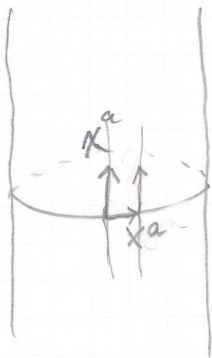


2. Four laws of black hole mechanics

Black hole thermodynamics	Ordinary thermodynamics
1 st The event horizon of a stationary black hole must be a Killing horizon.	A thermal state is an equilibrium state.
0 th $\kappa = \text{const.}$ over the horizon	$T = \text{const.}$ through out a thermal body
1 st $dM = \frac{1}{8\pi} \kappa dA + \Omega_H dJ + \Phi_H dQ$	$dE = T dS + P dV + \dots$
CONSERVATION LAW	
2 nd $\delta A \geq 0$ in any process	$\delta S \geq 0$ for an isolated thermal system
3 rd Impossible to achieve $\kappa = 0$ in a finite number of physical processes	$T \not\rightarrow 0$ ($S \rightarrow 0$ as $T \rightarrow 0$)

* Black hole $\mathcal{H}$ -dynamic 0th law:

$$\chi \cdot \nabla \chi_a = -\kappa \chi_a$$



i) $\kappa = \text{const.}$ along each orbit χ^a .

pf) $\mathcal{L}_\chi (\chi \cdot \nabla \chi_a) = -\mathcal{L}_\chi (\kappa \chi_a)$

$$\frac{1}{2} = -(\mathcal{L}_\chi \kappa) \chi_a - \kappa \mathcal{L}_\chi \chi_a$$

$$= 0 \because \mathcal{L}_\chi \chi_a = [\chi, \chi]_a$$

$$= \chi^b \nabla_b \chi_a - \chi_a \nabla_b \chi^b$$

$$\frac{1}{2} = \chi^b \nabla_b (\chi \cdot \nabla \chi_a) - (\chi \cdot \nabla \chi^b) \nabla_b \chi_a$$

$$= \chi^b \nabla_b \chi^c \nabla_c \chi_a + \chi^c \chi^b \nabla_b \nabla_c \chi_a$$

$$= \chi^c \chi^b \nabla_b \nabla_c \chi_a = R_{ac} \chi^b \chi^c \chi_d$$

$$= \chi^c \chi^b \chi^d R_{acbd}$$

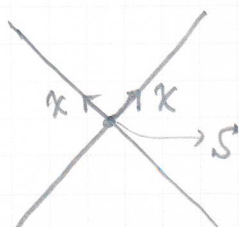
Symm.
in (bd)

anti-symm.
in (bd)

$$= 0$$

$$\therefore \mathcal{L}_\chi \kappa = \chi \cdot \nabla \kappa = 0$$

ii) $\kappa = \text{const.}$ on a bifurcate Killing horizon.



A pair of null surfaces which are generated by a null Killing vector field χ orthogonal to a (D-2) spacelike

pf) $\kappa^2 = -\frac{1}{2} \nabla_a \chi_b \nabla^a \chi^b$

$$\begin{aligned} x^c \nabla_c \kappa^2 &= 2\kappa x^c \nabla_c \kappa = -\frac{1}{2} (x^c \nabla_c \nabla_a \chi_b) \nabla^a \chi^b \times 2 \\ &= R_{bac}{}^d \chi_d \end{aligned}$$

$$\Rightarrow \kappa x \cdot \nabla \kappa = -\frac{1}{2} \chi_d x^c R_{bac}{}^d \nabla^a \chi^b \rightarrow 0 \text{ on } S,$$

Thus, κ is constant on the bifurcate surface S .

Since $\kappa = \text{const.}$ along each generator χ , it means that $\kappa = \text{const.}$ on the whole horizon.

iii) In GR, $\kappa = \text{const.}$ on any (not necessarily bifurcate) Killing horizon provided that the dominant energy condition is satisfied.

Tab  $k \cdot k < 0$

pf) See p. 332 in Wald's

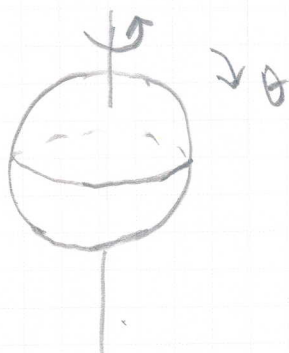
$-Tab k^b$: Energy-momentum 4-current density.

Future directed time-like or null vector for any future directed, timelike k^a .

Equivalently,

$$Tab v^a w^b \geq 0 \quad \forall \text{ future directed timelike}$$

For the Schwarzschild bh, $\kappa = \text{const.}$ is trivial due to the spherical symmetry. In the case of the Kerr bh, however, this property is rather surprising since the curvature actually varies along the polar direction.



* Black hole thermodynamic 1st law:

This law states a sort of energy conservation. When a stationary black hole is perturbed by some matter, the changes of parameters characterizing such black hole hold some relationship so that the energy is conserved.

Ex) For Schwarzschild bh,



$$A = 4\pi r_h^2 = 4\pi (2M)^2 = 16\pi M^2.$$

$$\delta A = 32\pi M \delta M \Rightarrow \delta M = \frac{1}{8\pi} \left(\frac{1}{4M} \right) \delta A.$$

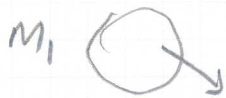
$$= \kappa \left(= \frac{1}{r_h^2} \right)$$

Note that $\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega_H \delta J + \dots$

ADM energy
defined at infinity

defined at the horizon

* Black hole θ -dynamic 2nd law:



$M_1 + M_2$



$$A \sim M_1^2 + M_2^2$$

$$A' \sim (M_1 + M_2)^2 = A + \underline{2M_1 M_2}$$

$$> A_1, A_2, A$$

Black hole area theorem: Hawking 1971

In GR, the area of the black hole event horizon never decrease provided that the cosmic censorship conjecture and the null energy condition hold.

→ No naked singularity

$$T_{ab} k^a k^b \geq 0 \quad \forall \text{ null vectors } k^a$$

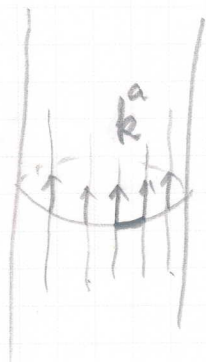
pt) $A = \oint d\vec{x} \sqrt{h}$

$$\frac{dA}{d\lambda} = \oint d\vec{x} \frac{d\sqrt{h}}{d\lambda} = \oint d\vec{x} \sqrt{h} \left(\frac{d\sqrt{h}/d\lambda}{\sqrt{h}} \right) = \oint d\vec{x} \sqrt{h} \theta$$

Expansion.

It can be shown that $\theta \geq 0$ under the conditions described above.

For a congruence of null geodesics on the horizon,



$$k \cdot \nabla \theta = \frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \underbrace{\hat{\sigma}_{ab}\hat{\sigma}^{ab}}_{\text{shear}} + \underbrace{\hat{\omega}_{ab}\hat{\omega}^{ab}}_{\text{twist}} - R_{ab}k^a k^b$$

Raychaudhuri's eq.

≤ 0 since $\hat{\sigma}_{ab}$ is a symmetric tensor

$$(R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}) k^a k^b$$

$$\Rightarrow R_{ab}k^a k^b - \frac{1}{2}R k \cdot k = 8\pi G T_{ab}k^a k^b$$

Thus, $R_{ab}k^a k^b \geq 0$ if $T_{ab}k^a k^b \geq 0$

null energy condition

Then $\frac{d\theta}{d\lambda}$

Raychaudhuri's eq. :

$$k \cdot \nabla \theta \equiv \frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^ak^b,$$

$\hat{\sigma}_{ab}$: shear, spacelike tensor $\Rightarrow \hat{\sigma}_{ab}\hat{\sigma}^{ab} \geq 0$.

$\hat{\omega}_{ab}$: twist. it can be set to be zero.

Einstein's eq.: $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}$

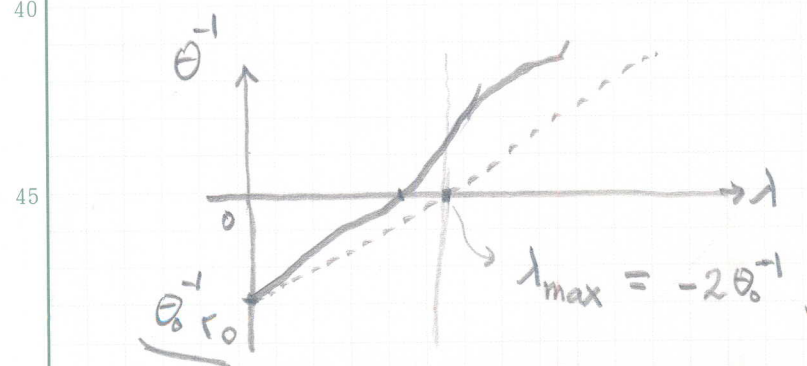
$$R_{ab}k^ak^b - \frac{1}{2}R k \cdot k \stackrel{0}{=} 8\pi G T_{ab}k^ak^b \geq 0 \text{ if the null energy condition holds for matters.}$$

$$\Rightarrow R_{ab}k^ak^b \geq 0.$$

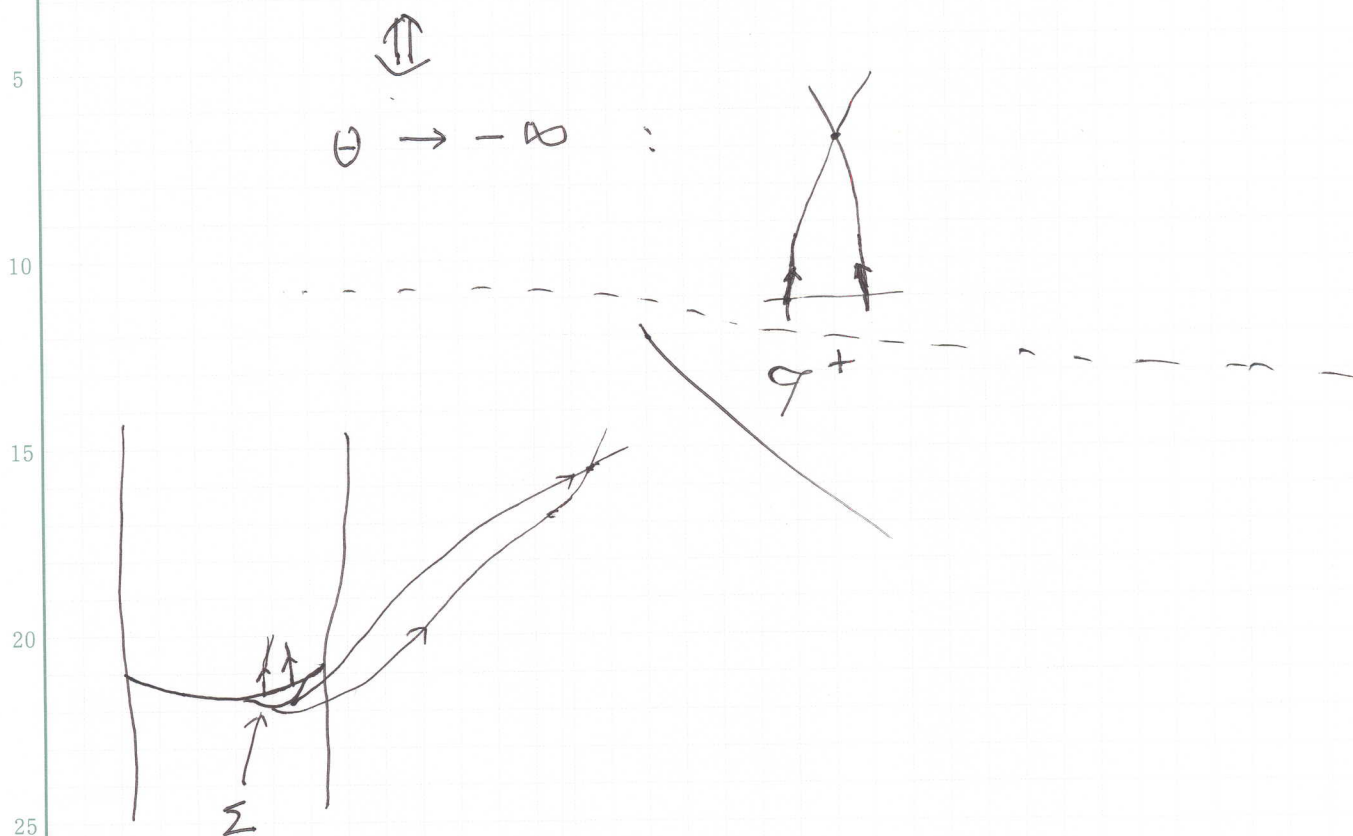
Then the RHS is negative-definite.

$$\Rightarrow \frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \quad \text{Or,} \quad \frac{d\theta^{-1}}{d\lambda} \geq \frac{1}{2}$$

$$\Rightarrow \theta^{-1}(\lambda) \geq \frac{1}{2}\lambda + \theta_0^{-1}.$$



Therefore, $\theta^{-1} \rightarrow 0$ in a finite affine time.



Now suppose $\theta < 0$ at some point on the horizon. Then one can deform a spacelike slice of the horizon slightly outward to obtain a compact spacelike surface Σ so that $\theta < 0$ still everywhere on Σ . If cosmic censorship is assumed, there should be some null geodesic orthogonal to Σ that remains on the boundary of the future of Σ all the way out to $\mathcal{H}^+$. However, we have a contradiction because, as explained above, there should exist a conjugate point on its path, which implies

it cannot stay on the boundary of $J^+(\Sigma)$.

therefore,

$\Theta \geq 0$ at any point on the horizon.

Namely,

$$\boxed{\frac{dA}{d\lambda} \geq 0.}$$

To summarize, these properties of a stationary black hole are very much analogous to those of an ordinary thermodynamic system.

Is this simply an analogy, not more than that, or it indicates some deep physical meaning to be understood?

$K \leftrightarrow T$: Black hole temperature?

$A \leftrightarrow S$: Black hole entropy?

WHAT DOES IT

MEAN ???

아!
저
나
만
그
때

내 영혼 쉬어가기 좋겠네.

IV. Black hole entropy

1. Black hole temperature

$$\delta E = T \delta S + P \delta V + \dots$$

$$\delta M = \frac{1}{8\pi} \kappa \delta A + 2\mu \delta J + \dots$$

$E = M$: Indeed the total energy of the system.

$$T_{BH} = \# \kappa$$

$$S_{BH} = \# A$$

Not known yet.

In Hawking radiation where a quantum matter is considered in a given black hole background, the particle emission rate is given by

$$\sim \frac{\Gamma(\omega)}{e^{2\pi\omega/\kappa} - 1} \Leftrightarrow \frac{1}{e^{E/kT} - 1} \quad \text{Thermal black body radiation at temperature } T.$$

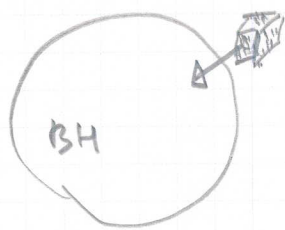
$$\Rightarrow \boxed{k T_{BH} = \frac{\hbar \kappa}{2\pi c}} \quad \left(T_{BH} = \frac{\kappa}{2\pi} \text{ w/ } \hbar = c = k = 1 \right).$$

Ex) Schwarzschild bh. $\kappa = \frac{1}{4M}$

$$T = \frac{1}{8\pi M} = \frac{M_{\odot}}{M} \times 6 \times 10^{-8} \text{ K. } \downarrow \text{ as } \Delta M > 0.$$

2. Bekenstein-Hawking entropy

* Bekenstein's conjecture



Without being assigned entropy to a black hole, one may easily construct a process in which

$$\Delta S_{\text{total}} < 0$$

$\Rightarrow$ Violation of the thermodynamic 2nd law ?

Bekenstein suggested that a black hole may possess entropy so that

$$\Delta S_{\text{tot}} = \Delta S_{\text{surround}} + \Delta S_{\text{BH}} \geq 0$$

* Rough estimation of such BH entropy:

When a star collapses to form a black hole, all information of the star is hidden by the event horizon. Assuming one bit of information per subatomic particle,

$$S_{\text{BH}} \sim \text{Total information lost} \approx \frac{M}{m_{\text{subatomic}}}$$

Classically, there would be no lower limit on the value of m . $S_{BH} \rightarrow \infty$ as $m \rightarrow 0$.

Being regarded as a quantum particle, however,



$$\lambda \sim \frac{h}{mc} \lesssim 2r_h \sim M.$$

$$\therefore S_{BH} \sim \frac{M}{\hbar} \sim \frac{M^2}{\hbar} \sim \frac{A}{\hbar}.$$

$$dM = \frac{1}{8\pi} \kappa \delta A + \dots$$

$$= \frac{\kappa}{2\pi} \delta \left(\frac{A}{4} \right)$$

$$\therefore S_{BH} = \frac{A}{4G} = \frac{\kappa c^3 A}{4G\hbar}$$

: Bekenstein-Hawking entropy.

$$S_{BH}^{\text{SOLAR}} \sim 10^{77} \gg S_{\text{SUN}} \sim 10^{58}$$

Enormously huge amount of entropy!

Note. In the 1st law of bh mechanics,

$$\delta M = \frac{\kappa}{2\pi} \cdot \delta \left(\frac{A}{4G} \right) + \dots$$

NO $\hbar$ APPEARED!

$$\Rightarrow (\hbar \kappa) \cdot \delta \left(\frac{A}{\hbar} \right)$$

$T_{BH} \sim \hbar \kappa \rightarrow 0$ in the classical limit of $\hbar = 0$.

3. Euclidean path integral method

The partition function for a system in thermal equilibrium at temperature $T = \beta^{-1}$ is given by

$$Z = \text{Tr} e^{-\beta H} = \sum_n e^{-\beta E_n} |\langle \phi_n | \phi \rangle|^2$$

$$= \sum_n \langle \phi | \phi_n \rangle \langle \phi_n | e^{-\beta H} | \phi_n \rangle \langle \phi_n | \phi \rangle$$

$$= \sum_{n,m} \langle \phi | \phi_n \rangle \langle \phi_n | e^{-\beta H} | \phi_m \rangle \langle \phi_m | \phi \rangle$$

$$= \langle \phi | e^{-\beta H} | \phi \rangle \quad \beta = i(t_2 - t_1)$$

$$= \langle \phi | e^{-i t_2 H} \cdot e^{i t_1 H} | \phi \rangle$$

$$= \langle \phi(t_2) | \phi(t_1) \rangle$$

$$= \int \mathcal{D}[\phi] e^{i S[\phi]}$$

$$t = -i\tau$$

$$= \int \mathcal{D}[\phi] e^{-S_E[\phi]} \rightarrow \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}_E[\phi]$$

$$\tau_2 = \tau_1 + \beta$$

$$\therefore Z = \int \mathcal{D}[\phi] e^{-S_E[\phi]}$$

where the integral is over all fields periodic in τ with periodicity β .

Similarly, one may obtain the partition function of a thermal system containing a black hole as follows;

$$Z = \int \mathcal{D}[g] e^{-I_E[g]}$$

$$I[g_{\text{BH}} + \delta g] = I[g_{\text{BH}}] + I_1[\delta g] + I_2[\delta g] + \dots$$

$$Z = e^{-I_E[g_{\text{BH}}]} \int \mathcal{D}[\delta g] e^{-I_{2E}[\delta g] - \dots}$$

$$\ln Z = \underbrace{-I_E[g_{\text{BH}}]}_{\text{Tree-level}} + \underbrace{\ln \left(\int \mathcal{D}[\delta g] e^{-I_{2E}[\delta g]} \right)}_{\text{one-loop}} + \dots$$

$$I[g] = -\frac{1}{16\pi} \int_M (R + 16\pi \mathcal{L}_m) - \frac{1}{8\pi} \oint_{\partial M} K$$

$$= - \int dt \left\{ \int_{\Sigma_t} [\pi^{ab} \dot{h}_{ab} - (N\mathcal{H} + N^a \mathcal{H}_a)] + \frac{1}{8\pi} \int_{\Sigma_t^\infty} (NK - N^a \pi_{ab} r^b) \right\}$$

$$= - \int dt \left\{ \int_{\Sigma_t} \pi^{ab} \dot{h}_{ab} - \underbrace{H}_{\downarrow} \right\}$$

For a stationary black hole, $\dot{h}_{ab} = 0$.

$$I_E[g_{BH}] = - \int d\tau \{ -H \} = "BH"$$

More precisely,

$$ds_{BH}^2 = -f(r) dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2.$$

$$t = -\lambda \tau.$$

$$f(r) = f(r_h) + f'(r_h)(r-r_h) + \dots$$

$$ds_E^2 = f(r) d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2.$$

$$r = r_h + \varepsilon \approx f'_\varepsilon d\tau^2 + \frac{d\varepsilon^2}{f'_\varepsilon} + r_h^2 d\Omega^2.$$

Recalling $\kappa = \frac{1}{2} f' \Big|_{r=r_h}$.

$$ds_E^2 \simeq 2\kappa\epsilon d\tau^2 + \left(\frac{d\epsilon^2}{2\kappa\epsilon} \right) + r_h^2 d\Omega^2.$$

$$= dp^2$$

$$dp = \frac{1}{\sqrt{2\kappa}} \left(\frac{d\epsilon}{\sqrt{\epsilon}} \right) = 2d\sqrt{\epsilon}.$$

$$= d\sqrt{\frac{2\epsilon}{\kappa}}$$

$$\therefore ds_E^2 \simeq \kappa^2 \rho^2 d\tau^2 + dp^2 + r_h^2 d\Omega^2.$$

$$\rho \equiv \left(\frac{2}{\kappa} \right)^{1/2} \sqrt{\epsilon}$$

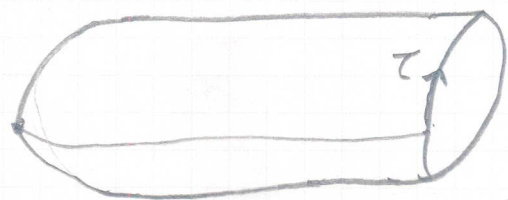
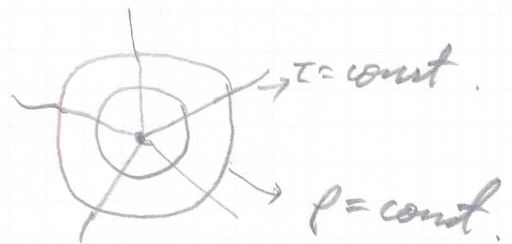
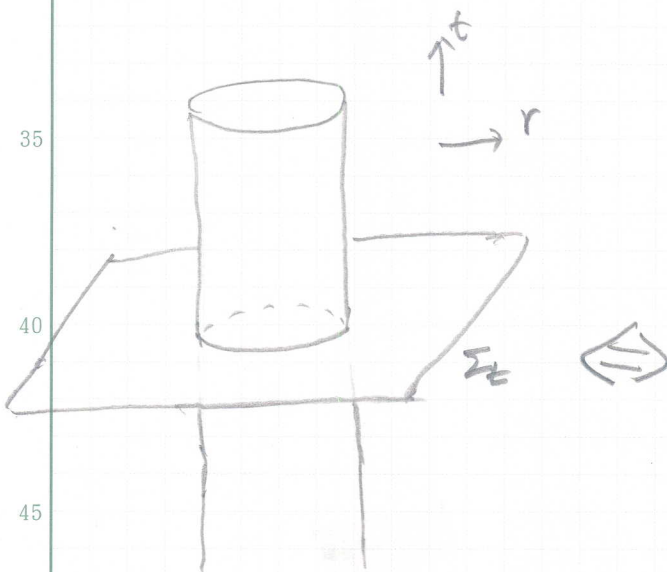
$$= \underbrace{dp^2 + \rho^2 d(\kappa\tau)^2}_{\text{}} + \underbrace{r_h^2 d\Omega^2}_{\text{}}$$

$$\epsilon = \frac{\kappa}{2} \rho^2.$$

$$\mathbb{R}^2 \times S^2 \Rightarrow$$

$(d\tau)^a$: Rotational Killing vector.

$\rho=0 \Leftrightarrow$ Horizon.



$r=r_h$

$r \rightarrow \infty$

Lorentzian vector

$\epsilon = \frac{\kappa}{2} \rho^2$

$\rho = \left(\frac{2}{\kappa} \right)^{1/2} \sqrt{\epsilon}$

$\epsilon = \frac{\kappa}{2} \rho^2$

Note. i) No Euclidean region corresponding to the black hole region.

ii) In order for the Euclideanized BH to be regular, τ must be periodic with periodicity $2\pi/\kappa$.

Thus, it is natural to take $\beta = 2\pi/\kappa$,

i.e., $T = \frac{\kappa}{2\pi}$.

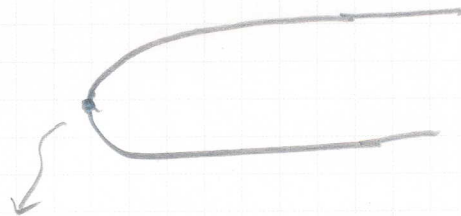


Equilibrium when

$$T = T_{BH}$$

$$\therefore T_{BH} = \frac{\kappa}{2\pi} \quad \text{AGAIN!}$$

Actually, near the horizon point, the Euclidean geometry is



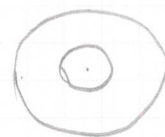
* $\otimes S^2$.

inner

The contribution of this "boundary" to $I_E[g_M]$ gives rise to

$$-\frac{1}{8\pi} \times 2\pi \chi \times A = -\frac{1}{4} A.$$

Euler number = $\begin{cases} 1 & \text{for a disk} \\ 0 & \text{for an annulus} \end{cases}$



$$\therefore I_E = \beta H - \frac{1}{4} A, \quad \ln Z = -I_E.$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = H = M_{\text{ADM}}.$$

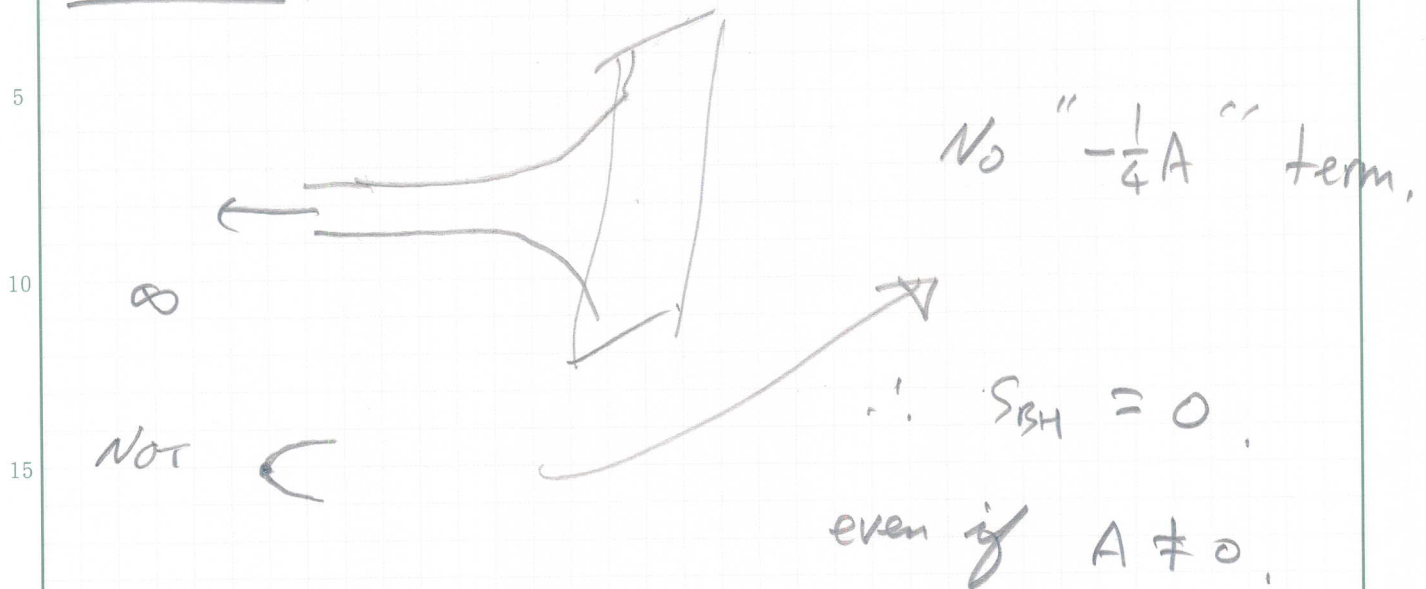
$$S = -(\beta \frac{\partial}{\partial \beta} - 1) \ln Z = \frac{A}{4}$$

$$\therefore S_{\text{BH}} = \frac{A}{4G} + \dots$$

Higher-order quantum corrections

REMARKS

For extremal bhs,

Prob. 4-1 In R-N bh,

- i) Show that the inner horizon coincides with the event horizon (i.e., $r_+ = r_-$) as $g \rightarrow M$.
- ii) Such bh is called an extremal bh.
 Show that $\kappa \rightarrow 0$ for the extremal bh.
- iii) Show that the proper distance between the extremal horizon and a certain point outside the event horizon is infinite.
- iv) Check $S_{BH} = 0$ for extremal R-N bh by using the formulae explained above.

4. Noether charge method.

The well-known entropy-area relation $S_{BH} = A/4$ was obtained for a stationary black hole solution in Einstein gravity whose action is given by

$$I = \int d^4x \sqrt{g} \left[\frac{1}{16\pi G} R + \mathcal{L}_{\text{matter}}(\phi, \nabla_a \phi, g_{ab}) \right].$$

Suppose the dynamics of gravitational fields is governed by more general action including higher-order derivative terms such as

$$I = \int d^Dx \sqrt{g} \mathcal{L}(g_{ab}, R, R_{abcd}, \nabla_a R_{bcde}, \dots, \nabla_{a_1} \dots \nabla_{a_n} R_{bcde}, \phi, \nabla_a \phi, \dots, \nabla_{a_1} \dots \nabla_{a_m} \phi).$$

Does the entropy-area formula still hold for a stationary black hole solution in this theory?

This question has its own interest. In addition, note that the low energy effective action of some candidate theory such as string theory contains higher-order curvature interactions.

It is indicative to see that the area of the event horizon could decrease during the Oppenheimer-Snyder collapse in Brans-Dicke theories of gravity.

$$\mathcal{L}_{BD} = \frac{1}{16\pi G} (\phi R - \frac{\omega}{\phi} \nabla_a \phi \nabla^a \phi) + \mathcal{L}_{matter}$$

(see Scheel, Shapiro & Teukolsky, PRD 51, 4236 (1995)
and G. Kang, PRD 54, 1483 (1996).)

$$\begin{aligned} \text{Since } \nabla_a \nabla_b T^{cd\dots} &= \nabla_{[a} \nabla_{b]} T^{cd\dots} + \underbrace{\nabla_{[a} \nabla_{b]} T^{cd\dots}}_{= -R_{ab}{}^c{}_e T^{ed\dots} - \dots}, \\ &= -R_{ab}{}^c{}_e T^{ed\dots} - \dots \end{aligned}$$

the most general covariant gravity action can be written as

$$I = \int d^D x \sqrt{g} \mathcal{L}(g_{ab}, R_{abcd}, \nabla_a R_{bcde}, \dots, \nabla_{[a_1} \dots \nabla_{a_n]} R_{bcde}, \phi, \nabla_a \phi, \dots, \nabla_{[a_1} \dots \nabla_{a_m]} \phi)$$

$\nwarrow$
 symmetrized covariant derivatives

Let $\mathbb{L} = \epsilon \mathcal{L}$: D-form

$\hookrightarrow$ D-dimensional volume element

$$\sqrt{g} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{D-1}$$

$$I = \int d^D x \sqrt{g} \mathcal{L} = \int \mathcal{L} \epsilon = \int \mathbb{L}$$

$$(g_{ab}, \phi) \rightarrow \psi$$

$$\delta L = \underbrace{E \delta \psi} + d\Theta[\delta \psi]$$

$$\left(= (E_g)^{ab} \delta g_{ab} + E_\phi \delta \phi \right)$$

$\Theta[\delta \psi]$: Symplectic potential (D-1)-form.

Determined only up to an additional closed (D-1)-form.

Let ξ^a be the infinitesimal generator of a diffeomorphism.

For $\delta \psi = \mathcal{L}_\xi \psi$,

$$\delta_\xi L = \mathcal{L}_\xi L = \underbrace{\xi \cdot dL}_{=0} + d(\xi \cdot L)$$

$$\therefore d(\Theta[\delta \psi] - \xi \cdot L) = -E \mathcal{L}_\xi \psi \Rightarrow 0 \text{ on-shell.}$$

Thus, defining the Noether current (D-1)-form

$$j \equiv \Theta[\mathcal{L}_\xi \psi] - \xi \cdot L,$$

we have $dj = 0$ modulo the EOM, i.e., j is closed when the eqn of motion is satisfied (i.e., $E=0$). It implies

$$j = dQ$$

Q : Noether charge (D-2)-form associated with ξ^a .

Consider an arbitrary variation $\delta\psi$ of the solution ψ (i.e., $E(\psi) = 0$).

$$\delta J = \delta \Theta[\psi, \xi\psi] - \underbrace{\xi \cdot \delta L}$$

$$= \underbrace{\xi \cdot E \delta\psi}_0 + \underbrace{\xi \cdot d\Theta[\psi, \delta\psi]}$$

$$= \underbrace{\xi \cdot \Theta[\psi, \delta\psi]}_0 - d(\xi \cdot \Theta[\psi, \delta\psi])$$

$$\Rightarrow \delta J = \underbrace{\delta \Theta[\psi, \xi\psi] - \xi \cdot \Theta[\psi, \delta\psi]} + d(\xi \cdot \Theta[\psi, \delta\psi])$$

$$\text{Or, } \omega[\psi, \delta\psi, \xi\psi] = \delta J - d(\xi \cdot \Theta[\psi, \delta\psi])$$

where $\omega[\psi, \delta\psi, \xi\psi] \equiv \delta \Theta[\psi, \xi\psi] - \xi \cdot \Theta[\psi, \delta\psi]$ is called the symplectic current (D-1)-form.

For any vector field ξ^a s.t. $\xi\psi = 0$ and the perturbation $\delta\psi$ satisfying the linearized $\mathcal{E}$, i.e.,

$$E[\psi + \delta\psi] = \underbrace{E(\psi)}_0 + \underbrace{E^{(1)}(\delta\psi)}_0 + \dots, \quad \delta J$$

one has $\omega[\psi, \delta\psi, \xi\psi=0] = 0$ and $\delta d\Theta[\psi, \xi\psi] = d\delta\Theta$.

Then,

$$0 = d[\delta Q[\xi] - \xi \cdot \Theta[4, \delta 4]]$$

Integrating it over a hypersurface C , we have

$$\oint_C \delta Q[\xi] - \xi \cdot \Theta[4, \delta 4] = 0.$$

For a stationary black hole solution with bifurcate Killing horizon, let ξ^a be the Killing generator that vanishes on the bifurcation $(D-2)$ -surface B .

$$\xi^a = (\partial_t)^a + \Omega_H^{(i)} (\partial_{\varphi_i})^a$$

stationary Killing field

$\partial_t \cdot \partial_t \rightarrow -1$ at infinity

axial Killing fields

Absent for $D=2$

$i=1$ for $D=3, 4$

$i=1, 2, \dots$ for $D \geq 5$.

Let C be an asymptotically flat spacelike hypersurface having B as its "interior boundary."



$$\int_B (\delta Q[\xi] - \xi \cdot \Theta[4, \delta 4])$$

$$= \int_{S^\infty} (\delta Q[\xi] - \xi \cdot \Theta[4, \delta 4])$$

Since $\xi^a \rightarrow 0$ on B , the left hand side becomes

$$\delta \int_B Q[\xi] \rightarrow \text{Noether charge density associated with the symmetry generated by } \xi^a.$$

Now suppose there exists a (D-1)-form B s.t.

$$\int_{S^\infty} \xi \cdot \Theta[\psi, \delta\psi] = \delta \int_{S^\infty} \xi \cdot B[\psi]$$

Then, we have

$$\begin{aligned} \delta \int_B Q[\xi] &= \delta \int_{S^\infty} (Q[\xi] - \xi \cdot B) \\ &= \delta \int_{S^\infty} (Q[\partial_t] - \partial_t \cdot B) + \sum_H^{(i)} \delta \int_{S^\infty} (Q[\partial_{\varphi_i}] - \partial_{\varphi_i} \cdot B) \end{aligned}$$

One can show that

$\overset{||}{M}$
 ADM mass (or energy)

$\overset{||}{-J_{(i)}}$
 angular momentum

$$\begin{aligned} \therefore \delta \int_B Q[\xi] &= \delta M - \sum_H^{(i)} \delta J_{(i)} \\ \text{w/ } \xi^a &= (\partial_t)^a + \sum_H^{(i)} \partial_{\varphi_i} \end{aligned}$$

One can show that

$$\delta \int_B Q[\xi] = \frac{\kappa}{2\pi} \delta \left(2\pi \int_B \Phi[\tilde{\xi}] \right).$$

pf) Since $\nabla_c \nabla_a \xi_b = -R_{abcd} \xi^d$ for a Killing vector ξ^a , one can always express $Q[\xi]$ as

$$Q[\xi] = \underbrace{\xi^a}_{\text{"}\xi^a\text{"}} + \underbrace{\nabla_a \xi_b}_{\text{"}\nabla_a \xi_b\text{"}}.$$

No contribution on B since $\xi^a = 0$ there.

$$\delta \int_B Q[\xi] = \delta \int_B \left(\underbrace{\chi^{ab} \nabla_a \xi_b}_{= \kappa \epsilon_{ab}} \right)$$

NEED MORE
RIGOROUS PROOF.

$$= \kappa \delta \int_B \chi^{ab} \nabla_a \tilde{\xi}_b$$

$\tilde{\xi}^a$: normalized to
have unit surface
gravity.

$$= \kappa \delta \int_B Q[\tilde{\xi}]$$

ie, $\tilde{\xi}^b \nabla_b \tilde{\xi}_a = -1 \cdot \tilde{\xi}_a$

Therefore, the black hole thermodynamic 1st law is established for a stationary bh solution in any arbitrarily diffeomorphism invariant gravity theory:

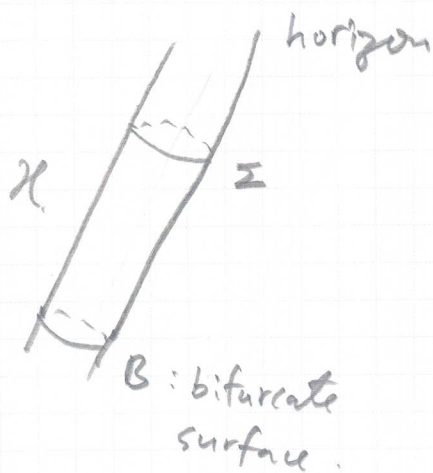
$$\delta M = \kappa \delta \int_B Q[\tilde{\xi}] + \sum_H^{(i)} \delta J_{(i)}.$$

Since $T_{BH} = \kappa/2\pi$, it naturally gives the black hole entropy formula in general:

$$S_{BH} = 2\pi \int_B \mathcal{Q}$$

Here, $\mathcal{Q}$ is the Noether charge density associated with the Killing field that is null on the horizon and normalized to have unit surface gravity.

For a stationary black hole horizon, it turns out that the Noether charge evaluated on the bifurcate surface is indeed same as that evaluated any other horizon cross section.



$$\int_{\mathcal{H}} d\mathcal{Q} = \int_B \mathcal{Q} + \int_{\Sigma} \mathcal{Q}$$

$$\int_{\mathcal{H}} \mathcal{Q} = \int_{\mathcal{H}} (\mathcal{Q}[\eta, \xi, \eta] - \xi \cdot L) = 0$$

Since ξ^a is tangent on $\mathcal{H}$ and so the pull back of $\xi \cdot L$ vanishes on $\mathcal{H}$.

* Explicite formulan for evaluating the Noether charge for a given arbitrary Lagrangian:

Ex) Einstein gravity: $\frac{1}{16\pi} \int d^4x \sqrt{g} R$.

$$\mathbb{L} = \frac{1}{16\pi} \mathbb{E} R \quad (\mathbb{L}_{abcd} = \frac{R}{16\pi} \mathbb{E}_{abcd})$$

$$\Theta_{abc} = \mathbb{E}_{dabc} \frac{1}{16\pi} g^{de} g^{fh} (\nabla_f \delta g_{eh} - \nabla_e \delta g_{fh})$$

$$\mathcal{Q}_{abc} = \mathbb{E}_{dabc} \frac{1}{8\pi} \nabla_e (\nabla^e \xi^{d3})$$

$$\mathcal{Q}_{ab} = \mathbb{E}_{abcd} \frac{-1}{16\pi} \nabla^c \xi^d \quad (\xi \cdot \nabla \xi^a = -\kappa \xi^a)$$

$$= \kappa \mathbb{E}^{cd} = \kappa \nabla^c \tilde{\xi}^d \quad w/ \tilde{\xi}^a = \kappa \xi^a$$

$$= \frac{1}{8\pi} \kappa \mathbb{E}_{ab}$$

$$\therefore S_{BH} = 2\pi \int \mathcal{Q}[\xi] = 2\pi \cdot \int \frac{1}{8\pi} \mathbb{E} = \frac{1}{4} \int d^2x \sqrt{h} = \frac{A}{4}$$

$$\frac{1}{16\pi} \int d^Dx \sqrt{g} \phi [R + \alpha R^2]$$

$$\Rightarrow S_{BH} = \frac{1}{4} \int d^{D-2}x \sqrt{h} \phi (1 + 2\alpha R)$$

In general,

$$S_{BH} = \frac{1}{4} \int d^p x \sqrt{-g} \left[R + F(\phi, R) + f(\phi) R_{\mu\nu} R^{\mu\nu} + \tilde{L}(\phi, \nabla_a \phi, g_{ab}, R_{abcd}, \nabla_a R_{bcde}, \dots) + \nabla^\mu \phi \nabla_\mu \phi \right]$$

$$S_{BH} = \frac{1}{4} \int d^p x \sqrt{-g} \left[1 + \frac{\partial F}{\partial R} + f(\phi) g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} (Y^{abcd} - \nabla_e Z^{e:abcd}) \epsilon_{ab} \epsilon_{cd} + 0 \right]$$

$$Y^{abcd} \equiv \frac{\partial \tilde{L}}{\partial R_{abcd}}, \quad Z^{e:abcd} \equiv \frac{\partial \tilde{L}}{\partial \nabla_e R_{abcd}}$$

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} = x^\mu l^\nu + x^\nu l^\mu$$


$l \cdot l = 0 = k \cdot k$
 $k \cdot l = 1$

Or,

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_a R_{bcde}, \dots, \nabla_{(a_1} \dots \nabla_{a_m)} R_{bcde}, \phi, \nabla_a \phi, \dots, \nabla_{(a_1} \dots \nabla_{a_n)} \phi)$$

$$E^{abcd} \equiv \frac{\partial \mathcal{L}}{\partial R_{abcd}} - \nabla_{a_1} \frac{\partial \mathcal{L}}{\partial (\nabla_{a_1} R_{abcd})} + \dots + (-1)^m \nabla_{(a_1} \dots \nabla_{a_m)} \frac{\partial \mathcal{L}}{\partial (\nabla_{(a_1} \dots \nabla_{a_m)} R_{abcd})}$$

$$\Rightarrow S_{BH} = 2\pi \int_{\Sigma} X^{cd} \epsilon_{cd}$$

binormal to Σ

$$(X^{cd})_{e_3 \dots e_p} = -E^{abcd} \epsilon_{abe_3 \dots e_p}$$

BLACK HOLES

MASS	RADIUS	TEMPERATURE	ENTROPY	LUMINOSITY	LIFETIME
M	$\frac{2M}{2\left(\frac{M}{M_{pl}}\right)\frac{\hbar}{M_{pl}c}}$	$\frac{1}{8\pi M}$ $\frac{1}{8\pi}\left(\frac{M_{pl}}{M}\right)\frac{M_{pl}c^2}{\hbar}$	$4\pi M^2$ $4\pi\left(\frac{M}{M_{pl}}\right)^2$	$\sim 10^{-3}/M^2$ $10^{-3}\left(\frac{M_{pl}}{M}\right)^2\frac{M_{pl}c^2}{t_{pl}}$	$\sim 10^3 M^3$ $10^3\left(\frac{M}{M_{pl}}\right)^3 t_{pl}$
$M_0 \approx 10^{57} \frac{\text{GeV}}{c^2}$ $\approx 2 \times 10^{32} \text{ g}$	10^6 cm	$10^{-7} \text{ }^\circ\text{K}$	10^{77}	$10^{-20} \text{ erg s}^{-1}$	10^{74} s
$M_E \approx 3 \times 10^{51} \frac{\text{GeV}}{c^2}$ $\approx 6 \times 10^{27} \text{ g}$	3 cm	$10^{-1} \text{ }^\circ\text{K}$	10^{66}	$10^{-9} \text{ erg s}^{-1}$	10^{57} s
$10^{26} \text{ g} \approx 10^{-7} M_0$	10^{-1} cm	3° K	10^{63}	$10^{-6} \text{ erg s}^{-1}$	10^{53} s
10^{14} g	10^{-14} cm	$10^{12} \text{ }^\circ\text{K}$	10^{39}	$10^{18} \text{ erg s}^{-1}$	10^{17} s
$10^7 \text{ g} = 10^4 \text{ kg}$ $\approx 10 \text{ tons}$	10^{-21} cm	$10^{19} \text{ }^\circ\text{K}$	10^{25}	$10^{33} \text{ erg s}^{-1}$	10^{-4} s
$M_p = 1 \frac{\text{GeV}}{c^2}$ $\approx 2 \times 10^{-24} \text{ g}$	10^{-51} cm	$10^{51} \text{ }^\circ\text{K}$ Background radiation 3° K	10^{-37}	$10^{94} \text{ erg s}^{-1}$ Luminosity of sun $10^{33} \text{ erg s}^{-1}$	10^{-97} s Age of universe 10^{17} s

$$M_{pl} = \sqrt{\frac{\hbar c}{G}} \approx 10^{19} \frac{\text{GeV}}{c^2} \approx 10^{-5} \text{ g} ; M_{pl} c^2 \approx 10^{16} \text{ erg} ; \frac{M_{pl} c^2}{\hbar} \approx 10^{32} \text{ }^\circ\text{K}$$

$$\frac{M_{pl} c^2}{t_{pl}} \approx 10^{59} \text{ erg s}^{-1} ; t_{pl} = \frac{\hbar}{M_{pl} c^2} \approx 10^{-43} \text{ s} ; \frac{\hbar}{M_{pl} c} \approx 10^{-33} \text{ cm}$$