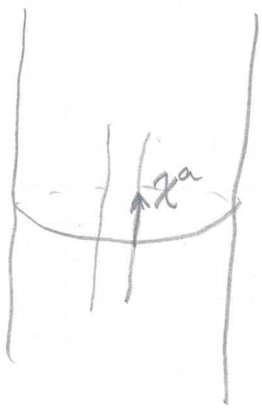
III. Black hole mechanics

1. Killing horizon, Surface gravity and area of the event horizon

* <u>Killing horizon</u>

An event horizon is called a Killing horizon when it possesses a Killing generator,



i.e., a null surface to which <u>a Killing field χ^a</u> is normal.

$\Downarrow$

$$\mathcal{L}_\chi g_{ab} = \nabla_a \chi_b + \nabla_b \chi_a = 0 .$$

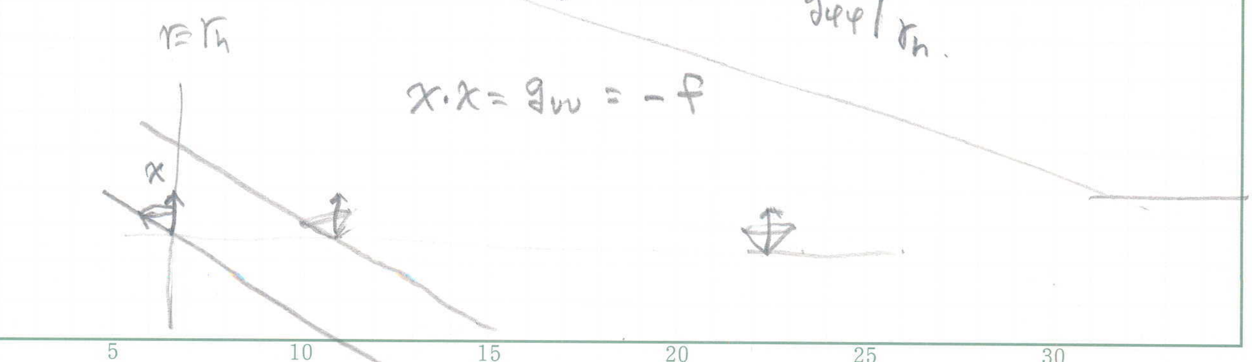
Ex) Schwarzschild b.h & R-N : $\chi^a = (\partial_t)^a$ in (t, r)

$= (\partial_v)^a$ in (v, r)

Kerr bh. $\chi^a = (\partial_t)^a + \Omega_H (\partial_\varphi)^a$

w/ $\Omega_H = -\left.\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_{r_h}$.

$\chi \cdot \chi = g_{vv} = -f$



* Surface gravity

Since $\chi\cdot\chi = 0 = \text{const.}$ on the horizon and χ^a is normal to it, $\nabla^a(\chi\cdot\chi) \propto \chi^a$.

$\Rightarrow \quad \nabla^a(\chi\cdot\chi) \equiv -2\kappa\,\chi^a$ at the horizon.

$$\nabla_a(\chi^b\chi_b) = 2\chi^b\nabla_a\chi_b = -2\chi^b\nabla_b\chi_a$$

$$\boxed{\therefore\ \chi\cdot\nabla\chi^a = \kappa\,\chi^a}$$

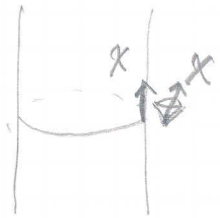
Geodesic equation in a non-affine parametrization.

How to evaluate κ for a given horizon?

i) Use the formula above in the vicinity of the horizon by using a regular coordinate system and take the limit of $r \to r_h$.

Prob. 3-1 Do it for the R-N b.h. in the Edington-Finkelstein coordinate system (v, r, θ, φ).

Answer: $\kappa = \frac{1}{2}\left(\frac{df}{dr}\right)_{r_h}$.

ii) 

$\chi\cdot\chi < 0$

$$u^a \equiv \frac{\chi^a}{\sqrt{-\chi\cdot\chi}}$$: 4-vector of a stationary observer.

Acceleration : $a^b = u\cdot\nabla u^b = \dfrac{\chi\cdot\nabla\chi^b}{-\chi\cdot\chi}$.

One can show that

$$\kappa^2 = -\frac{1}{2}(\nabla^a\chi^b)(\nabla_a\chi_b)\Big|_{r\to r_h}.$$

$$= \lim_{r\to r_h} \frac{\chi\cdot\nabla\chi^c\,\chi\cdot\nabla\chi_c}{-\chi\cdot\chi}$$

$$\therefore \boxed{\kappa = \lim_{r\to r_h} (Va)}$$

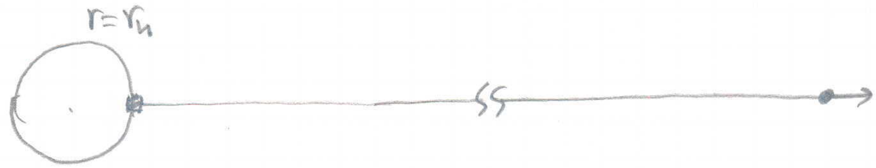
Here $a \equiv \sqrt{a^c a_c}$ & $V = \sqrt{-\chi\cdot\chi}$.

↓ for a static b.h. in which

redshift factor $\chi\cdot\chi \to -1$ at $r=\infty$.

$\Rightarrow \kappa =$ "redshifted proper acceleration" of the orbits of χ^a near the horizon.

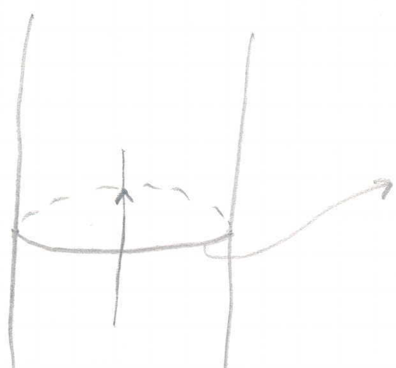
Or, force to be exerted at infinity in order to hold a unit test mass at the horizon.



$\frac{F_\infty}{m} = \kappa$.

Prob. 3-2. Solve the problem 4 in ch. 6 in Wald's.

* **Area of the horizon.**



2D spacelike hypersurface.

h_{ab} : induced metric

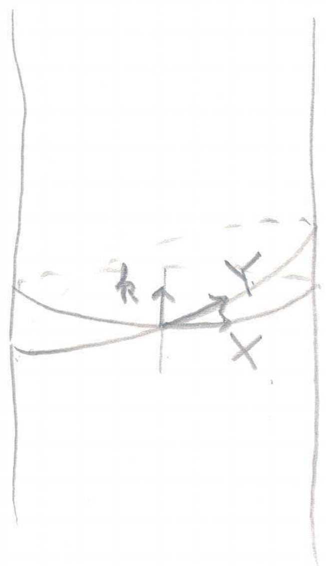
$$A \equiv \oint d^2x \sqrt{h}$$

Ex) $ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$\overset{r=r_h}{\Longrightarrow} r_h^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\therefore A = \int d\theta\, d\varphi \sqrt{g_{\theta\theta}\, g_{\varphi\varphi}} = \begin{cases} 4\pi r_h^2 & \text{for R-N bh.} \\ 4\pi(r_h^2 + a^2) & \text{for Kerr bh.} \end{cases}$$

Invariance of the value of the horizon area under Lorentz transformations:



k : null vector

$Y = X + \alpha k$.

$$Y\cdot Y = X\cdot X + 2\alpha \underbrace{X\cdot k}_{=0} + \alpha^2 \underbrace{k\cdot k}_{=0}.$$

$$\therefore Y\cdot Y = X\cdot X.$$