation as

$$\begin{aligned} \epsilon^\mu(\pm) &= \mp\frac{1}{\sqrt{2}}(0, 1, \pm i, 0), & (1.25a) \\ \epsilon^\mu(0) &= (0, 0, 0, 1). & (1.25b) \end{aligned}$$

29. Show that massless photon's wavefunction is a linear combination of $\epsilon^\mu(\pm)$. The longitudinal state $\epsilon^\mu(0)$ is not allowed.

30. Show the orthogonality conditions

$$\begin{aligned} \boldsymbol{\epsilon}^*(\lambda) \cdot \boldsymbol{\epsilon}(\lambda') &= \delta^{\lambda\lambda'}, & (1.26a) \\ \epsilon^*(\lambda) \cdot \epsilon(\lambda') &= -\delta^{\lambda\lambda'}. & (1.26b) \end{aligned}$$

31. Show that the spin sum of the polarization tensor is

$$\sum_{\lambda=\pm} \epsilon^{i*}(\lambda)\epsilon^j(\lambda) = \delta_\perp^{ij} = \delta^{ij} - \hat{\boldsymbol{z}}^i\hat{\boldsymbol{z}}^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \tag{1.27a}$$

if $k^\mu = (k, 0, 0, k)$.

32. Show that the spin sum of the polarization tensor is

$$\sum_{\lambda=\pm} \epsilon^{i*}(\lambda)\epsilon^{j}(\lambda) = \delta_{\perp}^{ij} = \delta^{ij} - \frac{\boldsymbol{k}^i \boldsymbol{k}^j}{\boldsymbol{k}^2}. \tag{1.28a}$$

if $k^\mu = (|\boldsymbol{k}|, \boldsymbol{k})$. The polarization sum is for the Coulomb gauge, where $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

33. Defining $n = (1, 0, 0, -1)$ when $p = (E, 0, 0, E)$, show that we may write it in a Lorentz covariant form

$$\sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda)\epsilon^{\nu}(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \tag{1.29a}$$

34. The formula

$$\Pi^{\mu\nu} = \sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda)\epsilon^{\nu}(\lambda) = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \tag{1.30}$$

is valid for any light-like vector $n^2 = 0$ satisfying $\epsilon \cdot p \neq 0$, $n \cdot p \neq 0$, and $n \cdot \epsilon = 0$. Show explicitly that

$$\Pi^{\mu}{}_{\mu} = -2, \tag{1.31a}$$
$$n_\mu \Pi^{\mu\nu} = 0, \quad n_\nu \Pi^{\mu\nu} = 0, \tag{1.31b}$$
$$p_\mu \Pi^{\mu\nu} = 0, \quad p_\nu \Pi^{\mu\nu} = 0. \tag{1.31c}$$

The polarization sum is for the light-cone gauge, where $n \cdot A = 0$ with $n^2 = 0$.

35. We can extend our formula to the case $n^2 \neq 0$ keeping $\epsilon \cdot p \neq 0$, $n \cdot p \neq 0$, and $n \cdot \epsilon = 0$. Derive

$$\Pi^{\mu\nu} = \sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda)\epsilon^{\nu}(\lambda) = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} - \frac{n^2 p^\mu p^\nu}{(p \cdot n)^2}. \tag{1.32}$$

from the conditions

$$\Pi^{\mu}{}_{\mu} = -2, \tag{1.33a}$$
$$n_\mu \Pi^{\mu\nu} = 0, \quad n_\nu \Pi^{\mu\nu} = 0, \tag{1.33b}$$
$$p_\mu \Pi^{\mu\nu} = 0, \quad p_\nu \Pi^{\mu\nu} = 0, \tag{1.33c}$$

The polarization sum is for the axial gauge, where $n \cdot A = 0$ with $n^2 \neq 0$.

1.3 Lagrangian

36. **Euler-Lagrange equation**: Action is defined by

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \tag{1.34}$$

Show that under the variation $\phi \to \phi + \delta\phi$, where ϕ and $\partial_\mu\phi$ are fixed at the end points

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L}(\phi, \partial_\mu\phi) = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi\right] \quad (1.35a)\\ &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi, \quad (1.35b)\end{aligned}$$

where we neglected the surface term

$$\int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] = \int da_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi. \quad (1.36a)$$

37. Show that $\delta S = 0$ if ϕ satisfies the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0. \quad (1.37)$$

38. **real scalar field** If

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \quad (1.38)$$

show that the Euler-Lagrange equation is the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0. \quad (1.39)$$

This leads to $p^2 = m^2$.

39. **Symmetry and conserved current** If the Lagrangian is invariant under a transformation $\phi \to \phi + \delta\phi$, show that

$$\begin{aligned} \delta\mathcal{L}(\phi, \partial_\mu\phi) &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi, \quad (1.40a)\\ &= \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \quad (1.40b)\\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] = 0. \quad (1.40c)\end{aligned}$$

If there is symmetry, there is a corresponding conserved quantity.

40. The conserved ($\partial \cdot J = 0$) current

$$J^\mu \propto \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi. \quad (1.41)$$

41. If the current is vanishing on a boundary surface, charge inside the surface is conserved

$$\frac{\partial Q}{\partial t} = -\int d^3\boldsymbol{x}\boldsymbol{\nabla}\cdot\boldsymbol{J} = -\int d\boldsymbol{a}\cdot\boldsymbol{J} = 0, \quad (1.42a)$$

where $Q = \int d^3\boldsymbol{x}J^0$.

42. Show that

$$-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{1.43a}$$

43. Show that

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) - \rho\phi + \boldsymbol{J}\cdot\boldsymbol{A}$$

44. Show that the Euler-Lagrange equation for the Lagrangian is the Maxwell's equations

$$\partial_\mu F^{\mu\nu} = J^\nu;\ (\partial^2 - \partial^\mu\partial_\nu)A^\nu = J^\nu \tag{1.44}$$

1.4 Classical charged-particle Lagrangian

45. Consider a particle with the charge q and mass m moving in an external electromagnetic field. The equation of motion is

$$\frac{dE}{dt} = q\boldsymbol{v}\cdot\boldsymbol{E}, \quad \frac{d\boldsymbol{p}}{dt} = q\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right). \tag{1.45a}$$

46. Let us define the four-velocity in Lorentz covariant notation

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(1, \boldsymbol{v}). \tag{1.46}$$

show that $u^2 = 1$.

47. Using the four-velocity in Lorentz covariant notation, show that

$$\frac{dp^\mu}{d\tau} = q\gamma\left(\boldsymbol{v}\cdot\boldsymbol{E}, \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right) = qF^{\mu\nu}u_\nu, \tag{1.47}$$

where $p^\mu = mu^\mu$ and m is the rest mass of the particle.

48. Show that

$$F^{\mu\nu}u_\nu = \gamma\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -v^1 \\ -v^2 \\ -v^3 \end{pmatrix} \tag{1.48a}$$

$$= \gamma\begin{pmatrix} v^1E^1 + v^2E^2 + v^3E^3 \\ E^1 + (v^2B^3 - v^3B^2) \\ E^2 + (v^3B^1 - v^1B^3) \\ E^3 + (v^1B^2 - v^2B^1) \end{pmatrix}^\mu. \tag{1.48b}$$

49. **Free particle Lagrangian**: Consider a free particle with a mass m. The action must be Lorentz invariant and the only available Lorentz invariant scalar is m.

$$S = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} \gamma L d\tau = f(m). \tag{1.49}$$

and $L\gamma$ must be a scalar. And the dimension must be order of mass. Let us try

$$L = -\frac{m}{\gamma} = -m\sqrt{1-\boldsymbol{v}^2}. \tag{1.50}$$

Show that the Euler-Lagrange equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0, \tag{1.51a}$$

$$\frac{\partial L}{\partial v^i} = \frac{mv^i}{\sqrt{1-\boldsymbol{v}^2}} = \gamma m v^i, \tag{1.51b}$$

$$\frac{\partial}{\partial t}(m\gamma v^i) = 0. \tag{1.51c}$$

Momentum is conserved! $\leftarrow$ free particle.

50. Show that corresponding Hamiltonian is

$$p^i = \frac{\partial L}{\partial v^i} = -\frac{\partial}{\partial v^i}\sqrt{1-\boldsymbol{v}^2} = \frac{mv^i}{\sqrt{1-\boldsymbol{v}^2}}, \tag{1.52a}$$

$$H = p^i v^i - L = \frac{m\boldsymbol{v}^2}{\sqrt{1-\boldsymbol{v}^2}} + m\sqrt{1-\boldsymbol{v}^2} \tag{1.52b}$$

$$= \frac{m}{\sqrt{1-\boldsymbol{v}^2}} - \frac{m(1-\boldsymbol{v}^2)}{\sqrt{1-\boldsymbol{v}^2}} + m\sqrt{1-\boldsymbol{v}^2} \tag{1.52c}$$

$$= \frac{m}{\sqrt{1-\boldsymbol{v}^2}} = \gamma m = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{1.52d}$$

51. **Charged particle Lagrangian**: In nonrelativistic quantum mechanics, $L_{\text{int}} = -V_{\text{int}}$. If we consider the electrostatic potential in nonrelativistic quantum mechanics,

$$L_{\text{int}} = -V_{\text{int}} = -q\phi. \tag{1.53}$$

52. Let us construct a Lorentz scalar. As we did for a charged particle, γL must be Lorentz invariant and in the nonrelativistic limit the Lagrangian must reduce the form shown above.

$$L_{\text{int}} = -\frac{qu \cdot A}{\gamma}, \tag{1.54}$$

where $u^\mu = \gamma(1, \boldsymbol{v})$ and $A^\mu = (\phi, \boldsymbol{A})$.

53. Therefore,

$$L = -\frac{m + qv \cdot A}{\gamma} = -m\sqrt{1-\boldsymbol{v}^2} - q\phi + q\boldsymbol{v}\cdot\boldsymbol{A} \tag{1.55}$$

Conjugate momentum is

$$P^i = \frac{\partial L}{\partial v^i} = \gamma m v^i + qA^i = (\boldsymbol{p} + q\boldsymbol{A})^i \;\rightarrow\; \boldsymbol{v} = \frac{\boldsymbol{P} - q\boldsymbol{A}}{\gamma m}, \tag{1.56a}$$

$$1 = \gamma^2(1 - \boldsymbol{v}^2) = \gamma^2 - \frac{(\boldsymbol{P} - q\boldsymbol{A})^2}{m^2}, \tag{1.56b}$$

$$\rightarrow \quad \gamma m = \sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2} \;\rightarrow\; \boldsymbol{v} = \frac{\boldsymbol{P} - q\boldsymbol{A}}{\sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2}}. \tag{1.56c}$$

54. Show that

$$(\boldsymbol{P} - q\boldsymbol{A}) \cdot \boldsymbol{v} = \frac{(\boldsymbol{P} - q\boldsymbol{A})^2}{\sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2}}, \tag{1.57a}$$

$$\frac{m}{\gamma} = \frac{m^2}{\gamma m} = \frac{m^2}{\sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2}}. \tag{1.57b}$$

Then the Hamiltonian is

$$H = \boldsymbol{P} \cdot \boldsymbol{v} - L = (\boldsymbol{P} - q\boldsymbol{A}) \cdot \boldsymbol{v} + \frac{m}{\gamma} + q\phi \tag{1.58a}$$

$$= \sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2} + q\phi \tag{1.58b}$$

55. The Hamiltonian for a charged particle

$$H = \sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2} + q\phi. \tag{1.59}$$

is same as that for a free particle

$$H = \sqrt{\boldsymbol{p}^2 + m^2} \tag{1.60}$$

when we substitute

$$H \rightarrow H - q\phi \tag{1.61a}$$

$$\boldsymbol{p} \rightarrow \boldsymbol{P} - q\boldsymbol{A}. \tag{1.61b}$$

56. Show that mass-shell condition still holds

$$p^2 = m^2, \quad p^\mu = (E - q\phi, \boldsymbol{P} - q\boldsymbol{A}), \tag{1.62}$$

where E is the total energy of the particle

57. **Nonrelativistic case**: Show that in the nonrelativistic(NR) limit

$$H = \sqrt{(\boldsymbol{P} - q\boldsymbol{A})^2 + m^2} + q\phi \rightarrow \frac{(\boldsymbol{P} - q\boldsymbol{A})^2}{2m} + q\phi. \tag{1.63}$$

We could have derived the form from the free particle equation

$$H = \frac{\boldsymbol{p}^2}{2m} \;\leftarrow\; [H \rightarrow H - q\phi,\; \boldsymbol{p} \rightarrow \boldsymbol{P} - q\boldsymbol{A}]. \tag{1.64}$$

58. **Schrödinger equation for a charged particle**: For a free particle, we replace

$$\boldsymbol{p} \to -i\boldsymbol{\nabla},\ H \to i\frac{\partial}{\partial t}, \tag{1.65}$$

which generate $\boldsymbol{p}$ and E once they act on the free-particle wavefunction $e^{-ip\cdot x}$, and apply the operator to the wavefunction ψ.

$$i\frac{\partial}{\partial t}\psi = \frac{(-i\boldsymbol{\nabla})^2}{2m}\psi \tag{1.66}$$

59. Show that the Schrödinger equation for a charged particle is

$$\begin{aligned}(H - q\phi)\,\psi &= \frac{(\boldsymbol{P} - q\boldsymbol{A})^2}{2m}\psi, &(1.67a)\\ i\left(\frac{\partial}{\partial t} + iqA^0\right)\psi &= -\frac{(-\boldsymbol{\nabla} + iq\boldsymbol{A})^2}{2m}\psi. &(1.67b)\end{aligned}$$

1.5 Gauge Invariance and Covariant Derivative

60. Let us define the covariant derivative

$$\mathcal{D}^\mu \equiv \partial^\mu + iqA^\mu = \left(D^0, -\boldsymbol{D}\right) = \left(\frac{\partial}{\partial t} + iqA^0, -\boldsymbol{\nabla} + iq\boldsymbol{A}\right). \tag{1.68}$$

Show that the shrödinger equation becomes

$$iD^0\psi = -\frac{\boldsymbol{D}^2}{2m}\psi. \tag{1.69}$$

61. Let us replace $\psi \to \psi' = U\psi$, where the transformation keeps the probability

$$\psi^\dagger\psi = \psi'^\dagger\psi' \to U^\dagger U = 1. \tag{1.70}$$

Therefore U is unitary.

62. We know physical observables are invariant under the gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\chi. \tag{1.71}$$

We will find there IS a gauge transformation that keeps the Shrödinger equation invariant under the unitary transformation $\psi \to \psi' = U\psi$.

$$\begin{aligned}D'^\mu &= \partial^\mu + iqA'^\mu = \partial^\mu + iq(A^\mu + \partial^\mu\chi) = D^\mu + iq\partial^\mu\chi &(1.72a)\\ D'^0 &= \partial^0 + iq(A^0 + \partial^0\chi) &(1.72b)\\ -\boldsymbol{D}' &= -\boldsymbol{\nabla} + iq(\boldsymbol{A} - \boldsymbol{\nabla}\chi). &(1.72c)\end{aligned}$$

Show, if $\partial^\mu\chi = \frac{i}{q}(\partial^\mu U)U^\dagger$, that

$$\begin{aligned}D'^0 U\psi &= UD^0\psi,\quad \boldsymbol{D}'U\psi = U\boldsymbol{D}\psi &(1.73a)\\ (\boldsymbol{D}')^2U\psi &= \boldsymbol{D}'(U\boldsymbol{D}\psi) = U\boldsymbol{D}^2\psi &(1.73b)\end{aligned}$$

Therefore, Schrödinger equation is invariant under the gauge transformation

$$A^\mu \to A'^\mu = UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger,\quad \psi \to \psi' = U\psi,\quad U^\dagger U = 1 \tag{1.74}$$

1.6 Scalar particle

63. **Scalar particle and EM interaction** Remember

$$H = \sqrt{\boldsymbol{p}^2 + m^2} \rightarrow H^2 = \boldsymbol{p}^2 + m^2. \tag{1.75}$$

Klein-Gordon equation is the wave equation for a scalar particle that can be obtained by the replacement $H \rightarrow i\partial^0$ and $\boldsymbol{p} = -i\boldsymbol{\nabla}$, that is $p^\mu \rightarrow i\partial^\mu$

$$(\partial^2 + m^2)\phi = 0. \tag{1.76}$$

We have checked that the Lagrangian for the equation of motion is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2). \tag{1.77}$$

If we introduce a complex scalar field, that can be constructed as a linear combination of two real scalar fields as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{1.78}$$

$\phi^* \neq \phi$.

64. Using the fact that ϕ_1 and ϕ_2 are satisfying the Klein-Gordon equation, show that ϕ and ϕ^* also satisfies the Klein-Gordon equation.

$$(\partial^2 + m^2)\phi = 0, \quad (\partial^2 + m^2)\phi^* = 0. \tag{1.79}$$

65. Show that the Lagrangian is

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi. \tag{1.80}$$

Now we can introduce the covariant derivative to the complex scalar field.

$$\mathcal{L} = (\mathcal{D}^\mu\phi)^\dagger(\mathcal{D}_\mu\phi) - m^2\phi^\dagger\phi, \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu. \tag{1.81}$$

66. Show that the Lagrangian is invariant under the gauge transformation

$$\phi \rightarrow U\phi, \quad U^\dagger U = 1, \tag{1.82a}$$

$$A^\mu \rightarrow UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger. \tag{1.82b}$$

67. The Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu. \tag{1.83a}$$

Show that the current J^μ is

$$J^\mu = +iq\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right], \tag{1.84}$$

by expanding the covariant derivative.

68. Show that the J^μ is conserved

$$\partial_\mu J^\mu = 0 \tag{1.85}$$

by using the Klein-Gordon equation.

69. Show that

$$Q = \int d^3\boldsymbol{x} J^0 = 0 \tag{1.86}$$

for a real scalar field $\phi^* = \phi$.

70. Show that the current becomes

$$J^\mu = q \times |N|^2 2p^\mu. \tag{1.87}$$

if we use the free-particle wavefunction $\phi = Ne^{-ip\cdot x}$ and $\phi^* = N^* e^{ip\cdot x}$, where N is the normalization factor.

71. Show that, if $\int d^3\boldsymbol{x} = V$,

$$Q = \int d^3\boldsymbol{x} J^0 = q \times |N|^2 2EV. \tag{1.88}$$

72. Choose the covariant normalization $N = 1/\sqrt{V}$ and show that the charge inside the volume $V = \int d^3\boldsymbol{x}$ is

$$Q = \int d^3\boldsymbol{x} J^0 = q \times 2E. \tag{1.89}$$

73. q is the charge of the particle ϕ.

74. Show that $2E$ is the number of particles in V.

75. Show that the charge Q in V is not Lorentz invariant.

76. The number of particles in V is $2m$ if the particle is at rest. Explain why the density is increasing with a factor E/m compared to that for the rest frame because of the length contraction.

77. **Negative energy solution to the Klein-Gordon equation**: Let us go back to the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0. \tag{1.90}$$

Show that there are two solutions

$$\phi_+ = Ne^{-ip\cdot x}, \quad \phi_- = Ne^{+ip\cdot x} \tag{1.91}$$

where $p = (E, \boldsymbol{p})$ with $E = \sqrt{\boldsymbol{p}^2 + m^2} > 0$.

78. Show that $J^\mu(\phi = \phi_+) = q \times 2p^\mu |N|^2$ and $J^\mu(\phi = \phi_-) = -q \times 2p^\mu |N|^2$.

79. Show that $J^\mu(\phi = \phi_-) = q \times 2(-p^\mu)$ means that a negative energy particle with charge $+q$ is flowing from the future to the past.

80. Show that $J^\mu(\phi = \phi_-) = (-q) \times 2p^\mu$ means that a positive energy particle with charge $-q$ is flowing from the past to the future.

81. Combining the two equivalent statements, we conclude as follows. Once we know how to deal with a scalar particle with charge $+q$, the wavefunction for a positive-energy scalar particle with charge $-q$ can be described in terms of the negative-energy scalar particle with charge $+q$ flowing backward!

1.7 Time-dependent perturbation theory

82. Consider

$$\begin{aligned} H &= H_0 + V, \quad H_0 = \frac{\boldsymbol{p}^2}{2m}, && (1.92a) \\ H_0\phi_n &= E_n\phi_n, \quad \int d^3\boldsymbol{x}\phi_m^*(\boldsymbol{x})\phi_n(\boldsymbol{x}) = \delta_{mn}. && (1.92b) \end{aligned}$$

Write the wavefunction ψ in terms of the eigenfunctions of the unperturbed Hamiltonian as

$$\psi = \sum_n a_n(t)\phi_n e^{-iE_nt}. \tag{1.93}$$

Solve a_n satisfying

$$i\frac{\partial}{\partial t}\psi = (H_0 + V)\psi \tag{1.94}$$

to get

$$i\sum_n \frac{\partial a_n(t)}{\partial t}\phi_n(\boldsymbol{x})e^{-iE_nt} = \sum_n a_n(t)V(t,\boldsymbol{x})\phi_n(\boldsymbol{x})e^{-iE_nt}. \tag{1.95}$$

83. Show that $a_n(t \to -\infty) = \delta_{ni}$ means the initial state is monochromatic

$$\psi(t = -\infty) = \phi_i(\boldsymbol{x})e^{-iE_it}. \tag{1.96}$$

84. Convoluting

$$\int d^3\boldsymbol{x}\phi_f^\dagger(\boldsymbol{x})e^{iE_ft}, \tag{1.97}$$

show that

$$a_{fi} = \delta_{fi} - i\int dtd^3\boldsymbol{x}\phi_f^\dagger(\boldsymbol{x})V(t,\boldsymbol{x})\phi_i(\boldsymbol{x})e^{i(E_f-E_i)t} \tag{1.98}$$

85. In short

$$\begin{aligned} \psi(t\to\infty) &= S\psi(t\to-\infty), && (1.99a) \\ S &= 1 + i\mathcal{T}, && (1.99b) \\ i\mathcal{T} &= -i\int d^4x\phi_f^*(x)V\phi_i(x) = i\int d^4x\mathcal{L}_{\text{int}} = L_{\text{int}}. && (1.99c) \end{aligned}$$

86. Show that, if $\mathcal{T}$ is Hermitian

$$\mathcal{T}^\dagger = \mathcal{T} \leftrightarrow L_{\text{int}}, \tag{1.100}$$

S is unitarity $S^\dagger S = 1$. Therefore,

$$\psi^\dagger\psi(t = \infty) = \psi^\dagger\psi(t = -\infty). \tag{1.101}$$

87. **Energy-momentum conservation**: If the potential is independent of time $V(t, \boldsymbol{x}) = V(\boldsymbol{x})$, iT_{fi} becomes

$$iT_{fi} = -iV_{fi} \int_{-\infty}^{\infty} dt e^{i(E_f - E_i)t} = -iV_{fi}(2\pi)\delta(E_f - E_i), \tag{1.102a}$$

$$V_{fi} = \int d^3\boldsymbol{x} \phi_f^\dagger(\boldsymbol{x}) V(\boldsymbol{x}) \phi_i(\boldsymbol{x}), \tag{1.102b}$$

and the energy is conserved.

$$\frac{\boldsymbol{p}_f^2}{2m} = \frac{\boldsymbol{p}_i^2}{2m}. \tag{1.103}$$

88. In general, $\boldsymbol{p}_f \neq \boldsymbol{p}_i$. Where is the lost momentum, $\boldsymbol{p}_i - \boldsymbol{p}_f$? It has been transferred to the potential because

$$V(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} V(\boldsymbol{k}) e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.104a}$$

$$\phi_f^\dagger(\boldsymbol{x}) = N_f e^{+i\boldsymbol{p}_f\cdot\boldsymbol{x}}, \tag{1.104b}$$

$$\phi_i(\boldsymbol{x}) = N_i e^{-i\boldsymbol{p}_i\cdot\boldsymbol{x}}. \tag{1.104c}$$

89. Show that

$$V_{fi} = N_f N_i \int d^3\boldsymbol{k} V(\boldsymbol{k}) \delta(\boldsymbol{k} + \boldsymbol{p}_f - \boldsymbol{p}_i) = N_f N_i V(\boldsymbol{p}_i - \boldsymbol{p}_f). \tag{1.105}$$

Therefore, momentum is conserved $\boldsymbol{k} + \boldsymbol{p}_f = \boldsymbol{p}_i$.

1.8 Propagator

90. **Propagator for a scalar particle**: Let us recall the transition matrix

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}}. \tag{1.106}$$

We choose $\mathcal{L}_{\text{int}} = -\phi^\dagger J$ which is analogous to the electromagnetic interaction lagrangian $-J_\mu A^\mu$. Resulting lagrangian for a scalar field is

$$\mathcal{L} = \phi^\dagger \left(-\partial^2 - m^2\right) \phi - \phi^\dagger J, \tag{1.107}$$

where we neglect the surface term. Then the wave equation becomes

$$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} = 0 \;\rightarrow\; \left(\partial^2 + m^2\right) \phi = -J. \tag{1.108}$$

Show that

$$\phi(x) = \int dy^4 \Delta_F(x - y) J(y), \tag{1.109}$$

if

$$\left(\partial^2 + m^2\right) \Delta_F(x) = -\delta(x). \tag{1.110}$$

Hint: Act $\partial^2 + m^2$ to both sides.

91. Show that

$$\begin{aligned} i\mathcal{T} &= i\int d^4x\mathcal{L}_{\text{int}} = -i\int d^4x J(x)\phi(x) && (1.111a) \\ &= -i\int d^4x d^4y J(x)\Delta_F(x-y)J(y) && (1.111b) \\ &= \int d^4x d^4y\,[-iJ(x)]\,[i\Delta_F(x-y)]\,[-iJ(y)] && (1.111c) \end{aligned}$$

92. $i\Delta_F(x-y)$ is the Feynman propagator. It describes the propagation of a scalar field from a space-time point y to x.

93. Show that

$$\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\frac{e^{-ip^0x^0}}{p^0-E+i\epsilon} = -i\;e^{-iEx^0}, \quad \text{if} \quad x^0 > 0, \tag{1.112}$$

Hint: Use Cauchy integral formula. Explain why the above integral is same as the complex integral along a contour closed on the lower-half plain.

94. Show that

$$\int\frac{dp^0}{2\pi}\frac{e^{-ip^0x^0}}{p^0+E-i\epsilon} = +i\;e^{iEx^0}, \quad \text{if} \quad x^0 < 0, \tag{1.113}$$

Explain why the above integral is same as the complex integral along a contour closed on the upper-half plain.

95. Show that

$$\begin{aligned} \int\frac{dp^0}{2\pi}\frac{e^{-ip^0x^0}}{p^2-m^2+i\epsilon} &= \int\frac{dp^0}{2\pi}\frac{e^{-ip^0x^0}}{(p^0)^2-(E^2-i\epsilon)} \\ &= \int\frac{dp^0}{2\pi}\frac{e^{-ip^0x^0}}{(p^0+E-i\epsilon)(p^0-E+i\epsilon)} \\ &= \frac{-i}{2E}\left(\theta(x^0)e^{-iEx^0}+\theta(-x^0)e^{+iEx^0}\right) && (1.114a) \end{aligned}$$

where $E=\sqrt{\boldsymbol{p}^2+m^2}>0$, $\epsilon\to 0^+$ and

$$\theta(x) = \begin{cases} 1 \text{ if } x>0 \\ 0 \text{ if } x<0 \end{cases} \tag{1.115}$$

96. Explain why the sign of $i\epsilon$ is important.

97. Explain what is wrong if ϵ is finite.

98. Show that

$$\begin{aligned} &\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot x} && (1.116a) \\ =& \int\frac{d^3\boldsymbol{p}}{2E(2\pi)^3}\left[\theta(x^0)e^{-i(Et-\boldsymbol{p}\cdot\boldsymbol{x})}+\theta(-x^0)e^{i(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\right] && (1.116b) \\ =& \int\frac{d^3\boldsymbol{p}}{2E(2\pi)^3}\left[\theta(x^0)e^{-ip\cdot x}+\theta(-x^0)e^{ip\cdot x}\right] && (1.116c) \end{aligned}$$

where $E=\sqrt{m^2+\boldsymbol{p}^2}>0$ and $p^0=E$ on the last line.

99. Show that the retarded term $\theta(x^0)$ is for the particle propagating to the future.

100. Show that the advanced term $\theta(-x^0)$ term is for the particle to the past.

101. Show that

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} \tag{1.117}$$

is the solution to the equation

$$\left(\partial^2 + m^2\right) \Delta_F(x) = -\delta(x) \tag{1.118}$$

1.9 Photon propagator

102. The $i\mathcal{T}$ matrix for a photon field is

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}} = -i \int d^4x J^\mu A_\mu, \tag{1.119}$$

where the equation of motion is Maxwell's equation

$$\partial_\nu F^{\nu\mu} = J^\mu; \quad \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu = J^\mu. \tag{1.120}$$

103. We want to find the solution to

$$\left(\partial^2 g^{\mu\alpha} - \partial^\mu \partial^\alpha\right) (D_F)_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x). \tag{1.121}$$

$$A^\mu(x) = \int dy^4 D_F^{\mu\nu}(x-y) J_\nu(y) \tag{1.122a}$$

$$\text{if} \qquad \left(\partial^2 g^{\mu\alpha} - \partial^\mu \partial^\alpha\right) (D_F)_{\alpha\nu} = g^\mu{}_\nu \delta(x). \tag{1.122b}$$

104. Show that

$$\begin{aligned}
\mathcal{T} &= i \int d^4x \mathcal{L}_{\text{int}} = -i \int d^4x J_\mu(x) A^\mu(x) && (1.123)\\
&= -i \int d^4x d^4y J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) && (1.124)\\
&= \int d^4x d^4y \left[-iJ_\mu(x)\right] \left[iD_F^{\mu\nu}(x-y)\right] \left[-iJ_\nu(y)\right] && (1.125)
\end{aligned}$$

$i\Delta_F^{\mu\nu}(x-y)$ is the Feynman propagator. It describes the propagation of a vector field from a space-time point y to x.

105. **Propagator in the Feynman gauge**: If we choose the Lorentz gauge, $\partial \cdot A = 0$, and we may neglect the term $\partial^\mu \partial^\nu$ terms in the wave equations. Maxwell's equation becomes

$$\partial^2 A^\mu = J^\mu. \tag{1.126}$$

Show that

$$D_F^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot x} = -g^{\mu\nu} \Delta_F(x) \text{ with } m = 0 \tag{1.127}$$

is the solution to the equation

$$\partial^2 g^{\mu\alpha} (D_F)_{\alpha\nu} = g^\mu{}_\nu \delta(x). \tag{1.128}$$

106. Show that the wave equation

$$\partial^2 A^\mu = J^\mu \tag{1.129}$$

is actually from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha} (\partial \cdot A)^2 \tag{1.130}$$

where $\alpha = 1$. The term added to the original Lagrangian is the gauge-fixing term.

107. The photon propagator

$$D_F^{\mu\nu}(x) = -g^{\mu\nu} \Delta_F(x) \tag{1.131}$$

is defined in the Feynman gauge, a special case of the Lorentz gauge.

108. **Propagator in the Lorentz gauge** Show that the equation of motion for the photon field in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha} (\partial \cdot A)^2 \tag{1.132}$$

is

$$\left[\partial^2 g^{\mu\alpha} + \left(\frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] A_\alpha = J^\mu \tag{1.133}$$

and the propagator D must satisfy

$$\left[\partial^2 g^{\mu\alpha} + \left(\frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x). \tag{1.134}$$

Note that we are using the Lorentz gauge $\partial \cdot A = 0$.

109. **Propagator in the Lorentz gauge** Show that

$$D^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\mu\nu} + (1-\alpha) \dfrac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} e^{-ip\cdot x} \tag{1.135}$$

$$= \int \frac{d^4p}{(2\pi)^4} \left[-g^{\mu\nu} + (1-\alpha) \frac{p^\mu p^\nu}{p^2} \right] \Delta_F(p) e^{-ip\cdot x} \tag{1.136}$$

$$\Delta_F(p) = \frac{1}{p^2 + i\epsilon} \tag{1.137}$$

is the solution to the equation

$$\left[\partial^2 g^{\mu\alpha} + \left(\frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x) \tag{1.138}$$

110. **Propagator in the axial gauge** We can choose the axial gauge $n \cdot A = 0$. In this case, gauge fixing term can be written in the form to have our Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha}(n \cdot A)^2 . \tag{1.139}$$

111. Show that

$$\begin{aligned} iD^{\mu\nu}(x) &= i\int \frac{d^4p}{(2\pi)^4}\left[-g^{\mu\nu} + \frac{n^\mu p^\nu + p^\mu n^\nu}{n\cdot p} - (n^2 + \alpha p^2)\frac{p^\mu p^\nu}{(n\cdot p)^2}\right]\Delta_F(p)e^{-ip\cdot x} && (1.140) \\ \Delta_F(p) &= \frac{1}{p^2 + i\epsilon} && (1.141) \end{aligned}$$

is the gluon propagator in the axial gauge.

112. **Massive spin-1 propagator** The Lagrangian for a massive spin-1 field is just like that for the photon except that the particle has nonvanishing mass. We insert the mass term $\frac{m^2}{2}B^\mu B_\mu$

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}B^\mu B_\mu - J^\mu B_\mu && (1.142) \\ F^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\nu && (1.143) \end{aligned}$$

113. Show that the equation of motion is

$$\left[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right]B_\nu = J^\mu. \tag{1.144}$$

114. The $\mathcal{T}$ matrix for a massive spin-1 field is

$$i\mathcal{T} = i\int d^4x\mathcal{L}_{\text{int}} = -i\int d^4x J^\mu B_\mu, \tag{1.145}$$

where the equation of motion is

$$\left[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right]B_\nu = J^\mu. \tag{1.146}$$

115. We want to find the solution to

$$\left[(\partial^2 + m^2)g^{\mu\alpha} - \partial^\mu\partial^\alpha\right]D_{\alpha\nu}(x) = g^\mu{}_\nu\delta(x). \tag{1.147}$$

$$\begin{aligned} B^\mu(x) &= \int dy^4 D^{\mu\nu}(x-y)J_\nu(y) && (1.148) \\ \text{if} \quad & \left[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu\right]D_{\alpha\nu}(x) = g^\mu{}_\nu\delta(x) && (1.149) \end{aligned}$$

Show that

$$\begin{aligned} i\mathcal{T} &= i\int d^4x\mathcal{L}_{\text{int}} = -i\int d^4x J_\mu(x)B^\mu(x) && (1.150) \\ &= -i\int d^4x d^4y J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) && (1.151) \\ &= \int d^4x d^4y \left[-iJ_\mu(x)\right]\left[iD^{\mu\nu}(x-y)\right]\left[-iJ_\nu(y)\right] && (1.152) \end{aligned}$$

116. Show that

$$D^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{-g^{\mu\nu} + \dfrac{p^\mu p^\nu}{m^2}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot x} \tag{1.153}$$

$$= \int \frac{d^4p}{(2\pi)^4} \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right] \Delta_F(p) e^{-ip\cdot x} \tag{1.154}$$

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} \tag{1.155}$$

is the solution to the equation

$$\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu\right] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x) \tag{1.156}$$

Note that $p^2 \neq m^2$ in general.

1.10 Feynman rules

Momentum-space Feynman rule Let us go back to the $\mathcal{T}$ matrix

$$i\mathcal{T} = i\int d^4x \mathcal{L}_{\text{int}} \tag{1.157}$$

$$= \int d^4x d^4y \, [-iJ(x)] \cdot [iD(x-y)] \cdot [-iJ(y)] \tag{1.158}$$

where indices are suppressed for vector particles.

117. Expand the currents as

$$J(y) = \int \frac{d^4k}{(2\pi)^4} J(k) e^{-ik\cdot y} \tag{1.159}$$

and show that

$$i\mathcal{T} = i\int d^4x \mathcal{L}_{\text{int}} \tag{1.160}$$

$$= \int d^4x d^4y \, [-iJ(x)] \cdot [iD(x-y)] \cdot [-iJ(y)] \tag{1.161}$$

$$= \int \frac{d^4p}{(2\pi)^4} \, [-iJ(-p)] \cdot [iD(p)] \cdot [-iJ(p)] \, . \tag{1.162}$$

118. Consider monochromatic currents

$$J(x) = J(p_1) = N^2 \hat{J}(p_1) e^{-ip_1\cdot x} \tag{1.163}$$

$$J(y) = J(p_2) = N^2 \hat{J}(p_2) e^{-ip_2\cdot y} \tag{1.164}$$

where N^2 came from the normalization of the initial and final states involving the current. Show that in this case

$$i\mathcal{T} = i\int d^4x \mathcal{L}_{\text{int}} = N^4 (2\pi)^4 \delta^4(p_1 + p_2) \mathcal{M} \tag{1.165}$$

$$\mathcal{M} = \left[-i\hat{J}(p_2)\right] \cdot [iD(p = p_1 = -p_2)] \cdot \left[-i\hat{J}(p_1)\right] \tag{1.166}$$

119. Show that the propagators for a scalar field and massless/massive vector fields are even functions;

$$D(x) = D(-x), \quad D(p) = D(-p) \tag{1.167}$$

Hint: Look into the partial differential equation for the propagator.

120. Show that

$$\begin{aligned} |\mathcal{T}|^2 &= \left| i \int d^4x \mathcal{L}_{\text{int}} \right|^2 = N^8 \left[(2\pi)^4 \delta^4(p_1 + p_2) \right]^2 |\mathcal{M}|^2 & (1.168) \\ &= N^8 \times VT \times (2\pi)^4 \delta^4(p_1 + p_2) |\mathcal{M}|^2 & (1.169) \\ \mathcal{M} &= \left[-i\hat{J}(p_2) \right] \cdot \left[iD(p = \pm p_1 \text{ or } \pm p_2) \right] \cdot \left[-i\hat{J}(p_1) \right] & (1.170) \end{aligned}$$

Hint: $(2\pi)^4\delta^4(0) = \int dt d^3\boldsymbol{x}$.

121. Show that the probability of the transition per unit volume per unit time is

$$\begin{aligned} P &= \frac{|\mathcal{T}|^2}{TV} = \frac{1}{V^4} \times (2\pi)^4 \delta^4(p_1 + p_2) |\mathcal{M}|^2 & (1.171) \\ \mathcal{M} &= \left[-i\hat{J}(p_2) \right] \cdot \left[iD(p = \pm p_1 \text{ or } \pm p_2) \right] \cdot \left[-i\hat{J}(p_1) \right] & (1.172) \end{aligned}$$

where we used $N = 1/\sqrt{V}$.

122. **Cross section** Show that the number density of a particle with energy E in V is

$$\frac{2E}{V}. \tag{1.173}$$

123. Show that the flux of the two colliding particle is

$$\begin{aligned} F &= |\boldsymbol{v}_a - \boldsymbol{v}_b| \times \frac{2E_a}{V} \times \frac{2E_b}{V} & (1.174) \\ &= 4\left(|\boldsymbol{p}_a| E_b + |\boldsymbol{p}_b| E_a \right) / V^2 & (1.175) \\ &= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} / V^2 & (1.176) \end{aligned}$$

where $\boldsymbol{v}_i$, $\boldsymbol{p}_i$, and E_i are the velocity, momentum and energy of the $i-$th particle.

124. Show that the number of states in the final state is

$$\begin{aligned} dN_{\text{final states}} &= V\theta(p_1^0)\delta(p_1^2 - m_1^2) \frac{d^4p_1}{(2\pi)^4} V\theta(p_2^0)\delta(p_2^2 - m_2^2) \frac{d^4p_2}{(2\pi)^4} \\ &= \frac{Vd^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{Vd^3\boldsymbol{p}_2}{2E_2(2\pi)^3} & (1.177) \end{aligned}$$

if there are two final particles.

125. Cross section is defined by

$$\begin{aligned}
d\sigma &= \frac{P}{F} \times dN_{\text{final states}} \\
&= \frac{1}{V^4}(2\pi)^4 \delta(p_a + p_b - p_1 - p_2)|\mathcal{M}|^2 \\
\times & \frac{V^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} \times \frac{V d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{V d^3\boldsymbol{p}_2}{2E_2(2\pi)^3} \\
&= \frac{|\mathcal{M}|^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}(2\pi)^4 \delta(p_a + p_b - p_1 - p_2) \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}. \qquad (1.178)
\end{aligned}$$

126. We usually define the phase space $d\Phi$ after including the energy-mpmentum delta function

$$d\Phi = (2\pi)^4 \delta\left(P - \sum_i p_i\right) dN_{\text{final states}}, \qquad (1.179)$$

where P is the sum of initial momenta p_i is the momentum of the $i-$th final-state particle.

127. We find V dependence exactly cancels. If we redefine the phase space

$$d\Phi = (2\pi)^4 \delta(p_a + p_b - p_1 - p_2) \prod_{i=1}^{n} \frac{d^3\boldsymbol{p}_i}{2E_i(2\pi)^3} \qquad (1.180)$$

we have

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} d\Phi \qquad (1.181)$$

128. Show that the mass term $\frac{1}{2}m^2 A^\mu A_\mu$ for a gauge field is NOT invariant under gauge transformation

$$A^\mu \to U A^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger, \quad \psi \to U\psi, \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu \qquad (1.182)$$

This guarantees that the gauge field is massless.

129. Show that gauge field is travelling with the speed of light.

1.11 Dirac equation

130. We would like to construct a relativistically covariant theory for a fermion. If we ignore the spin, the equation must reduce to the Klein-Gordon equation. But we want to have an equation which has linear time derivative instead of $\partial/\partial t^2$, which appears in the Klein-Gordon equation.

$$i\frac{\partial \psi}{\partial t} = H\psi, \quad H = -i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m \qquad (1.183)$$

131. Show that ψ must include the four states

$$|1\rangle = |\uparrow, E > 0\rangle,\ |2\rangle = |\downarrow, E > 0\rangle, \tag{1.184}$$
$$|3\rangle = |\uparrow, E < 0\rangle,\ |4\rangle = |\downarrow, E < 0\rangle \tag{1.185}$$

132. Show that H is Hermitian.

133. Show that

$$\langle m|H|n\rangle = \sqrt{\boldsymbol{p}^2 + m^2}, \quad \text{if} \quad m = n = 1, 2 \tag{1.186}$$
$$\langle m|H|n\rangle = -\sqrt{\boldsymbol{p}^2 + m^2}, \quad \text{if} \quad m = n = 3, 4 \tag{1.187}$$
$$\langle m|H|n\rangle = 0, \quad \text{if} \quad m \neq n \tag{1.188}$$
$$\sum_n \langle n|H|n\rangle = 0 \tag{1.189}$$

134. Show that α^1, α^2, α^3, and β are Hermitian.

135. If the Dirac equation is equivalent to the Klein-Gordon equation if we neglect the spin dependence and the sign of the energy, show

$$H^2 = -\boldsymbol{\nabla}^2 + m^2 \quad \rightarrow \quad (\partial^2 + m^2)\psi = 0 \tag{1.190}$$

136. Show that the condition requires

$$\frac{1}{2}\left(\alpha^i\alpha^j + \alpha^j\alpha^i\right) = \delta^{ij} \tag{1.191}$$
$$\alpha^i\beta + \beta\alpha^i = 0 \tag{1.192}$$
$$\beta^2 = 1 \tag{1.193}$$

137. Show that the conditions $(\alpha^i)^2 = 1$ and $\beta^2 = 1$ require that the eigenvalues for the matirces are ± 1.
From now on we will use the notation

$$\{A, B\} \equiv AB + BA. \tag{1.194}$$

138. Let us choose the basis so that the first two components are positive-energy components and the other two are negative-energy components. By taking $\boldsymbol{p} = 0$, show that

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.195}$$

139. From now on we express any 4×4 matrix in spinor space in terms of 2×2 block matrices.

140. Show that $\beta^2 = 1$.

141. Let us try arbitrary 4×4 matrices

$$\alpha^i = \begin{pmatrix} a^i & b^i \\ c^i & d^i \end{pmatrix} \tag{1.196}$$

142. Show that the condition $\{\beta, \alpha^i\} = 0$ requires

$$\alpha^i = \begin{pmatrix} 0 & b^i \\ c^i & 0 \end{pmatrix} \tag{1.197}$$

143. Show that the condition $\alpha^\dagger = \alpha$ requires $c = b^\dagger$.

144. Show that the condition $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$ requires

$$b^i(b^j)^\dagger + b^j(b^i)^\dagger = 2\delta^{ij} \tag{1.198}$$
$$(b^i)^\dagger b^j + (b^j)^\dagger b^i = 2\delta^{ij} \tag{1.199}$$

145. Choose the term $i = j$ to find b^i is unitary

$$b^i(b^i)^\dagger = (b^i)^\dagger b^i = 1 \tag{1.200}$$

The solution is the $3 = 2^2 - 1$ SU(2) generators, Pauli matrices; $b^i = \sigma^i$, where Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.201}$$

Therefore, we have

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{1.202}$$

146. Show that

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{1.203}$$
$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \tag{1.204}$$
$$\left[\sigma^i, \sigma^j\right] \equiv \sigma^i\sigma^j - \sigma^j\sigma^i = +2i\epsilon^{ijk}\sigma^k \tag{1.205}$$

147. Show that

$$\boldsymbol{a} \cdot \boldsymbol{\sigma}\ \boldsymbol{b} \cdot \boldsymbol{\sigma} = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{\sigma} \tag{1.206}$$

148. Show that

$$\boldsymbol{a} \cdot \boldsymbol{\alpha}\ \boldsymbol{b} \cdot \boldsymbol{\alpha} = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{\Sigma}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{1.207}$$

149. Defining $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma}) = (\beta, \beta\boldsymbol{\alpha})$, show that

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{1.208}$$

150. Show that the anticommutation relations for β and α^i's reduce to a single formula

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1.209}$$

where the 4×4 identity matrix

151. Let us define

$$\not{a} \equiv \gamma^\mu a_\mu \tag{1.210}$$

Show that

$$\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b, \quad \not{a}^2 = a^2 \tag{1.211}$$

152. Show that

$$(\gamma^\mu)^\dagger = \gamma_\mu, \quad (\gamma_\mu)^\dagger = \gamma^\mu \tag{1.212}$$

153. Show that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \quad \gamma^0(\gamma_\mu)^\dagger\gamma^0 = \gamma_\mu \tag{1.213}$$

Multiply β to the original Dirac equation and find

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{1.214}$$

Taking the Hermitian adjoint and find

$$-i\partial_\mu\bar{\psi}\gamma^\mu - \bar{\psi}m = 0 \tag{1.215}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$.

154. Show that the Euler-Lagrange equation for the ψ field in the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{1.216}$$

gives the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{1.217}$$

155. Show that the Lagrangian can also be written as

$$\mathcal{L} = -i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi \tag{1.218}$$

if we neglect the surface term.

156. Show that the Euler-Lagrange equation for $\psi^\dagger$ field is

$$-i\partial_\mu\bar{\psi}\gamma^\mu - \bar{\psi}m = 0 \tag{1.219}$$

157. Show that the Lagrangian for the Quantum electrodynamics

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} && (1.220)\\ &= -i\left(\mathcal{D}_\mu\psi\right)^\dagger \gamma^0\gamma^\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} && (1.221)\end{aligned}$$

is invariant under the gauge transformation

$$\psi \to U\psi,\ A^\mu \to UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger,\ \mathcal{D}^\mu = \partial^\mu + iqA^\mu \tag{1.222}$$

158. Introducing the covariant derivative $\mathcal{D}^\mu = \partial^\mu + iqA^\mu$, show that

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - J^\mu A_\mu && (1.223)\\ J^\mu &= q\bar{\psi}\gamma^\mu\psi && (1.224)\end{aligned}$$

159. Making use of the Dirac equations

$$i\gamma^\mu\partial_\mu\psi = m\psi, \qquad -i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu = m\bar{\psi} \tag{1.225}$$

show that the current is conserved

$$\partial_\mu J^\mu = 0 \tag{1.226}$$

1.12 Spinor

160. Let us consider an electron with momentum p^μ and $z-$compenent spin s, where $p^0 = E > 0$. Show that the wavefunction $\psi(x) = u(p,s)e^{-ip\cdot x}$ should be normalized to be

$$\psi^\dagger(x)\psi(x) = 2E \to u^\dagger(p,s)u(p,s') = 2E\delta_{ss'}. \tag{1.227}$$

161. Using the length contraction, show that the number of electron in the whole space of volume V is 1.

162. Show that

$$\bar{u}(p)u(p) = 2m \tag{1.228}$$

using $u^\dagger(p,s)u(p,s') = 2m\delta_{ss'}$ and Lorentz covariance only.

163. Remind the fact that $\bar{\psi}\gamma^\mu\psi$ is transforming like a four vector. Using the result $u^\dagger(p,s)u(p,s) = 2E$ and Lorentz covariance only, show that

$$\bar{u}(p)\gamma^\mu u(p) = 2p^\mu. \tag{1.229}$$

164. Using spinors for an electron at rest $p = (m, \mathbf{0})$, show that

$$\sum_s u(p,s)\bar{u}(p,s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+\gamma^0). \tag{1.230}$$

165. Using Lorentz covariance, show that

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{\not{p}+m}{2m}. \tag{1.231}$$

166. Let us consider a negative-energy electron with momentum $-p^\mu$ and $z-$compenent spin s, where $p^0 = E > 0$. Show that the wavefunction $\psi(x) = u(-p,s)e^{+ip\cdot x}$ should be normalized to be

$$\psi^\dagger(x)\psi(x) = 2E \to u^\dagger(-p,s)u(-p,s') = 2E\delta_{ss'}. \tag{1.232}$$

167. Explain why it is not proportional to $-2E < 0$ but proportional to $2E > 0$.

168. Using the length contraction, show that the number of electron in the whole space of volume V is 1.

169. Show that

$$\bar{u}(-p,s)u(-p,s') = -2m\delta_{ss'} \tag{1.233}$$

using $u^\dagger(-p,s)u(-p,s') = 2m\delta_{ss'}$ and Lorentz covariance only.

170. Show that

$$\sum_s u(-p,s)\bar{u}(-p,s) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}(-1+\gamma^0). \tag{1.234}$$

171. Using Lorentz covariance, show that

$$\sum_s u(-p,s)\bar{u}(-p,s) = \frac{\not{p}-m}{2m}. \tag{1.235}$$

172. Explain why this formula cannot be obtained if we substitute $p \to -p$ to the postive energy case

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{\not{p}+m}{2m}. \tag{1.236}$$

173. Following the results for the charged scalar field, we would like to make use of the negative-energy solution for the positive energy antiparticle with momentum $-(-p) = p$. Then the wavefunction for the antiparticle with momentum p can be $\bar{v}(p) \equiv \bar{u}(-p)$. Show that

$$\begin{aligned} v^\dagger(p,s)v(p,s') &= 2E\delta_{ss'}, && (1.237a)\\ \bar{v}(p,s)v(p,s') &= -2m\delta_{ss'}, && (1.237b)\\ \sum_s v(p,s)\bar{v}(p,s) &= \frac{\not{p}-m}{2m}, && (1.237c) \end{aligned}$$

where $p = (E, \boldsymbol{p})$ and $E = \sqrt{\boldsymbol{p}^2+m^2} > 0$.

174. Wavefunction for a positive-energy antiparticle with momentum p^μ is

$$\bar{v}(p)e^{-ip\cdot x}. \tag{1.238}$$

175. For the final state, we use

$$v(p)e^{+ip\cdot x}. \tag{1.239}$$

176. Using Dirac equation, show that

$$\begin{aligned}
(\not{p}-m)u(p) &= 0, && (1.240a)\\
\bar{u}(p)(\not{p}-m) &= 0, && (1.240b)\\
(\not{p}+m)v(p) &= 0, && (1.240c)\\
\bar{v}(p)(\not{p}+m) &= 0. && (1.240d)
\end{aligned}$$

177. Using the fact that $-p \to p$ in the replacement $v(p) = u(-p)$, explain why the spin-up positive-energy positron state is expressed in terms of spin-down negative-energy electron state.

178. Formal way to prove this is involving charge-conjuagtion operation with the transformation matrix

$$i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.241}$$

to the spin-half Lagrangian.

179. Replacing the derivative in Dirac equation by the covariant derivative $\mathcal{D}^\mu = \partial^\mu + iqA^\mu$, show that

$$(i\not{\partial} - q\not{A} - m)\,\psi = 0. \tag{1.242}$$

Taking complex conjugate, show that

$$(-i\not{\partial}^* - q\not{A}^* - m)\,\psi^* = 0 \to [-\gamma^{\mu*}\,(i\partial_\mu + qA_\mu) - m]\,\psi^* = 0, \tag{1.243}$$

where $A_\mu^* = A_\mu$.

180. Using the fact

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{1.244}$$

and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.245}$$

show that

$$-\gamma^{\mu*} = \begin{cases} -\gamma^\mu & \text{if } \mu \neq 2, \\ +\gamma^\mu & \text{if } \mu = 2. \end{cases} \tag{1.246}$$

181. Show for

$$U = i\gamma^2, \tag{1.247}$$

that

$$U^2 = 1 \to U^{-1} = U^\dagger = U, \tag{1.248a}$$
$$U^{-1}\gamma^\mu U = -\gamma^\mu \to U\gamma^\mu = -\gamma^\mu U \text{ if } \mu \neq 2, \tag{1.248b}$$
$$U^{-1}\gamma^\mu U = \gamma^\mu \to U\gamma^\mu = \gamma^\mu U \text{ if } \mu = 2, \tag{1.248c}$$

Therefore,

$$i\gamma^2\left(-\gamma^{\mu*}\right) = \gamma^\mu\left(i\gamma^2\right). \tag{1.249}$$

182. Show that

$$i\gamma^2\left[-\gamma^{\mu*}\left(i\partial_\mu + qA_\mu\right) - m\right]\psi^* = 0 \to \left[\gamma^\mu\left(i\partial_\mu + qA_\mu\right) - m\right]\left(i\gamma^2\psi^*\right) = 0. \tag{1.250}$$

Therefore, $i\gamma^2\psi^*$ is the wavefunction for the antiparticle.

1.13 Fermion propagator

183. Show that the Dirac

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = J \tag{1.251}$$

Be careful with the sign in front of the source term on the right-hand side; $H = i\partial^0$.

184. Show that the propagator satisfies the equation

$$\left(i\gamma^\mu\partial_\mu - m\right)S_F(x) = I\delta(x) \tag{1.252}$$

where I is the 4×4 identity matix in spinor space.

185. Replacing $S_F(x) = \left(i\gamma^\mu\partial_\mu + m\right)f(x)$, show that $f(x)$ satisfies the equation

$$\left(\partial^2 + m^2\right)f(x) = -\delta(x) \tag{1.253}$$

186. Show that the solution to the above equation is

$$S_F(x) = \left(i\gamma^\mu\partial_\mu + m\right)\Delta_F(x) \tag{1.254}$$

187. Show that

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4}\frac{\left(\not{p} + m\right)e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} \tag{1.255}$$

188. Show that the Dirac

$$\bar{\psi}\left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right) = J \tag{1.256}$$

where $A \overleftarrow{\partial}_\mu \equiv \partial_\mu A$. Be careful with the sign in front of the source term on the right-hand side; $H = i\partial^0$.

189. Show that the propagator satisfies the equation

$$S'_F(x)\left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right) = I\delta(x) \tag{1.257}$$

where S'_F is the propagator for the $\bar{\psi}$ field. We will show that $S'_F(x) \neq S_F(x)$

190. Replacing $S'_F(x) = g(x)\left(-i\gamma^\mu \overleftarrow{\partial}_\mu + m\right)$, show that $g(x)$ satisfies the equation

$$\left(\partial^2 + m^2\right) g(x) = -\delta(x) \tag{1.258}$$

191. Show that the solution to the above equation is

$$S'_F(x) = \Delta_F(x)\left(-i\gamma^\mu \overleftarrow{\partial}_\mu + m\right) \tag{1.259}$$

192. Show that

$$\begin{aligned} S'_F(x) &= \int \frac{d^4p}{(2\pi)^4}\frac{(-\not{p}+m)\,e^{-ip\cdot x}}{p^2-m^2+i\epsilon} && (1.260)\\ &= \int \frac{d^4p}{(2\pi)^4}\frac{(\not{p}+m)\,e^{ip\cdot x}}{p^2-m^2+i\epsilon} = S_F(-x) \neq S_F(x) && (1.261)\end{aligned}$$

$(e^{-ip\cdot x} \neq e^{ip\cdot x})$

193. Show that S_F is not an even function

$$S_F(-x) \neq S_F(x), S_F(-p) \neq S_F(p) \tag{1.262}$$

194. Show that

$$\begin{aligned} \psi(x) &= \int d^4y S_F(x-y)J(y) = \int d^4y S_F(x-y)[q\gamma^\mu A_\mu(y)]\psi(y), && (1.263)\\ \bar{\psi}(x) &= \int d^4y J(y)S_F(y-x) = \int d^4y \bar{\psi}(y)[q\gamma^\mu A_\mu(y)]S_F(y-x) && (1.264)\end{aligned}$$

195. Show that the positive-energy component propagates as

$$\begin{aligned} S_F(x) &= \theta(x^0)S_{\text{ret.}}(x) + \theta(-x^0)S_{\text{adv.}}(x) && (1.265)\\ S_{\text{ret.}}(x) &= -i\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\not{p}+m}{2E}e^{-ip\cdot x} && (1.266)\\ S_{\text{adv.}}(x) &= -i\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{-\not{p}+m}{2E}e^{ip\cdot x} && (1.267)\end{aligned}$$

where $p^0 = E = \sqrt{m^2+\boldsymbol{p}^2} > 0$ From now on we neglect the normalization factor $N = 1/\sqrt{V}$ which does not affect the invariant measurables like cross section.

196. **Transition amplitude** $\mathcal{M}$ Once we know the transition amplitude $\mathcal{M}$ in momentum space, we can calculate the cross section of a process.

197. For a scalar-exchange process

$$\mathcal{M} = \left[-i\hat{J}(-p)\right]\frac{i}{p^2-m^2+i\epsilon}\left[-i\hat{J}(p)\right] \tag{1.268}$$

198. For a photon-exchange process

$$\mathcal{M} = \left[-i\hat{J}_\mu(-p)\right] \frac{i\left[-g^{\mu\nu} + (1-\alpha)\dfrac{p^\mu p^\nu}{p^2}\right]}{p^2 + i\epsilon} \left[-i\hat{J}_\nu(p)\right] \tag{1.269}$$

199. For a massive-vector-exchange process

$$\mathcal{M} = \left[-i\hat{J}_\mu\right] \frac{i\left[-g^{\mu\nu} + \dfrac{p^\mu p^\nu}{m^2}\right]}{p^2 - m^2 + i\epsilon} \left[-i\hat{J}_\nu\right] \tag{1.270}$$

200. For a fermion-exchange process

$$\mathcal{M} = \left[-i\hat{J}\right] \frac{i\left(\not{p} + m\right)}{p^2 - m^2 + i\epsilon} \left[-i\hat{J}\right] \tag{1.271}$$

Chapter 2

QCD Lagrangian

2.1 QED Summary

1. In the previous chapter we wrote the QED interaction Lagrangian, derived Feynman rules, and learned how to write the amplitude. QED Lagrangian is given by

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

where the covariant derivative is defined by

$$\mathcal{D}^\mu = \partial^\mu + iqA^\mu \tag{2.2}$$

with $q = Qe$, $e = \sqrt{4\pi\alpha} > 0$ and $Q = -1$ for the electron.

2. Wavefunctions for incoming positive-energy electron and positron with momentum p are

$$e^- : u(p)e^{-ip\cdot x}, \quad e^+ : \bar{v}(p)e^{-ip\cdot x}. \tag{2.3}$$

3. Wavefunctions for outgoing positive-energy electron and positron with momentum p are

$$e^- : \bar{u}(p)e^{+ip\cdot x}, \quad e^+ : v(p)e^{+ip\cdot x}. \tag{2.4}$$

4. Wavefunction for incoming and outgoing photons with momentum k are

$$\text{in} : \epsilon^\mu(k)e^{-ik\cdot x}, \quad \text{out} : \epsilon^{\mu*}(k)e^{+ik\cdot x}. \tag{2.5}$$

5. Propagator for the electron with momentum p is

$$iS_F(p) = \frac{i}{\not{p} - m + i\epsilon}. \tag{2.6}$$

6. Propagator for the photon has various forms depending on the gauge, which does not change the observables. In the Feynman gauge, the propagator for a photon with momentum k is

$$iD_F^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2 + i\epsilon}. \tag{2.7}$$

7. The vertex factor can be read from the interaction Lagrangian $i\mathcal{L}_{\text{int}}$ as

$$\bar{e}A^\mu e = +iq\gamma^\mu, \ q = -e. \tag{2.8}$$

8. Show that once the coupling q is defined by the covariant derivative, the coupling is same inclusing sign for both electron and positron.

9. We can re-express the field strength tensor in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{1}{iq}\left[\mathcal{D}^\mu, \mathcal{D}^\nu\right], \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu. \tag{2.9}$$

10. Check the gauge invariance of the Lagrangian.

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{4(iq)^2}\text{Tr}\left(\left[\mathcal{D}^\mu, \mathcal{D}^\nu\right]\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right]\right), \tag{2.10}$$

under the gauge transformation

$$\psi \to U\psi, \quad \mathcal{D}^\mu \to U\mathcal{D}^\mu U^\dagger. \tag{2.11}$$

Note that the trace is for the 1×1 matrix U.

2.2 QCD Lagrangian

11. It is known that there are three ($N_c = 3$) color states for a quark, which is a spin-half particle. The color is independent of spin and momentum.

12. We can extend QED to treat this new degree of freedom by introducing gauge fields mediating the color force between any two quarks. This can be done by declaring the spinor field ψ has the wavefunction of the form

$$\psi_{\text{quark}} = \psi_{\text{quark}}(\text{spin}) \times \psi_{\text{quark}}(\text{color}). \tag{2.12}$$

13. We can introduce the gauge transform for the quark just like that for the electron as

$$\psi \to U\psi, \quad \mathcal{D}^\mu \to U\mathcal{D}^\mu U^\dagger. \tag{2.13}$$

Note that the matrix U is now a 3×3 matrix and acts only on the color wavefunction.

$$U\psi_{\text{quark}} = \psi_{\text{quark}}(\text{spin}) \times U\psi_{\text{quark}}(\text{color}). \tag{2.14}$$

14. We know that U must be unitary. Show that the matrix U can be parametrized by

$$U = e^{-i\sum_{a=1}^{N_c^2-1} T^a\alpha^a}, \tag{2.15}$$

where α^a is real and T^a's are the SU(N_c) generators.

15. Show that T^a is traceless and Hermitian.

16. Show that the number of generators for the SU(N_c) is $N_c^2 - 1$.
17. The covariant derivative can be generalized to the SU(N_c) as

$$\mathcal{D}^\mu = \partial^\mu + ig_s A^\mu, \tag{2.16}$$

where A^μ is the matrix-valued gluon field

$$A^\mu = A_a^\mu T^a. \tag{2.17}$$

Note that g_s is the strong coupling. Therefore, gluons can have $N_c^2 - 1 = 8$ different color states.

18. We can imagine the QCD Lagrangian should be of the form

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a, \tag{2.18}$$

because there are 8 gluons. Note that $G_a^{\mu\nu}$ is the field strength tensor for the gluon with color a, which is a QCD analogy of the photon field strength tensor.

19. Let us check the gauge invarince of the Lagrangian. We can use the fact

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}, \tag{2.19}$$

to derive

$$-\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a = -\frac{1}{2}\mathrm{Tr}\left(G^{\mu\nu}G_{\mu\nu}\right). \tag{2.20}$$

Again, $G^{\mu\nu}$ is a matrix

$$G^{\mu\nu} = G_a^{\mu\nu}T^a. \tag{2.21}$$

20. Show that the covariant derivative transforms as

$$\mathcal{D}^\mu \to U\mathcal{D}^\mu U^\dagger, \quad \mathcal{D}^\mu\mathcal{D}^\nu \to U\mathcal{D}^\mu\mathcal{D}^\nu U^\dagger, \tag{2.22}$$

under the gauge transformation

$$\psi \to U\psi, \quad A^\mu \to UA^\mu U^\dagger - \frac{1}{ig_s}(\partial^\mu U)U^\dagger. \tag{2.23}$$

21. Show that the QCD Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{2}\mathrm{Tr}\left(G^{\mu\nu}G_{\mu\nu}\right) \tag{2.24}$$

is invariant under the gauge transformation. We have constructed the QCD Lagrangian, which is gauge invariant and Lorentz invariant.

22. In the previous section, we have written the QED Lagrangian as the following.

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a. \tag{2.25}$$

Show that

$$G^{\mu\nu} = \frac{1}{ig_s}\left[\mathcal{D}^\mu, \mathcal{D}^\nu\right] = \partial^\mu A^\nu - \partial^\nu A^\mu + ig_s\left[A^\mu, A^\nu\right]. \tag{2.26a}$$

23. Show that
$$G_a^{\mu\nu} = 2\mathrm{Tr}\left(G^{\mu\nu}T^a\right) = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + 2ig_s A_b^\mu A_c^\nu \mathrm{Tr}\left(T^a[T^b, T^c]\right). \tag{2.27}$$

24. Using the identity $[T^b, T^c] = if^{abc}T^a$, show that
$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_s f^{abc} A_b^\mu A_c^\nu. \tag{2.28}$$

25. Show that there exists three-gluon coupling in QCD.

26. Show that there exists four-gluon coupling in QCD.

2.3 Gauge Fixing in QED

27. We learned that we need gauge-fixing term such as
$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \tag{2.29}$$
in the Lorentz gauge in order to derive the photon propagator.

28. Show in QED that the gauge transform changes the photon field as
$$A^\mu + \partial^\mu \chi, \tag{2.30}$$
if we choose
$$U = e^{-iq\chi}. \tag{2.31}$$

29. Show that the gauge condition $\partial_\mu A^\mu = 0$ transforms as
$$\partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0. \tag{2.32}$$
There appears unpleasant term $\partial_\mu \partial^\mu \chi$.

30. Show that
$$\partial_\mu \partial^\mu \chi = 0, \tag{2.33}$$
if we want to keep the gauge condition $\partial_\mu A^\mu = 0$. Under the transform, gauge field is shifted to another gauge field which still satisfies the condition $\partial_\mu A^\mu = 0$.

31. Once we introduce a gauge-fixing term, our gauge transform cannot be completely general. The transform must keep the gauge condition in order for the Lagrangian to be gauge invariant. The gauge invariance is valid only in the group of gauges satisfying the same gauge condition.

32. Show that
$$\partial_\mu \partial^\mu \chi = 0, \tag{2.34}$$
is the equation of motion for the Lagrangian
$$\mathcal{L} = \partial^\mu \chi^\dagger \partial_\mu \chi. \tag{2.35}$$
We can include this term into the Lagrangian in order to correct the gauge invariance of $\partial_\mu A^\mu$.

33. Show that inclusion of this scalar field does not make any physical contribution in reality because it does not have any interaction term with any other fields.

2.4 Gauge-Fixing and Ghost Terms in QCD

34. We can introduce the same kind of gauge-fixing term like that of the Lorentz gauge in QED. We can try the QCD analogy of this

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\alpha}\left(\partial_\mu A_a^\mu\right)^2. \tag{2.36}$$

Note that there are $N_c^2 - 1$ conditions $\partial_\mu A_a^\mu = 0$ for a =1, 2, $\cdots$, $N_c^2 - 1$.

35. We learned that in QCD the gauge transform changes the gluon field as

$$A^\mu \to U A^\mu U^\dagger - \frac{1}{ig_s}\left(\partial^\mu U\right)U^\dagger, \tag{2.37}$$

where $A^\mu = A_a^\mu T^a$. Let us find how each gluon field A_a^μ transforms in QCD.

36. It is convenient to choose the parametrization as

$$U = e^{-ig_s\alpha^a T^a}. \tag{2.38}$$

Note that α^a should be real to preserve U unitary. Expanding the matrix U in powers of α upto corrections of order α^2, show that

$$\begin{aligned} A^\mu &\to (1 - ig_s\alpha^c T^c)\, A_x^\mu T^x\, (1 + ig_s\alpha^c T^c) + \partial^\mu \alpha_a T^a && (2.39a)\\ &= (A_a^\mu + \partial^\mu\alpha_a)\, T^a + ig_s A_x^\mu [T^x, T^c]\alpha_c && (2.39b)\\ &= (A_a^\mu + \partial^\mu\alpha_a)\, T^a + ig_s A_x^\mu\, (if^{axc})\, \alpha_c T^a. && (2.39c)\end{aligned}$$

Therefore, A_a^μ transforms as

$$A_a^\mu \to A_a^\mu + \left[\partial^\mu\delta_{ac} + ig_s A_x^\mu\,(if^{axc})\right]\alpha_c. \tag{2.40}$$

37. Let us introduce a matrix in the adjoint representation

$$\left(t^x\right)_{ac} \equiv if^{axc}. \tag{2.41}$$

Show that

$$A_a^\mu \to A_a^\mu + \left[\partial^\mu\delta_{ac} + ig_s A_x^\mu\left(t^x\right)_{ac}\right]\alpha_c. = A_a^\mu + \widetilde{\mathcal{D}}_{ac}^\mu \alpha_c, \tag{2.42}$$

where the covariant derivative $\tilde{\mathcal{D}}_{ac}$ in the adjoint representation has the same form

$$\widetilde{\mathcal{D}}_{ac}^\mu \equiv \partial^\mu\delta_{ac} - g_s A_x^\mu f^{axc} = \partial^\mu\delta_{ac} + ig_s\tilde{A}_{ac}^\mu, \quad \tilde{A}_{ac}^\mu \equiv A_x^\mu\left(t^x\right)_{ac} = A_x^\mu\left(if^{axc}\right). \tag{2.43}$$

as that in the fundamental represenation.

38. The problem is resolved if we include the ghost term to the Lagrangian. One can find detailed discussion in most field theory text book such as Peskin.

$$\mathcal{L}_{\text{ghost}} = (\partial_\mu\bar{\eta}_a)\,\widetilde{\mathcal{D}}_{ac}^\mu\eta_c = (\partial_\mu\bar{\eta}_a)\left(\partial^\mu\delta_{ac} - g_s f^{axc}A_x^\mu\right)\eta_c. \tag{2.44}$$

39. Ghost η is a complex scalar field. However, it behaves like a fermion in statistical sense. We will find later.

2.5 SU(N_c) algebra summary

SU(N_c) Generator T^a, $a = 1, \cdots, N_c^2 - 1$

$$\begin{aligned} [T^a, T^b] &= i f^{abc} T^c & (2.45)\\ (t^b)_{ac} &= i f^{abc} & (2.46)\\ F_{ij} &= F^a T^a_{ij} \leftarrow F^a = 2\ \mathrm{Tr}(F T^a), \quad \mathrm{Tr}(T^a, T^b) = \frac{1}{2}\delta^{ab} & (2.47)\\ f^{axy} f^{bxy} &= N_c \delta^{ab} \\ &= -(t^a)_{xy}(t^b)_{xy} = (t^a)_{xy}(t^b)_{yx} = \mathrm{Tr}(t^a t^b) & (2.48) \end{aligned}$$

2.6 QCD Feynman rules

40. we are ready to derive QCD Feynman rules from the QCD Lagrangian.

$$\mathcal{L} = \overline{\psi}\left(i\gamma_\mu \mathcal{D}^\mu - m\right)\psi - \frac{1}{4}\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f^{abc} A^\mu_b A^\nu_c\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{apq} A^p_\mu A^q_\nu\right). \quad (2.49)$$

where the covariant derivatives are defined by

$$\begin{aligned} \mathcal{D}^\mu_{ij} &= \delta_{ij}\partial^\mu + i g_s A^\mu_a T^a_{ij} & (2.50)\\ \mathcal{D}^\mu_{ac} &= \delta_{ab}\partial^\mu + i g_s A^\mu_b (t^b)_{ac} = \delta_{ab}\partial^\mu - g_s G^\mu_b f^{abc} \leftarrow (t^b)_{ac} = i f^{abc} & (2.51) \end{aligned}$$

where indices $i, j, k, \cdots$ and $a, b, c, \cdots$ denote color indices for quark and gluon, respectively.

41. Let us recall the followings. Field strength tensor from fundamental representation

$$\begin{aligned} G^{\mu\nu} &= -\frac{i}{g_s}[\mathcal{D}^\mu, \mathcal{D}^\nu] = -\frac{i}{g_s}[\partial^\mu + i g_s A^\mu_a T_a, \partial^\nu + i g_s A^\nu_b T_b] \\ &= \partial^\mu A^\nu_a T_a - \partial^\nu A^\mu_a T_a + i g_s A^\mu_x A^\nu_y [T_x, T_y] \\ &= \partial^\mu A^\nu_a T^a - \partial^\nu A^\mu_a T^a - g_s A^\mu_x A^\nu_y f_{xya} T^a \leftarrow [T^a, T^b] = i f^{abc} T^c & (2.52)\\ G^{\mu\nu}_a &= 2\mathrm{Tr}(G^{\mu\nu} T^a) = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f_{abc} A^\mu_b A^\nu_c & (2.53)\\ (t^b)_{ac} &= i f^{abc} & (2.54) \end{aligned}$$

42. Derivative

$$\partial_\mu \to -i p_\mu (\text{incoming line}), \quad \partial_\mu \to +i p_\mu (\text{outgoing line}) \quad (2.55)$$

43. Vertex

$$+i\ \mathcal{L} \quad (2.56)$$

44. Wavefunctions for incoming positive-energy quark and antiquark with momentum p are

$$q : u(p) e^{-ip\cdot x}, \quad \bar{q} : \bar{v}(p) e^{-ip\cdot x}, \quad (2.57)$$

where we have suppressed the color wavefunction. The color wavefunction is of the form

$$c_i,\ c_i^\dagger c_j = \delta_{ij},\ i, i = 1, 2, 3. \quad (2.58)$$

45. Wavefunctions for outgoing positive-energy quark and antiquark with momentum p are

$$q : \bar{u}(p)e^{+ip\cdot x}, \quad \bar{q} : v(p)e^{+ip\cdot x}. \tag{2.59}$$

46. Wavefunction for incoming and outgoing gluons with momentum k are

$$\text{in} : \epsilon^{\mu}(k)e^{-ik\cdot x}, \quad \text{out} : \epsilon^{\mu *}(k)e^{+ik\cdot x}. \tag{2.60}$$

The color wavefunction for the gluon with color index a is

$$c_a, \ c_a^{\dagger}c_b = \delta_{ab}, \ i, i = 1, 2, \cdot, 8. \tag{2.61}$$

47. Propagator for the quark with momentum p with initial and final color indices i and j is

$$iS_F(p) \times \delta_{ji} = \frac{i\delta_{ji}}{\not{p} - m + i\epsilon}. \tag{2.62}$$

The factor shows the color is preserved δ_{ji}.

48. Propagator for the photon has various forms depending on the gauge, which does not change the observables. In the Feynman gauge, the propagator for a photon with momentum k is

$$iD_F^{\mu\nu}(k) \times \delta_{ba} = \frac{-ig^{\mu\nu}\delta_{ba}}{k^2 + i\epsilon}. \tag{2.63}$$

49. The $\bar{q}_j A_a^{\mu} q_i$ vertex factor can be read from the interaction Lagrangian $i\mathcal{L}_{\text{int}}$ as

$$\bar{q}_j A_a^{\mu} q_i : -ig_s T_{ji}^{a} \gamma^{\mu}. \tag{2.64}$$

50. Coupling g_s is universal for any quark.

2.6.1 Gluon Vertices

51. From the QCD Lagrangian

$$\begin{aligned} \mathcal{L} \ = \ & -\frac{1}{4}(\partial^{\mu}A_a^{\nu} - \partial^{\nu}A_a^{\mu})^2 - \frac{1}{2\alpha}(\partial \cdot A^a)^2 \\ & +\frac{1}{2}g_s f^{abc}(\partial^{\mu}A_a^{\nu} - \partial^{\nu}A_a^{\mu})A_{\nu}^{b}A_{\nu}^{c} - \frac{1}{4}g_s^2 f^{xab}f^{xcd}A_a^{\mu}A_b^{\nu}A_{\mu}^{c}A_{\nu}^{d}, \end{aligned} \tag{2.65}$$

derive the following Feynman rules.

52. gluon propagrator

$$g \ : \ -i\,\frac{g^{\mu\nu} - \frac{\lambda-1}{\lambda}p^{\mu}p^{\nu}/p^2}{p^2 + i\epsilon} \tag{2.66}$$

53. Three-gluon vertex $g_{c_1}^{\mu_1}(p_1 : \text{in}) - g_{c_2}^{\mu_2}(p_2 : \text{in}) - g_{c_3}^{\mu_3}(p_3 : \text{in})$ is

$$ggg \ : \ - \ g_s f^{c_1c_2c_3} \ \left((p_1 - p_2)^{\mu_3}g^{\mu_1\mu_2} + (p_2 - p_3)^{\mu_1}g^{\mu_2\mu_3} + (p_3 - p_1)^{\mu_2}g^{\mu_3\mu_1}\right). \tag{2.67}$$

54. Solution:

$$
\begin{aligned}
i\mathcal{L}_{ggg} &= i\left[-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\right]_{ggg} = igf^{abc}A^\mu_b A^\nu_c \partial_\mu A^a_\nu \qquad (2.68)\\
&= g\sum_{\text{perm.}\{1,2,3\}} f^{abc}\int d^4p_1 d^4p_2 d^4p_3 e^{-i(p_1+p_2+p_3)\cdot x}(p_1\cdot A^a_2)(A^b_1\cdot A^c_3) \qquad (2.69)\\
&= -g\sum_{\text{perm.}\{1,2,3\}} f^{abc}\int d^4p_1 d^4p_2 d^4p_3 e^{-i(p_1+p_2+p_3)\cdot x}\\
&\quad \times\left[-(p_1\cdot A^a_2)(A^b_1\cdot A^c_3)+(p_3\cdot A^a_2)(A^b_1\cdot A^c_3)\cdots\right]\\
&= \int d^4p_1 d^4p_2 d^4p_3 e^{-i(p_1+p_2+p_3)\cdot x} A^a_{\mu_1}(p_1)A^b_{\mu_2}(p_2)A^c_{\mu_3}(p_3)\\
&\quad \times(-gf^{abc})\cdot\left[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3}+g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1}+g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\right]. \qquad (2.70)
\end{aligned}
$$

$(abc) = (c_1c_2c_3)$ If we choose the momentum direction into the vertex(annihilation at x), momentum dependence is $e^{-ik\cdot x}$ style and derivative can be replaced by $-ik$ in momentum space.

55. Following the above way, derive the four-gluon vertex $g^{\mu_1}_{c_1}(p_1 : \text{in}) - g^{\mu_2}_{c_2}(p_2 : \text{in}) - g^{\mu_3}_{c_3}(p_3 : \text{in}) - g^{\mu_4}_{c_4}(p_4 : \text{in})$ is

$$
\begin{aligned}
gggg \;:\; -\, ig_s^2\;\Big[\; &+f^{c_1c_2x}f^{c_3c_4x}\left(g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right)\\
&-f^{c_1c_3x}f^{c_2c_4x}\left(g^{\mu_1\mu_4}g^{\mu_2\mu_3}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}\right)\\
&+f^{c_1c_4x}f^{c_2c_3x}\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_3}g^{\mu_2\mu_4}\right)\;\Big]. \qquad (2.71)
\end{aligned}
$$

56. Show the Bose symmetry of three- and four-gluon vertices explicitly.

2.6.2 Ghost Vertices

57. From the ghost term

$$\mathcal{L} = -\delta^{ac}\overline{\eta}_a\partial^2\eta_c + g_s f_{abc}\overline{\eta}_a\partial_\mu A^\mu_b\eta_c, \qquad (2.72)$$

derive the following rules

58. $ghost(p)$ propagator

$$gh \;:\; \frac{i}{p^2+i\epsilon} \qquad (2.73)$$

59. $ghost(p_f\text{out}, c_f) - g(\mu, c_g) - ghost(p_i\text{in}, \text{color} = c_i)$

$$gh_f - g - gh_i \;:\; +p^\mu_f g_s f^{c_fc_gc_i} \qquad (2.74)$$

Chapter 3

SU(N)

In this chapter, we review the properties of the special unitary group, SU(N), which is useful in calculating color factors in various QCD processes.

3.1 Generators and structure constants

1. Consider a transform $N \times N$ matrix U which transforms a matrix $\mathcal{O}$ and a vector ψ in complex field as

$$\psi \to \psi' = U\psi \quad \text{and} \quad \mathcal{O} \to \mathcal{O}' = U\mathcal{O}U^{-1}. \tag{3.1}$$

   Show that $\psi^\dagger\psi$ is invariant under this transformation

$$\psi^\dagger \mathcal{O}\psi \to \psi^\dagger U^\dagger U\mathcal{O}U^{-1}U\psi = \psi^\dagger\mathcal{O}\psi, \tag{3.2}$$

   if the transformation operator is unitary:

$$U^{-1} = U^\dagger. \tag{3.3}$$

   When the transform is infinitesimal, U may be expressed as

$$U = e^{-i\epsilon_a T_a} = 1 - i\epsilon_a T_a, \tag{3.4}$$

   where ϵ_a's are the infinitesimal real parameters and T_a's are the generators of the transformation.

2. Show that iT^a must be antihermitian.

3. Show that T^a must be hermitian.

4. Show that T^a must be traceless. Since U is unitary, T_a's are Hermitian. If we restrict $\det U = +1$ as the case of the identity transformation, the T_a matrices are restricted to be traceless as

$$\begin{aligned}
\det U &= \epsilon_{i_1 i_2,\ldots,i_N} U_{1i_1} U_{2i_2} \ldots U_{Ni_N} \\
&= \epsilon_{i_1 i_2,\ldots,i_N} (\delta_{1i_1} - i\epsilon_a T_a^{1i_1})(\delta_{2i_2} - i\epsilon_a T_a^{2i_2}) \ldots (\delta_{Ni_N} - i\epsilon_a T_a^{1i_N}) \\
&= \epsilon_{12,\ldots,N} - i\epsilon_a \left(\epsilon_{i_1 i_2,\ldots,i_N} (T_a^{1i_1}\delta_{1i_1} \ldots \delta_{Ni_N} + \ldots + \delta_{1i_1} \ldots \delta_{N-1 i_{N-1}} T_a^{1i_1})\right) \\
&= 1 - i\epsilon_a \left(\epsilon_{i_1 2,\ldots,N} T_a^{1i_1} + \ldots \epsilon_{12,\ldots,i_N} T_a^{Ni_N}\right) \\
&= 1 - i\epsilon_a \mathrm{Tr} T_a \quad \to \mathrm{Tr} T_a = 0 \quad \leftarrow \quad \det U = +1.
\end{aligned} \tag{3.5}$$

5. Show that there are N_c^2-1 free real parameters for ϵ^a. Consider how many T_a's are independent. Since a transformation matrix U is an $N \times N$ complex matrix, there are $2N^2$ parameters at first. Hermitian constraint discards N^2 parameters and traceless condition does one more parameter so that we now have N^2-1 independent T_a's.

6. If $N_2 = 2$, the problem is the same as that for a spin-1/2 particle. Show that the generators for the SU(2) are Pauli sigma matrices. Are there $N_c^2 - 1 = 3$ generators?

7. Consider a commutator $-i\,[T_a, T_b]$. It is a traceless and hermitian operator. Therefore it can be expressed as a linear combination of the generators as

$$[T_a, T_b] = i f_{abc} T_c, \tag{3.6}$$

where f^{abc} is the ab anti-symmetric evidently from the definition using commutator.

8. Show that f^{abc} is totally anti-symmetric.

$$\begin{aligned} & f^{abc} = f^{bca} = f^{cab} \\ = \quad & -f^{bac} = -f^{cba} = -f^{acb} = \frac{1}{3!}\left(f^{abc} + f^{bca} + f^{cab} - f^{bac} - f^{cba} - f^{acb}\right). \end{aligned} \tag{3.7a}$$

You can prove it if you use the relations

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA] \rightarrow \mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB], \tag{3.8}$$

which are valid for any matrices.

9. Since the trace of any two generator product is symmetric under the exchange of the color indices,

$$\mathrm{Tr}[T^aT^b] = \mathrm{Tr}[T^bT^a] = \frac{1}{2}\left(\mathrm{Tr}[T^aT^b] + \mathrm{Tr}[T^bT^a]\right), \tag{3.9}$$

$\mathrm{Tr}[T^aT^b]$ is proportional to δ^{ab} and the normalization can be chosen freely. Conventionally λ is set to 1/2.

$$\mathrm{Tr}\,[T_aT_b] = \frac{1}{2}\delta_{ab}. \tag{3.10}$$

10. We can infer that the antisymmetric term in T_aT_b is proportional to $if_{abc}T_c$ from the commutation relation. Therefore we can express the product of any two generators as

$$T_aT_b = \frac{1}{2}\left[(Af_{abc} + Bd_{abc})T_c + C\delta_{ab}\mathrm{I}\right], \tag{3.11}$$

where d_{abc} is ab-symmetric and A, B and C are to be determined as follows. Directly from the commutation relation, we can find $A = i$ and from the normalization condition, $\mathrm{Tr}[T_aT_b] = \delta_{ab}/2$, we can find $C = 1/N$. Setting $B = 1$ also, we get

$$\begin{aligned} T_aT_b &= \frac{1}{2}\left[(if_{abc} + d_{abc})T_c + \frac{1}{N}\delta_{ab}\mathrm{I}\right], \\ \{T_a, T_b\} &= d_{abc}T_c + \frac{1}{N}\delta_{ab}\mathrm{I}. \end{aligned} \tag{3.12}$$

11. Then we find that d_{abc} is fully symmetric as

$$\begin{aligned}
\mathrm{Tr}\left[\{T_a, T_b\}\, T_c\right] &= d_{abd}\mathrm{Tr}\left[T_d T_c\right] = \frac{1}{2}d_{abc} \quad \leftarrow \quad ab \ - \ \text{symmetric} \\
&= \mathrm{Tr}\left[T_a T_b T_c + T_b T_a T_c\right] = \mathrm{Tr}\left[T_b T_c T_a + T_b T_a T_c\right] \\
&= \mathrm{Tr}\left[T_b\, \{T_a, T_c\}\right] \quad \leftarrow \quad ac \ - \ \text{symmetric.}
\end{aligned} \tag{3.13}$$

12. And these are the useful relations

$$\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}, \tag{3.14}$$

$$\mathrm{Tr}[T^a T^b T^c] = \frac{1}{4}[d^{abc} + i f^{abc}], \tag{3.15}$$

$$d^{abc} = 2\mathrm{Tr}[\{T^a, T^b\}T^c], \tag{3.16}$$

$$f^{abc} = -2i\mathrm{Tr}[[T^a, T^b]T^c], \tag{3.17}$$

$$\mathrm{Tr}[T^a T^b T^c T^d] = \frac{1}{4}\left[\frac{1}{N}\delta^{ab}\delta^{cd} + \frac{1}{2}(d^{abe} + i f^{abe})(d^{cde} + i f^{cde})\right]. \tag{3.18}$$

3.2 Derivation of completeness relation

Any $N \times N$ traceless and hermitian matrix $\mathcal{A}$ can be expressed in the linear combination of the generators as

$$\mathcal{A} = \sum_{a=1}^{N^2-1} \alpha^a T^a, \tag{3.19}$$

where α^a's are real. There are N^2 degrees of freedom in arbitrary $N \times N$ hermitian matrices and there are $N^2 - 1$ generators, T^a's. Therefore, we need one more $N \times N$ matrix to form a basis of the hermitian matrix other than the $N^2 - 1$ generators, T^a's. Since identity matrix is an hermitian and it is independent of the generators, we can form a basis by adding the identity matrix.

13. Show that any $N \times N$ Hermitian matrix is expressed in a linear combination of the identity matrix and SU(N) generators.

$$\mathcal{H} = \alpha^0 \mathrm{I} + \sum_{a=1}^{N^2-1} \alpha^a T^a, \tag{3.20}$$

where α^i for i =0, 1, $\cdots$, $N^2 - 1$ are all real. Then we can obtain useful relation from this completeness condition. By choosing the normalization

$$\mathrm{Tr}\left[T_a T_b\right] = \frac{1}{2}\delta_{ab}, \tag{3.21}$$

we obtain the explicit values of the coefficients as

$$\mathrm{Tr}\,\mathcal{H} = \alpha^0 N \quad \text{and} \quad \mathrm{Tr}[T^a \mathcal{H}] = \frac{1}{2}\alpha^a. \tag{3.22}$$

14. Rewriting the hermitian matrix $\mathcal{H}$

$$\mathcal{H} = \frac{1}{N}\mathrm{Tr}[\mathcal{H}]\,\mathrm{I} + 2\sum_{a=1}^{N^2-1}\mathrm{Tr}[T^a\mathcal{H}]\,T^a, \tag{3.23}$$

in matrix representation, show that

$$\begin{aligned}\mathcal{H}_{\mu\nu} &= \frac{1}{N}\mathcal{H}_{\alpha\alpha}\delta_{\mu\nu} + 2\sum_{a=1}^{N^2-1}T^a_{\mu\nu}T^a_{\alpha\beta}\mathcal{H}_{\beta\alpha}, \\ \mathcal{H}_{\beta\alpha}(\delta_{\mu\beta}\delta_{\nu\alpha}) &= \mathcal{H}_{\beta\alpha}\left(\frac{1}{N}\delta_{\alpha\beta}\delta_{\mu\nu} + 2\sum_{a=1}^{N^2-1}T^a_{\mu\nu}T^a_{\alpha\beta}\right).\end{aligned} \tag{3.24}$$

15. Using the fact that $\mathcal{H}_{\mu\nu}$ is an arbitrary complex number for any $\mu > \nu$ and an arbitrary real number for any $\mu = \nu$, prove the completeness relation

$$\sum_{a=1}^{N^2-1}T^a_{\mu\nu}T^a_{\alpha\beta} = \frac{1}{2}\left(\delta_{\mu\beta}\delta_{\nu\alpha} - \frac{1}{N}\delta_{\mu\nu}\delta_{\alpha\beta}\right). \tag{3.25}$$

With this relation, we can calculate any color factor involving SU(N) gauge theory.

3.3 Useful trace formulas

We can derive various trace formulas and relations among the structure constants which are very useful in practical calculations concerning perturbative QCD. In this section, we derive these practically useful relations in detail, by using the results shown in previous sections.

From now on, we use summation convention, where any two repeated indices are assumed to be summed over color indices.

16. By using the completeness relation, we can show that the sum of squared generators is proportional to the identity matrix as

$$\begin{aligned}\left(\sum_{a=1}^{N^2-1}T^aT^a\right)_{\mu\nu} &= \sum_{a=1}^{N^2-1}T^a_{\mu\alpha}T^a_{\alpha\nu} = \frac{1}{2}\left(\delta_{\mu\nu}\delta_{\alpha\alpha} - \frac{1}{N}\delta_{\mu\alpha}\delta_{\alpha\nu}\right) \\ &= \frac{1}{2}\left(N\delta_{\mu\nu} - \frac{1}{N}\delta_{\mu\nu}\right), \\ \rightarrow \sum_{a=1}^{N^2-1}T^aT^a &= \frac{N^2-1}{2N}\mathrm{I}.\end{aligned} \tag{3.26}$$

For SU($N=3$),

$$C_F = \frac{N^2-1}{2N} = \frac{4}{3}. \tag{3.27}$$

The color factor appears in the quark wavefunction renormalization factor.

17. When we calcualte gluon-loop corrections to ferimion-gluon-fermion vertices, we need to evaluate the matrix such as

$$\sum_a T^a T^b T^a. \tag{3.28}$$

Using the completeness relation to re-order the matrix product and using the formula

$$\sum_a T^a T^a = \frac{N^2-1}{2N}\mathrm{I}, \tag{3.29}$$

show that

$$\begin{aligned}
T^a T^a T^b &= \frac{N^2-1}{2N} T^b, &(3.30)\\
T^a T^b T^a &= -\frac{1}{2N} T^b, &(3.31)\\
T^a T^a T^b T^b &= \frac{(N^2-1)^2}{4N^2}\mathrm{I}, &(3.32)\\
T^a T^b T^a T^b &= -\frac{N^2-1}{4N^2}\mathrm{I}, &(3.33)\\
T^a T^b T^b T^a &= \frac{(N^2-1)^2}{4N^2}\mathrm{I}. &(3.34)
\end{aligned}$$

18. We can derive the following trace formulas using the same method used above. Here are the SU(N) trace formulas up to 4 pairs of indices

$$\begin{aligned}
\mathrm{Tr}[T^a T^b] &= \frac{1}{2}\delta^{ab}, &(3.35)\\
\mathrm{Tr}[T^a T^a T^b] &= 0, &(3.36)\\
\mathrm{Tr}[T^a T^b]\mathrm{Tr}[T^a T^c] &= \frac{1}{4}\delta^{bc}, &(3.37)\\
\mathrm{Tr}[T^a T^a T^b T^c] &= \frac{N^2-1}{4N}\delta^{bc}, &(3.38)\\
\mathrm{Tr}[T^a T^b T^a T^c] &= -\frac{1}{4N}\delta^{bc}, &(3.39)\\
\mathrm{Tr}[T^a T^b T^c]\mathrm{Tr}[T^a T^b T^d] &= -\frac{1}{4N}\delta^{cd}, &(3.40)\\
\mathrm{Tr}[T^a T^b T^c T^a T^b T^d] &= \frac{N^2+1}{8N^2}\delta^{cd}, &(3.41)\\
\mathrm{Tr}[T^a T^b T^c]\mathrm{Tr}[T^b T^a T^d] &= \frac{N^2-2}{8N}\delta^{cd}, &(3.42)\\
\mathrm{Tr}[T^a T^b T^c T^b T^a T^d] &= \frac{1}{8N^2}\delta^{cd}, &(3.43)\\
\mathrm{Tr}[T^a T^a T^b T^b T^c T^d] &= \frac{(N^2-1)^2}{8N^2}\delta^{cd}, &(3.44)\\
\mathrm{Tr}[T^a T^a T^b T^c T^b T^d] &= -\frac{N^2-1}{8N^2}\delta^{cd}, &(3.45)\\
\mathrm{Tr}[T^a T^b T^a T^b T^c T^d] &= -\frac{N^2-1}{8N^2}\delta^{cd}. &(3.46)
\end{aligned}$$

3.4 Adjoint representation

In this section we investigate the properties of the structure constant f^{abc}'s and symmetric d^{abc}'s. And we will look into the adjoint representation which is made up of the structure constant f^{abc} itself.

Compare the above identity with the one driven in previous section,

$$T^a T^b = \frac{1}{2}\left[\frac{1}{N}\delta^{ab}\,\mathrm{I} + (d^{abc} + i f^{abc})T^c\right]. \tag{3.47}$$

19. If we multiply δ^{ab} and sum over color indices, then we get the properties of the structure constant and symmetric d^{abc} as

$$f^{aab} = 0 \quad \text{and} \quad d^{aab} = 0. \tag{3.48}$$

20. If we use Eq.(3.47) two times we get

$$\begin{aligned} T^a T^b T^c &= \frac{1}{4N}[d^{abc} + i f^{abc}]\,\mathrm{I} \\ &+ \frac{1}{2}\left[\frac{1}{N}\delta^{ab}\delta^{ce} + \frac{1}{2}(d^{abd} + i f^{abd})(d^{dce} + i f^{dce})\right] T^e. \end{aligned} \tag{3.49}$$

Let us consider triple product $T^a T^b T^c$. By using Eq. (3.26) we get

$$\begin{aligned} T^a T^a T^b &= \frac{N^2-1}{2N}T^b \quad \text{and} \quad T^a T^b T^a = -\frac{1}{2N}T^b \\ \longrightarrow \quad T^a[T^a, T^b] &= \frac{N}{2}T^b \quad \text{and} \quad T^a\{T^a, T^b\} = \frac{N^2-2}{2N}T^b. \end{aligned} \tag{3.50}$$

Comparing the result by using the Eqs.(3.6), (3.12)

$$\begin{aligned} T^a[T^a, T^b] &= T^a i f^{abc} T^c = \frac{i}{2} f^{abc}[T^a, T^c] = \frac{i}{2} f^{abc} i f^{ace} T^e = \frac{1}{2} f^{acb} f^{ace} T^e, \\ T^a\{T^a, T^b\} &= T^a\left(d^{abc}T^c + \frac{1}{N}\delta^{ab}\mathrm{I}\right) = \frac{1}{2}d^{abc}\{T^a, T^c\} + \frac{1}{N}T^b \\ &= \frac{1}{2}d^{abc}d^{ace}T^e + \frac{1}{N}T^b, \end{aligned} \tag{3.51}$$

show that

$$f^{abc}f^{abd} = N\delta^{cd} \tag{3.52a}$$

$$f^{abc}d^{abd} = 0 \tag{3.52b}$$

$$d^{abc}d^{abd} = \frac{N^2-4}{N}\delta^{cd}. \tag{3.52c}$$

21. With the identities and Eqs. (3.16) and (3.17) we obtain other relations

$$\begin{aligned} d^{abc}f^{abd} &= 0, \\ d^{abc}d^{abc} &= \frac{(N^2-1)(N^2-4)}{N}, \\ f^{abc}f^{abc} &= N(N^2-1). \end{aligned} \tag{3.53}$$

22. The structure constants f^{abc} themselves define adjoint representation

$$F^b_{ac} \equiv i f_{abc}. \tag{3.54}$$

23. Show that F^b's are traceless, hermitian, and anti-symmetric $(N^2-1)\times(N^2-1)$ matrices.

24. By using the Jacobi identity, show that

$$\begin{aligned}
[[T^a,T^b],T^c]+[[T^b,T^c],T^a]+[[T^c,T^a],T^b] &= 0,\\
if^{abd}[T^d,T^c]+if^{bcd}[T^d,T^a]+if^{cad}[T^d,T^b] &= 0,\\
if^{abd}if^{dce}T^e+if^{bcd}if^{dae}T^e+if^{cad}if^{dbe}T^e &= 0,\\
(-if^{abd})(-if^{dce})+(-if^{bcd})(-if^{dae})+(-if^{cad})(-if^{dbe}) &= 0,\\
(-if^{abd})(-if^{dce})-(-if^{adc})(-if^{ebd}) &= if^{aed}(-if^{dbc}),\\
(F^aF^e)_{bc}-(F^eF^a)_{bc} &= if^{aed}F^d_{bc}\\
\rightarrow \quad [F^a,F^b]=if^{abc}F^c. &
\end{aligned} \tag{3.55}$$

25. Defining the fully symmetric d^{abc} likewise($D^a_{bc}=d^{abc}$), we get

$$\begin{aligned}
[\{T^a,T^b\},T^c]+[\{T^b,T^c\},T^a]+[\{T^c,T^a\},T^b] &= 0,\\
d^{abd}[T^d,T^c]+d^{bcd}[T^d,T^a]+d^{cad}[T^d,T^b] &= 0,\\
d^{abd}if^{dce}T^e+d^{bcd}if^{dae}T^e+d^{cad}if^{dbe}T^e &= 0,\\
d^{abd}if^{edc}-d^{adc}if^{ebd} &= -d^{bcd}if^{dae},\\
-(D^aF^e)_{bc}+(F^eD^a)_{bc}=if^{ead}D^d_{bc}, &\\
\rightarrow \quad [F^a,D^b]=if^{abc}D^c. &
\end{aligned} \tag{3.56}$$

26. Formulas related to the structure constants can be rewritten in terms of adjoint representation as

$$f^{aab} = 0 \rightarrow \mathrm{Tr}\,F^b=0, \tag{3.57}$$

$$d^{aab} = 0 \rightarrow \mathrm{Tr}\,D^b=0, \tag{3.58}$$

$$f^{abc}f^{abd} = (-if^{cab})(-if^{dba}) = \mathrm{Tr}[F^cF^d] = N\delta^{cd}, \tag{3.59}$$

$$f^{abc}f^{abd} = (-if^{acb})(-if^{abd}) = \sum_{a=1}^{N^2-1}(F^aF^a)_{cd} = N\mathrm{I}_{cd}, \tag{3.60}$$

$$d^{abc}f^{abd} = 0 \rightarrow \mathrm{Tr}[D^cF^d]=0, \tag{3.61}$$

$$d^{abc}f^{abd} = 0 \rightarrow \sum_{a=1}^{N^2-1}(D^aF^a)_{cd} = 0_{cd}, \tag{3.62}$$

$$d^{abc}d^{abd} = d^{cab}d^{dba} = \mathrm{Tr}[D^cD^d] = \frac{N^2-4}{N}\delta^{cd}, \tag{3.63}$$

$$d^{abc}d^{abd} = d^{acb}d^{abd} = \sum_{a=1}^{N^2-1}(D^aD^a)_{cd} = \frac{N^2-4}{N}\mathrm{I}_{cd}. \tag{3.64}$$

Now we summarize the results concerning the two representations. Regardless of the two representations

$$T(R)\ \delta_{ab} = \mathrm{Tr}\,[T_a T_b] \quad \text{and} \quad C_2(R)\ \mathrm{I} = \sum_{a=1}^{N^2-1} T_a^{(R)} T_a^{(R)}, \tag{3.65}$$

where I is the identity matrix in the representation R and the label R is given as

$$T_a^{(F)} = T_a \quad \text{and} \quad T_a^{(A)} = F_a. \tag{3.66}$$

We can set the normalization of the generators by setting the value of $T(R)$. Standard normalizations are given as

$$T(R) = \frac{1}{2} \quad \text{and} \quad T(R) = N. \tag{3.67}$$

Then the values of $C_2(R)$ are fixed as

$$C_2(R) = \frac{N^2-1}{2N} \quad \text{and} \quad C_2(R) = N. \tag{3.68}$$

3.5 SU(3) Clebsch-Gordan coefficients

27. As the meson formed from $q\overline{q}$ into singlet and octet in flavor SU(3), the color state formed by Q and $\overline{Q}$ with color i and j form a color singlet and a color octet states as

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{3.69}$$

Obviously the singlet state is the identity matrix and octet states are the eight generators of SU(3) in the fundamental representation up to normalization as

$$\langle 3i; \overline{3}j | 1 \rangle = N_1 \delta_{ij} \quad \text{and} \quad \langle 3i; \overline{3}j | 8a \rangle = N_8 T^a_{ij}. \tag{3.70}$$

The octet coefficient is proportional to the physical vertex ; gluon with color a goes to a color i quark and a color j antiquark. One should be cautious not to confuse the order of indices i and j in above equations.

Normalizing the nine states as

$$\begin{aligned}
\langle 1|1\rangle &= \sum_{ij} \langle 1|3i;\overline{3}j\rangle\langle 3i;\overline{3}j|1\rangle \\
&= |N_1|^2 \sum_{ij} \delta_{ji}\delta_{ij} \\
&= |N_1|^2 \mathrm{Tr}[II] = 3|N_1|^2 = 1, \qquad (3.71)\\
\langle 8a|8a\rangle &= \sum_{ij} \langle 1|3i;\overline{3}j\rangle\langle 3i;\overline{3}j|1\rangle \quad (\textit{not sum over index } a) \\
&= |N_8|^2 \sum_{ij} T^{a*}_{ij} T^a_{ij} \\
&= |N_8|^2 \sum_{ij} T^a_{ji} T^a_{ij} \\
&= |N_8|^2 \mathrm{Tr}[T^a T^a] = \frac{1}{2}|N_8|^2 = 1. \qquad (3.72)
\end{aligned}$$

If we perform the same derivation for the case of SU(N), then we get

$$\langle 3i;\bar{3}j|1\rangle = \frac{1}{\sqrt{N}}\delta_{ij} \quad \text{and} \quad \langle 3i;\bar{3}j|8a\rangle = \sqrt{2}T^a_{ij}, \tag{3.73}$$

provided that the generators of the fundamental representation are normalized as

$$\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}. \tag{3.74}$$

28. To find the color factor for singlet and octet in this case, we use this kind of method: Since any product of color matrices are expressed as

$$T_n \equiv T^{a_1}T^{a_2}T^{a_3}\cdots\cdot T^{a_n} \;=\; A(a_1,a_2,\ldots,a_n) + B^a(a_1,a_2,\ldots,a_n)T^a \tag{3.75}$$

$$\begin{aligned} A(a_1,a_2,\ldots,a_n) &= \frac{1}{N}\mathrm{Tr}(T_n) && (3.76)\\ B^a(a_1,a_2,\ldots,a_n) &= 2\mathrm{Tr}(T^aT_n) && (3.77)\end{aligned}$$

Then the total color factor is

$$\begin{aligned} C_T \equiv \mathrm{Tr}T_nT_n^\dagger &= A(a_1,a_2,\ldots,a_n)A^*(a_1,a_2,\ldots,a_n)\mathrm{Tr}(\mathrm{I})\\ &+ B^a(a_1,a_2,\ldots,a_n)B^{*b}(a_1,a_2,\ldots,a_n)\mathrm{Tr}(T^aT^b)\\ &= A(a_1,a_2,\ldots,a_n)A^*(a_1,a_2,\ldots,a_n)N\\ &+ B^a(a_1,a_2,\ldots,a_n)B^{*b}(a_1,a_2,\ldots,a_n)\frac{1}{2}\delta^{ab}\\ &= NA(a_1,a_2,\ldots,a_n)A^*(a_1,a_2,\ldots,a_n)\\ &+ \frac{1}{2}B^a(a_1,a_2,\ldots,a_n)B^{*a}(a_1,a_2,\ldots,a_n) && (3.78)\end{aligned}$$

And the singlet and octet factors are

$$\begin{aligned} C_1 &= A(a_1,a_2,\ldots,a_n)A(a_1,a_2,\ldots,a_n)\\ &= \frac{1}{N^2}\mathrm{Tr}(T_n)\mathrm{Tr}(T_n) && (3.79)\\ C_8\delta^{ab} &= B^a(a_1,a_2,\ldots,a_n)B^b(a_1,a_2,\ldots,a_n)\\ &= \frac{\delta^{ab}}{N^2-1}B^a(a_1,a_2,\ldots,a_n)B^a(a_1,a_2,\ldots,a_n) && (3.80)\\ C_8 &= \frac{1}{N^2-1}B^a(a_1,a_2,\ldots,a_n)B^a(a_1,a_2,\ldots,a_n)\\ &= \frac{4}{N^2-1}\mathrm{Tr}(T^aT_n)\mathrm{Tr}(T^aT_n) && (3.81)\end{aligned}$$

Therefore there is a relation among the total color factor, singlet and octet color factor

$$C_T \;=\; NC_1 + \frac{1}{2}(N^2-1)C_8 \tag{3.82}$$

3.6 Examples

29. In calculations of QCD amplitudes such as $gg \to gg$, we have to calculate color factors such as

$$f^{abc}f^{axy}f^{dbx}f^{dcz}, \tag{3.83}$$

which is very involved.

$$\begin{aligned}
f^{abc}f^{axy}f^{dbx}f^{dcz} &= +\frac{1}{2}N^2\delta^{yz} && (3.84)\\
d^{abc}f^{axy}f^{dbx}f^{dcz} &= 0 && (3.85)\\
d^{abc}d^{axy}f^{dbx}f^{dcz} &= +\frac{1}{2}(N^2-4)\delta^{yz} && (3.86)\\
d^{abc}f^{axy}d^{dbx}f^{dcz} &= -\frac{1}{2}(N^2-4)\delta^{yz} && (3.87)\\
d^{abc}d^{axy}d^{dbx}f^{dcz} &= 0 && (3.88)\\
d^{abc}d^{axy}d^{dbx}d^{dcz} &= +\frac{1}{2N^2}(N^2-4)(N^2-12)\delta^{yz} && (3.89)
\end{aligned}$$

where the last formula is not derived directly from the relations given above, instead, with

$$\begin{aligned}
&4^4 \times \mathrm{Tr}(T^aT^bT^c)\mathrm{Tr}(T^aT^xT^y)\mathrm{Tr}(T^dT^bT^x)\mathrm{Tr}(T^dT^cT^z)\\
&= (d^{abc}+if^{abc})(d^{axy}+if^{axy})(d^{dbx}+if^{dbx})(d^{dcz}+if^{dcz})
\end{aligned} \tag{3.90}$$

and the remaining formulas. And the relation is invariant under cyclic rotations such as

$$\begin{aligned}
A^{abc}B^{axy}C^{dbx}D^{dcz} &= B^{abc}C^{axy}D^{dbx}A^{dcz}\\
&= C^{abc}D^{axy}A^{dbx}B^{dcz}\\
&= D^{abc}A^{axy}B^{dbx}C^{dcz}
\end{aligned} \tag{3.91}$$

30. The completeness relation is the most powerful tool in calculating color factors. You are advised to transform any color factors into the fundamental representation first. Then using the completeness relation to reorder the color matrices so that you can express the color factor as a linear combination of products of color factors made of

$$\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}. \tag{3.92}$$

Then the remaining calculations are products of

$$\delta^{aa} = N^2 - 1 \;\text{ and/or }\; \delta^{ii} = N. \tag{3.93}$$

Chapter 4

Tree-Level Calcualtion

4.1 $e^+e^- \to \mu^+\mu^-$

1. Neglecting masses, show that

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}. \tag{4.1}$$

4.2 $e^+e^- \to q + \bar{q}$

2. Neglecting masses, show that

$$\sigma(e^+e^- \to q\bar{q}) = N_c \times e_q^2 \times \sigma(e^+e^- \to \mu^+\mu^-), \tag{4.2}$$

where e_q is the fractional charge of the quark.