The 5th Summer Institute for Theoretical Physics
Aug. 8 - 19, 2005 at APCTP Headquarters at POSTECH

Lectures on Black Hole Thermodynamics

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Abstract

A series of lectures on black hole thermodynamics is given at the 5th Summer Institute for Theoretical Physics held at Pohang during Aug. 8 - 19, 2005. Some of main topics covered in the lecture includes basic concepts on black hole, ADM quantities, black hole mechanics, black hole entropy, and physical justifications.

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I. INTRODUCTION TO BLACK HOLES

1. Laplace's dark heavenly body: 1798
2. Schwarzschild and Reissner-Nordstrom black holes
3. Event horizon
4. Curvature singularity
5. Penrose diagrams

II. ADM QUANTITIES

1. 'Energy' of the gravitational field
2. ADM mass
3. Angular momentum
4. Charges

III. BLACK HOLE MECHANICS

1. Killing horizon, Surface gravity and area of the event horizon
2. Four laws of black hole mechanics
   - 0^{th}: $\kappa =$ constant
   - 1^{st}: Conservation law
   - 2^{nd}: Area theorem
   - 3^{rd}: Impossible to reach $\kappa = 0$ in a finite number of physical processes

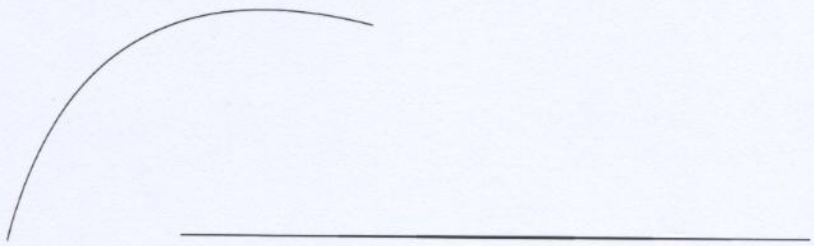
IV. BLACK HOLE ENTROPY

1. Black hole temperature: Hwaking radiation, Regularity of the Euclideanized black hole
2. Bekenstein-Hawking entropy
3. Euclidean path integral method
4. Noether charge method
5. Various other methods

V. PHYSICAL JUSTIFICATIONS FOR BLACK HOLE THERMODYNAMICS

1. Hawking radiation
2. QFT in curved spacetimes
3. Quantum gravity: String theory, Loop gravity

VI. SUMMARY



I. Introduction to black holes

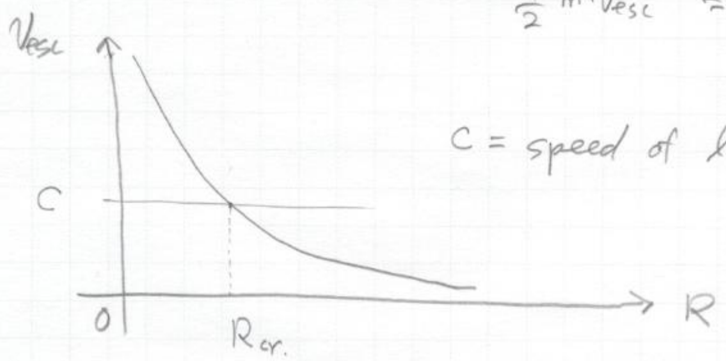
1. Laplace's dark heavenly body : 1798

$v_0 \geq v_{esc}$: It reaches to the infinity.

$v_0 < v_{esc}$: It cannot.

$$\frac{1}{2} m v_{esc}^2 = G \frac{Mm}{R} \Rightarrow \boxed{v_{esc} = \sqrt{\frac{2GM}{R}}}.$$

C = speed of light indep. of m!



Thus, if $R < R_{cr}$, the light cannot reach to an observer at infinity. The stellar object would be seen to be a dark body. Note that

$$R_{cr} = \frac{2GM}{C^2},$$

which is the same as the horizon radius of the Schwarzschild black hole.

Prob. 1-1. Applying the same argument above, show that a body of about the same density as the sun but 250 times its radius would exert a strong gravitational field so that no light could escape from its surface.

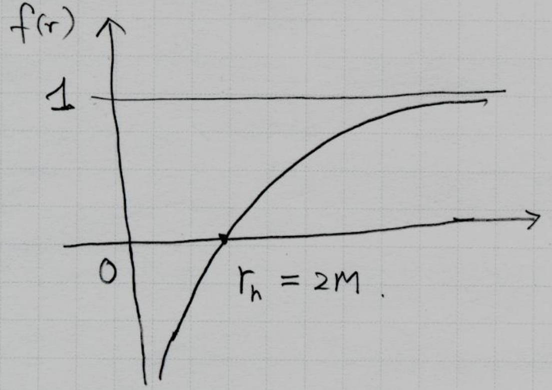
2. Schwarzschild and Reissner-Nordström black holes

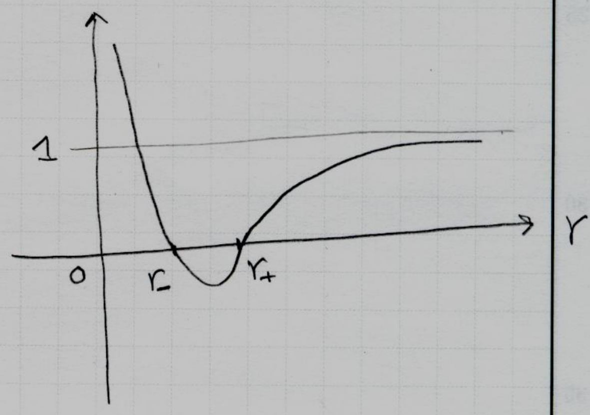
$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

$$f(r) = \begin{cases} 1 - \dfrac{2M}{r} & : \text{Schwarzschild b.h.} \quad \text{with } G=c=1. \\ 1 - \dfrac{2M}{r} + \dfrac{q^2}{r^2} & : \text{Reissner-Nordström b.h.} \end{cases}$$

M = ADM mass

q = Electric charge





$$r_\pm = M \pm \sqrt{M^2 - q^2}.$$

* Causal structure

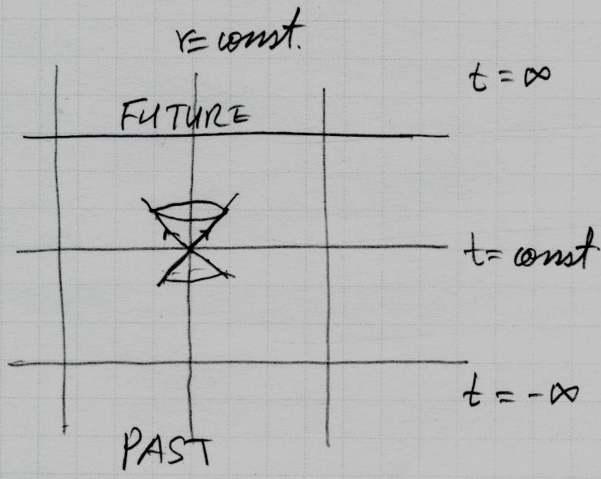
$$ds^2 \overset{r\to\infty\ (r \gg r_\pm)}{\simeq} -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Metric for the flat spacetime.

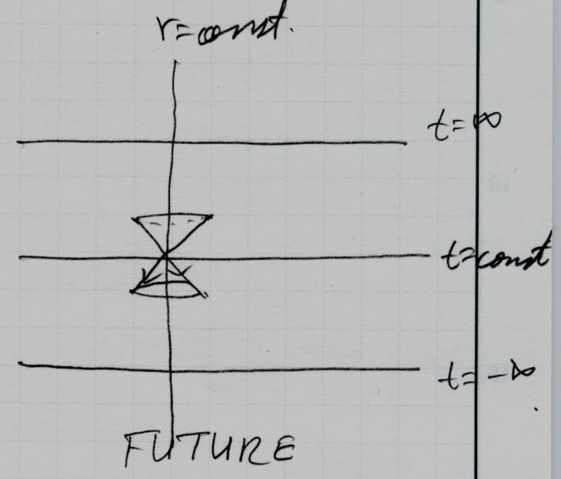
t : "time" coordinate r : radial coordinate.

At $\theta, \varphi =$ const., $ds^2 \simeq -dt^2 + dr^2 = 0$.

$\Rightarrow \quad dt = \pm dr$.



Or



FUTURE: $\Delta t > 0$ $\Delta t < 0$.

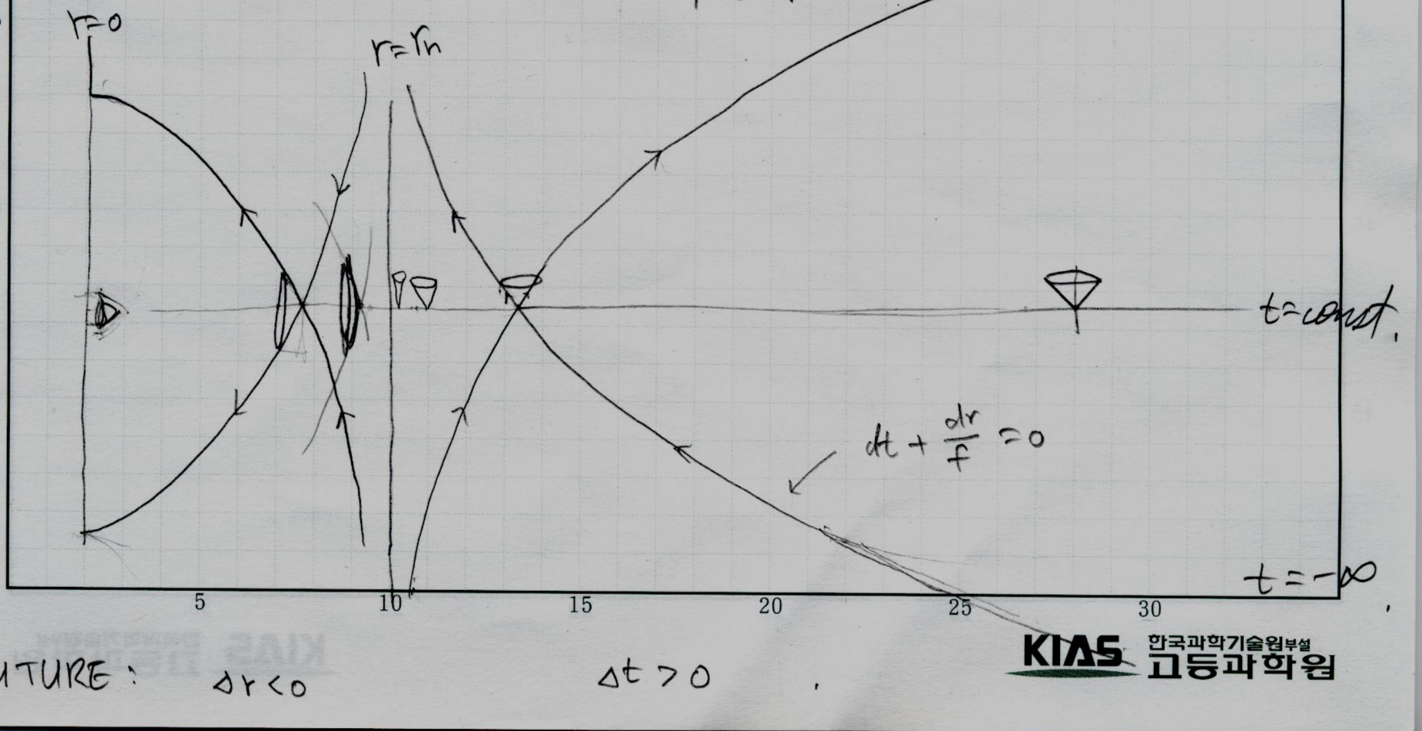
As $r \to 0$ or $r \to r_h$,

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f}.$$

$r > r_h$: $-$ $+$ $\Rightarrow$ t : "time" coordinate.

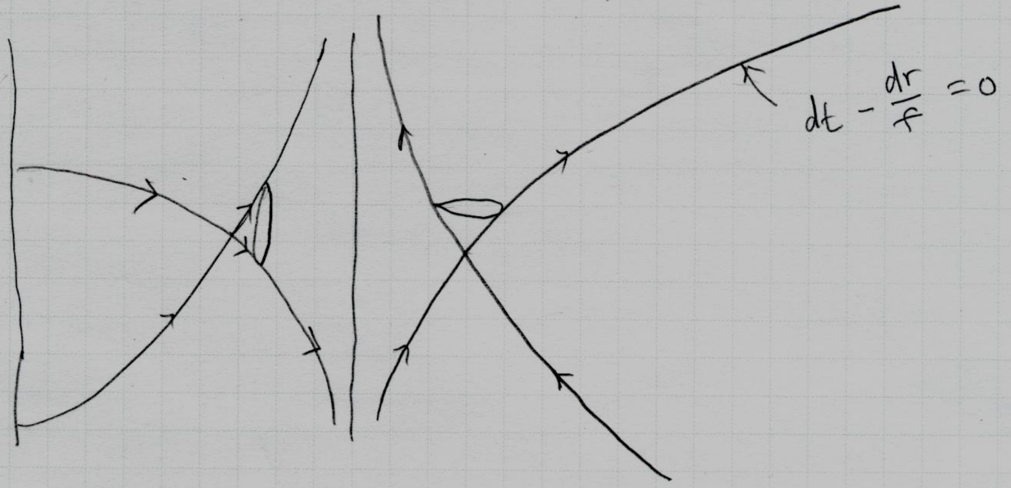
$0 < r < r_h$: $+$ $-$ $\Rightarrow$ r : " ".

$ds^2 = 0 \quad \Rightarrow \quad dt = \pm \dfrac{dr}{f(r)}$.

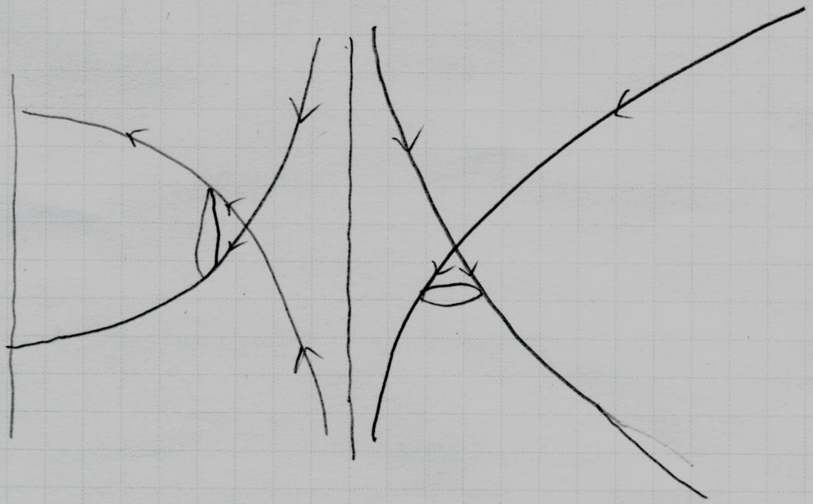


i) FUTURE: $\Delta r < 0$ $\Delta t > 0$.

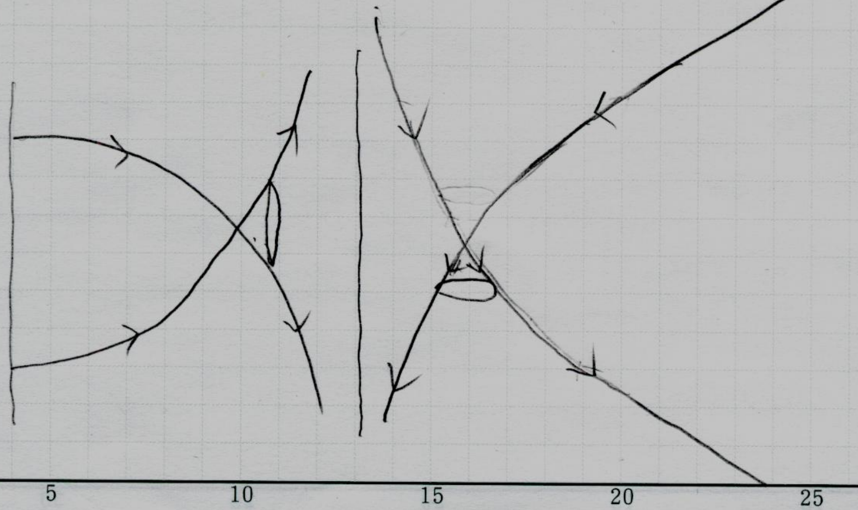
ii) FUTURE : $\Delta r > 0$ & $\Delta t > 0$



iii) $\Delta r < 0$ & $\Delta t < 0$



iv) $\Delta r > 0$ & $\Delta t < 0$



Let us use the so-called Edington-Finkelstein coordinates.

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} = -f\left(dt^2 - \frac{dr^2}{f^2}\right)$$

$$= -f\left(dt + \frac{dr}{f}\right)\left(dt - \frac{dr}{f}\right).$$

Defining $dv = dt + \frac{dr}{f}$, i.e., $dt = dv - \frac{dr}{f}$,

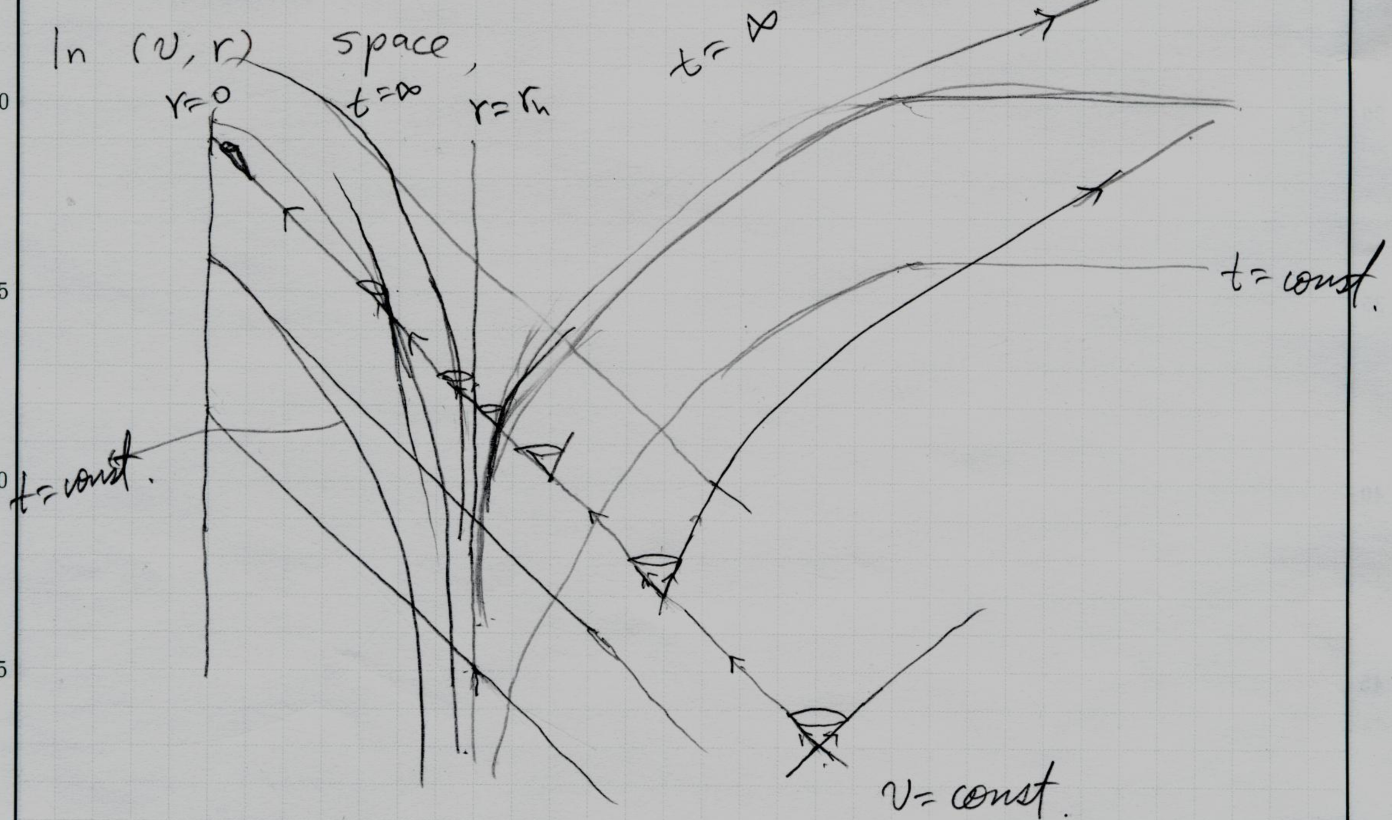
$$ds^2 = -f\,dv\left(dv - 2\frac{dr}{f}\right) = -f\,dv^2 + 2\,dv\,dr.$$

$$(t,r) \longrightarrow (v,r) \quad \text{with} \quad v = t + \underbrace{r + 2m\ln\left|\frac{r}{2m}-1\right|}_{=r_*}.$$

$= r_*$: Regge-Wheeler tortoise coordinate.

Note that $v = \text{const}$ for an ingoing null trajectory, i.e., $dv = 0$.

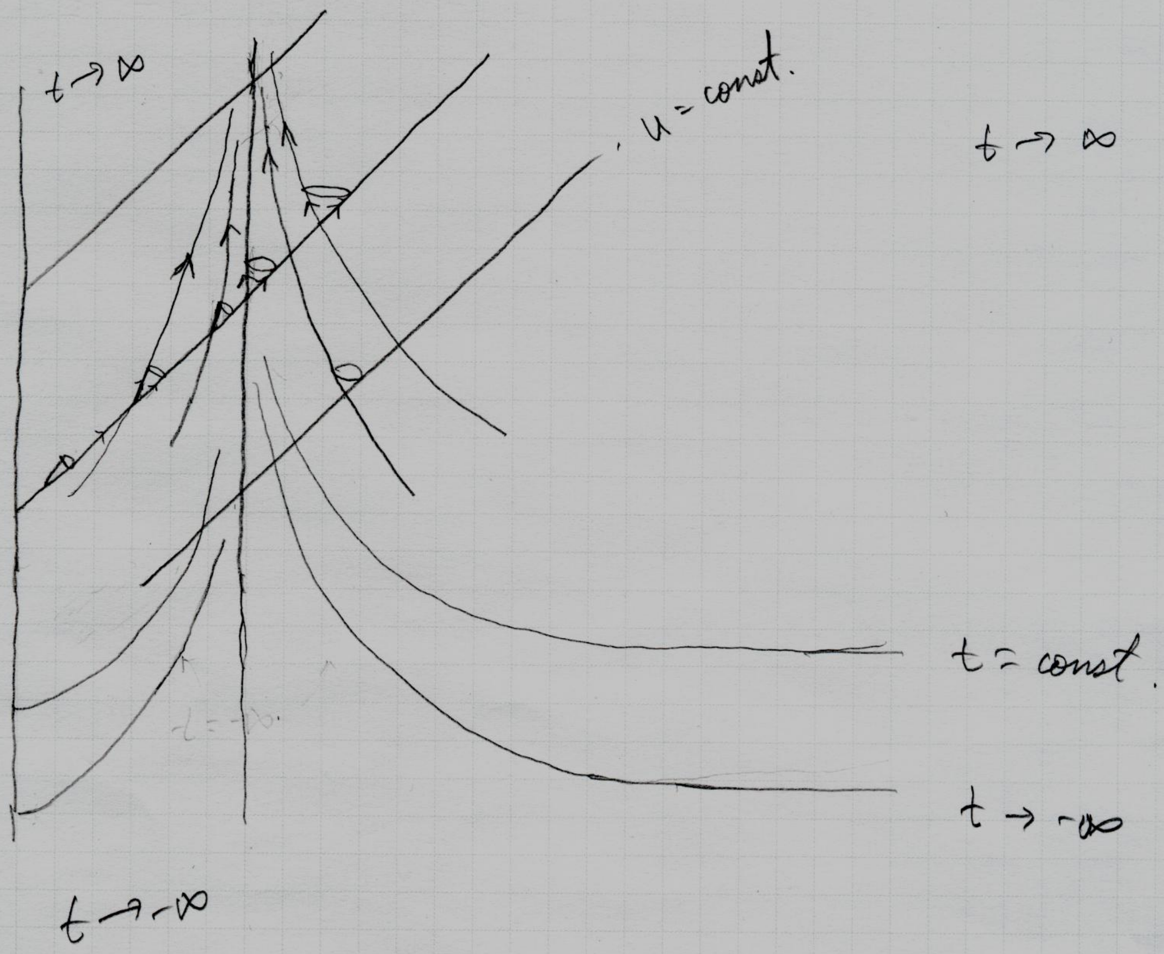
In (v,r) space,



Case ii)

$$du \equiv dt - \frac{dr}{f}.$$

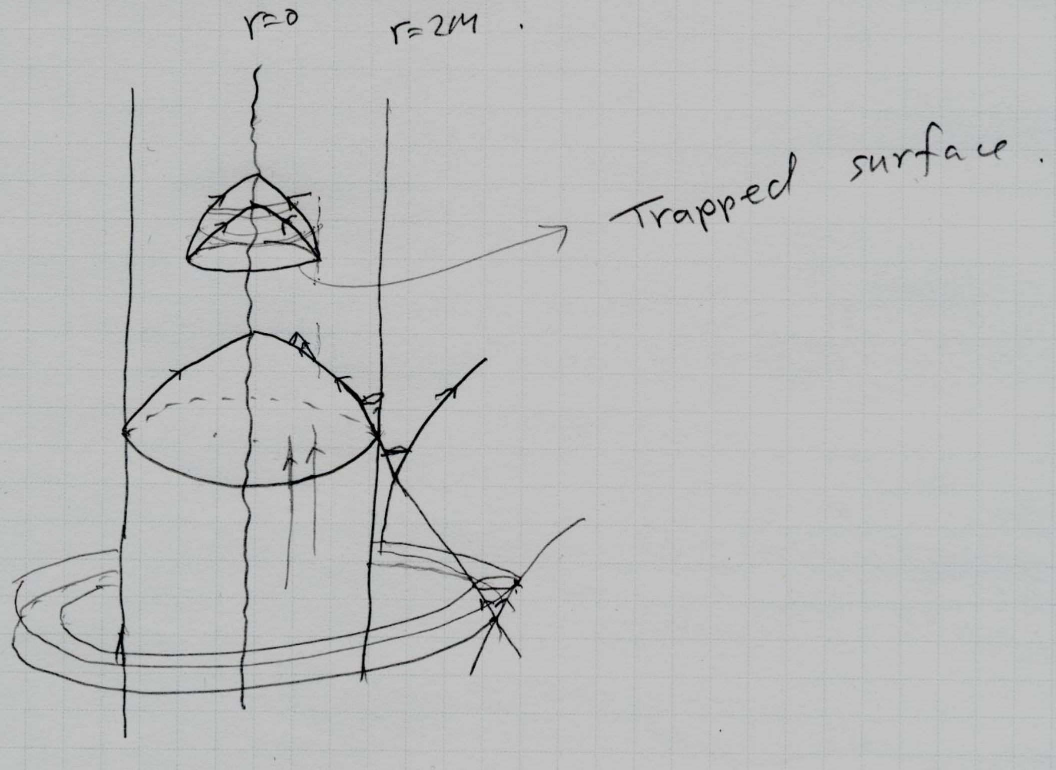
$$ds^2 = -f\left(du + 2\frac{dr}{f}\right)du = -f\,du^2 - 2\,du\,dr.$$



3. Event horizon

* <u>Black hole region</u> : "A region of spacetime from which nothing, not even light, can escape to infinity"

* <u>Event horizon</u> : "The boundary of a black hole region"



4. Curvature singularity

Although $r = 2M$ turns out to be a coordinate singularity, there exists a real singularity at $r=0$.

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = f''^2 + \frac{4}{r^2}f'^2 + \frac{4}{r^4}f^2 - \frac{8}{r^4}f + \frac{4}{r^4}$$

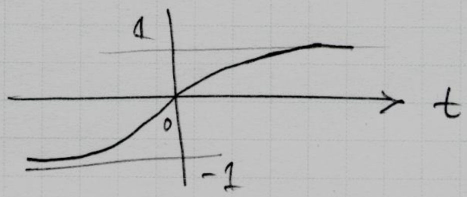
$$\overset{R\text{-}N}{=} \frac{8(6M^2r^2 - 12M^2g^2r + 7g^4)}{r^8} \overset{\text{Schw}}{\Longrightarrow} \frac{48M^2}{r^6}.$$

5. Penrose diagram

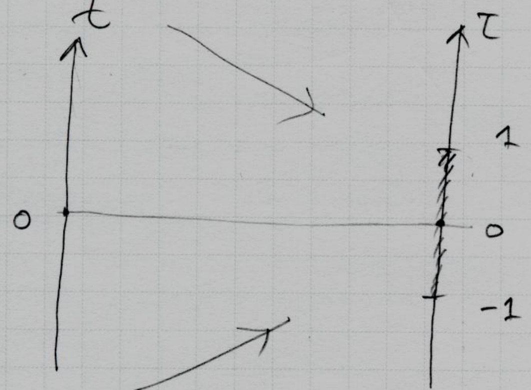
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \Longrightarrow d\tilde{s}^2 = \Omega^2(x)\, ds^2.$$

Ex) $ds^2 = -dt^2$ with $t: -\infty \sim \infty$

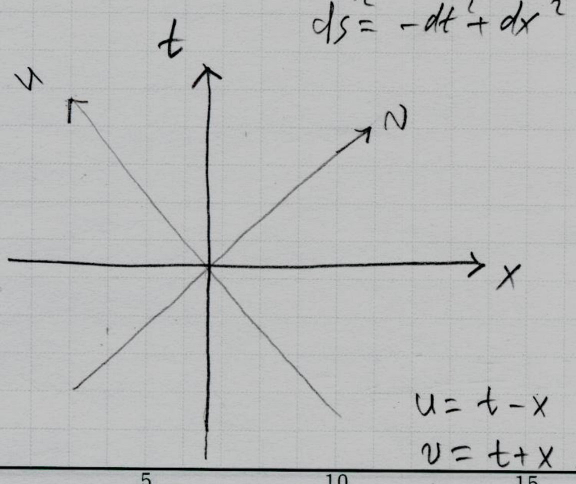
Let $\tau = \tanh t$.



$$d\tilde{s}^2 = -d\tau^2 = \mathrm{sech}^4 t\, ds^2.$$

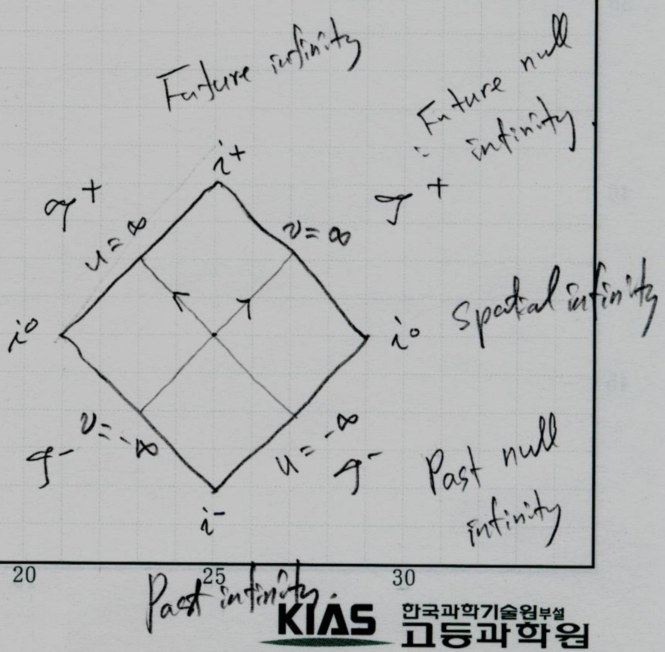


$$ds^2 = -dt^2 + dx^2$$

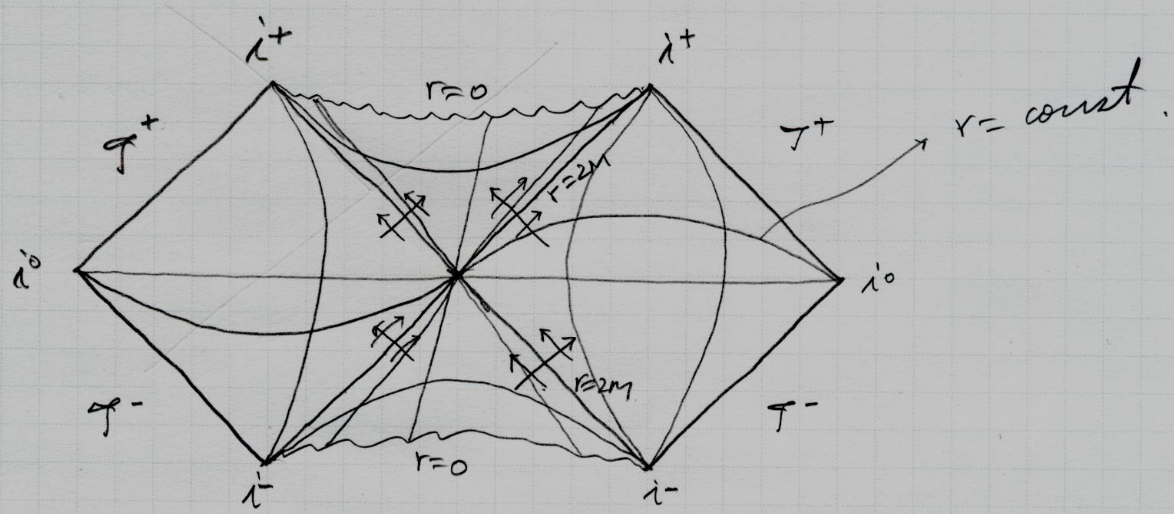


$u = t - x$

$v = t + x$.



* Maximally extended Schwarzschild black hole



Black hole region : $M - J^-(\mathcal{J}^+)$

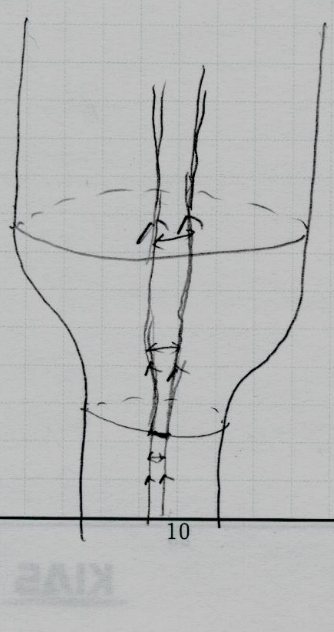
Event horizon : $\dot{J}^-(\mathcal{J}^+) \cap M$ (Future).

↓

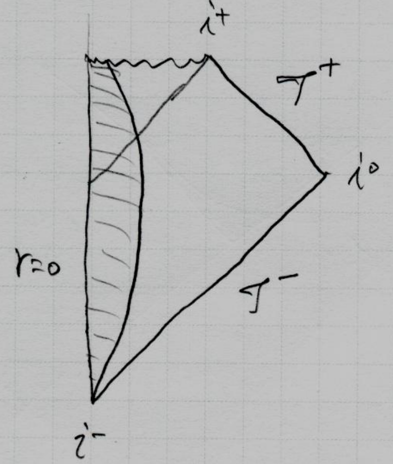
Global nature

Apparent horizon : $\theta = 0$.

$$\theta \equiv \frac{d\delta A/d\lambda}{\delta A}$$

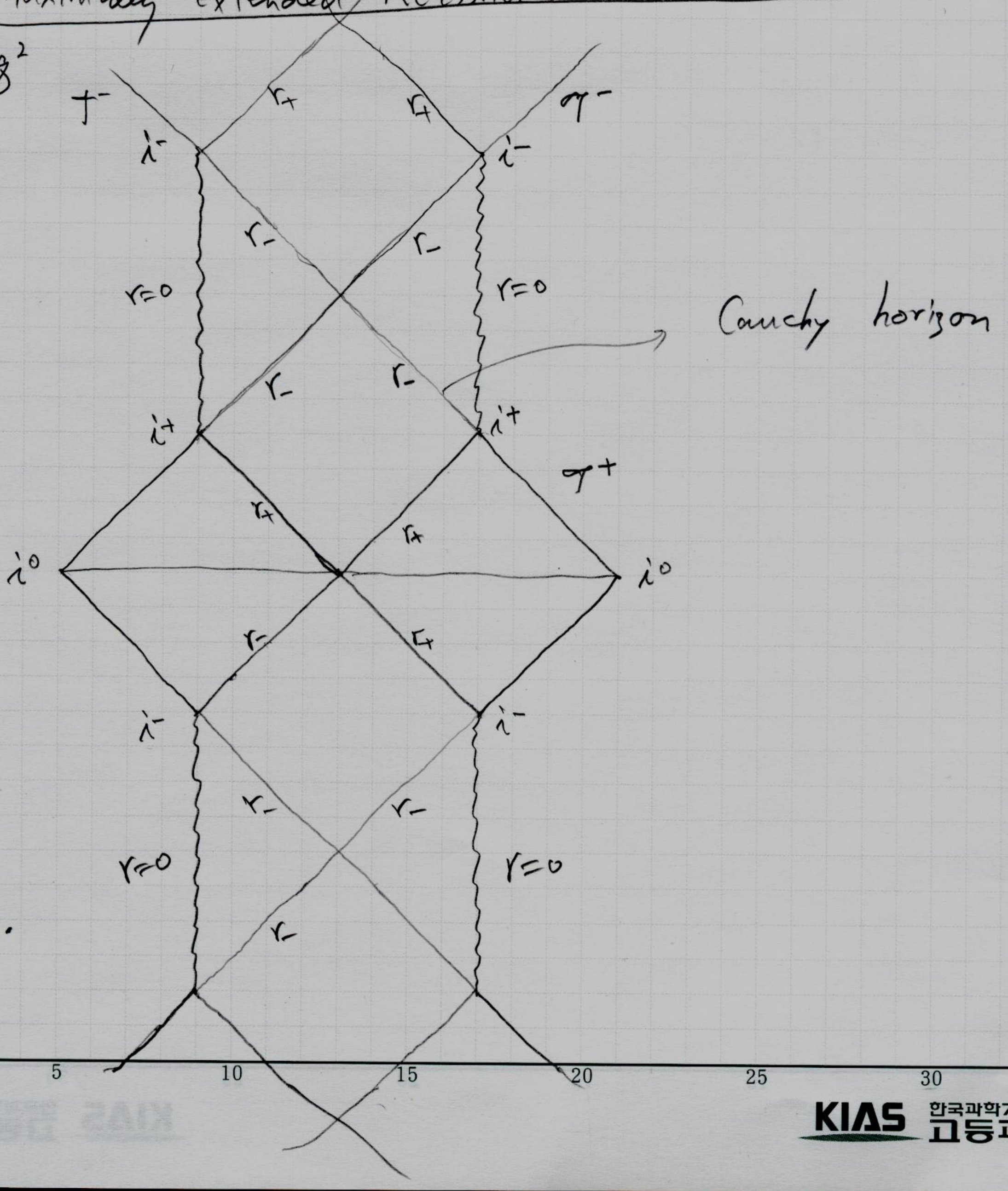


* Realistic collapsed black hole:

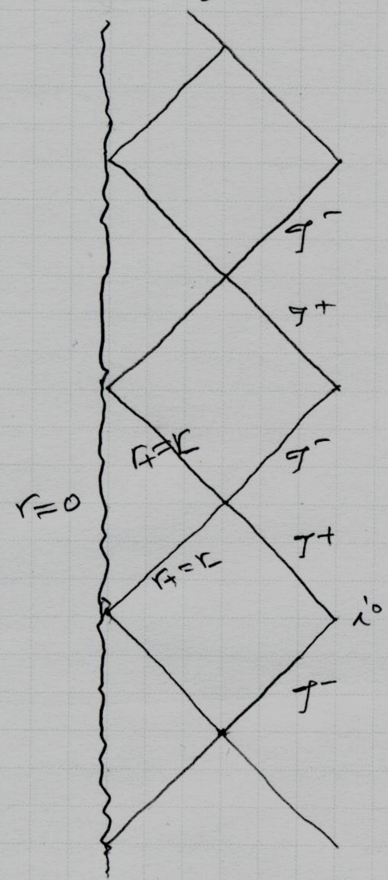


* Maximally extended Reissner-Nordström black holes.

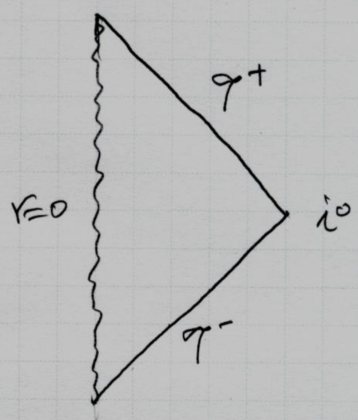
i) $M^2 > g^2$



ii) $M^2 = g^2$



iii) $M^2 < g^2$



<u>Prob. 1-2</u> : What is the longest proper time elapsed for a particle falling inwards starting from the event horizon at rest in the Schwarzschild b.h.? Note that the corresponding trajectory is that of the free falling particle. Any acceleration exerted on the particle simply increases the elapsed time.

Prob. 1-3 Kruskal extension ('60)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

$(t, r) \longrightarrow (T, X)$ such that

$$\left(\frac{r}{2M}-1\right)e^{r/2M} = X^2 - T^2,$$

$$\frac{t}{2M} = \ln\frac{X+T}{X-T} = \frac{2}{\tanh(T/X)}.$$

i) Show that

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Notice that the metric component is regular except for $r=0$.

ii) Draw the lines of $r=2M$, $r=0$, $r=$ const., $t=\pm\infty$, and $t=$ const. in the T-X plane.

iii) Show that the static killing field $\xi^a = (\partial_t)^a$ vanishes at $X=T=0$.

II. ADM quantities ⟨ 1. 'Energy' of the gravitational field

In special relativity where the spacetime is regarded as a fixed background, not affected by matter, the energy of matter fields is well-defined and it is conserved globally.

$$\partial^a T_{ab} = 0$$: "Local conservation of the energy-momentum of matter"

$$E = \int_\Sigma T_{00}\, d^3x = \int_\Sigma T_{ab} t^b d\Sigma^a \qquad t^a = (\partial_t)^a \uparrow \quad \Sigma$$

$$(d\Sigma^a = n^a d^3x .)$$

$$t\cdot\nabla E = \int_\Sigma \partial_t T_{00}\, d^3x = \int_\Sigma \partial_i T_{i0}\, d^3x = \oint_\infty T_{i0}\, dS^i = 0$$

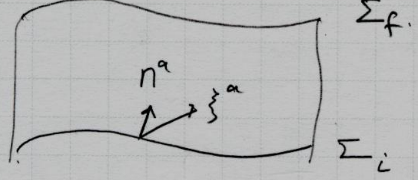
$$\partial^a T_{a0} = -\partial_t T_{00} + \partial_i T_{i0} = 0 .$$

In general relativity where the spacetime could be curved by matter contents, we still have

$$\nabla^a T_{ab} = 0 . \qquad (R_{ab} - \tfrac{1}{2} g_{ab} R = 8\pi G\, T_{ab}).$$

If the spacetime possesses a Killing vector field ξ^a, i.e., $\nabla_a \xi_b + \nabla_b \xi_a = 0$, we find

$$\nabla^a (T_{ab} \xi^b) = \nabla^a T_{ab} \xi^b + T_{ab} \nabla^a \xi^b = 0 .$$



$$0 = \int_M \nabla^a (T_{ab}\xi^b)$$

$$= \int_{\Sigma_i} T_{ab}\xi^b n^a + \int_{\Sigma_f} + \cancelto{0}{\int_\infty}$$

$$\Rightarrow \quad \int_{\Sigma_i} T_{ab}\xi^b n^a = \int_{\Sigma_f} T_{ab}\xi^b n^a$$

$\therefore \quad E \equiv \int_\Sigma T_{ab}\xi^b n^a \, d\Sigma$ is a conserved quantity.

For a time-like vector field t^a in general, however,

$$\nabla^a (T_{ab} t^b) = T_{ab}\nabla^a t^b \neq 0 \, . \qquad \int_\Sigma T_{ab} t^b n^a \neq \text{const.}$$

Thus, the local conservation of the energy-momentum of matter does not lead to a global conservation law. On physical grounds, this is not surprising since some energy of matter can be transfered into the gravitational field and vice versa.

$$T_{ab} \longrightarrow \underset{\substack{\uparrow \\ \text{matter}}}{T_{ab}} + \underset{\substack{\uparrow \\ \text{gravity}}}{t_{ab}}$$

Ex) Landau-Lifshitz "pseudotensor"

(162)

* <u>How to define the "energy" of g_{ab}?</u>

In a region of weak gravitation fields,

$$g_{ab} = \eta_{ab} + \epsilon \gamma^{(1)}_{ab} + \epsilon^2 \gamma^{(2)}_{ab} + \cdots$$

$$G_{ab}[g_{cd}] = R_{ab} - \frac{1}{2} g_{ab} R = 0$$

$$G_{ab}[\eta_{cd} + \gamma_{cd}] = \underbrace{G_{ab}[\eta_{cd}]}_{\overset{\|}{0}} + \epsilon \underbrace{G^{(1)}_{ab}[\gamma^{(1)}_{cd}]}_{\substack{=0 \\ \text{Linear order}}} + \epsilon^2 \underbrace{\left(G^{(1)}_{ab}[\gamma^{(2)}_{cd}] + G^{(2)}_{ab}[\gamma^{(1)}_{cd}]\right)}_{\substack{=0 \\ \text{Quadratic order}}} + \cdots$$

$$\Rightarrow \quad G^{(1)}_{ab}[\gamma^{(2)}_{cd}] = - G^{(2)}_{ab}[\gamma^{(1)}_{cd}] = 8\pi G\, t_{ab} ,$$

where

$$t_{ab} \equiv - \frac{1}{8\pi G} G^{(2)}_{ab}[\gamma^{(1)}_{cd}] = -\frac{1}{8\pi G}\Big[\frac{1}{2} \gamma^{(1)cd} \partial_a \partial_b \gamma^{(1)}_{cd} + \frac{1}{4} \partial_a \gamma^{(1)}_{cd} \partial_b \gamma^{(1)cd} + \cdots \Big]$$

Linearized grav. fields $\gamma^{(1)}_{ab}$ play like a sort of effective matter fields producing next order correction $\gamma^{(2)}_{ab}$ in gravi fields.

$$t_{ab} = t_{ba} \quad \& \quad \partial^a t_{ab} = 0 .$$

However, i) Not unique : $\tilde{t}_{ab} = t_{ab} + \partial^c \partial^d u_{acbd}$ with $u_{acbd} = u_{[ac]bd} = u_{ac[bd]} = u_{bdac}$

ii) Not gauge invariant.

But the total energy

$$E = \int_\Sigma t_{00}\, d^3x \quad \text{is well-defined.}$$

Actually, in general relativity there is no known meaningful notion of local energy density of the gravitational field.

Basic reason for this:

g_{ab} : "Background spacetime structure"

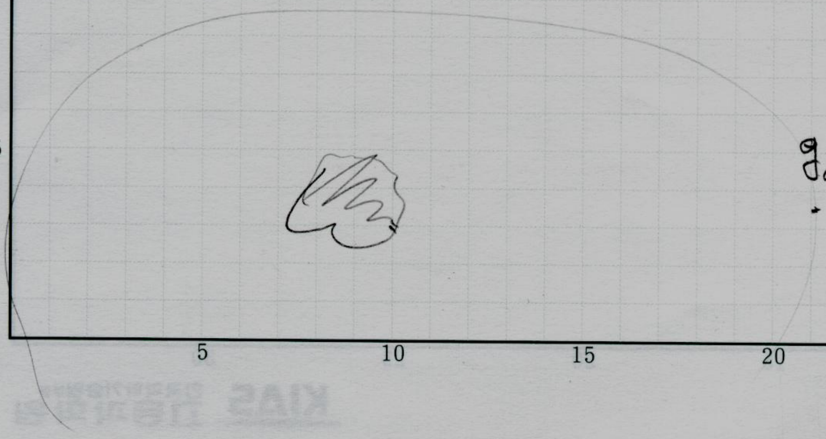
$+$

"Dynamical aspects of the grav. field"

"Energy" should be attributed to

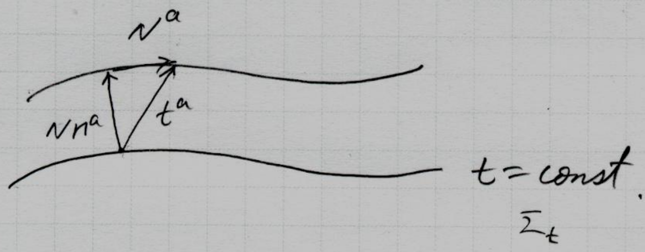
However, precisely decomposing g_{ab} into those two parts has not been known so far.

For an isolated system, however, the total energy can be defined well.



2. ADM mass

* Hamiltonian formulation



Define a time function t such that the manifold is a foliation of $t = \text{const}$ 3-D spacelike hypersurfaces.

$$t^a \cdot \nabla_a t = 1 .$$

$$t^a = N n^a + N^a .$$

Lapse function : $N = -g_{ab} t^a n^b = (n \cdot \nabla t)^{-1}$.

Shift vector : $N^a = h^a{}_b t^b$

where $h_{ab} = g_{ab} + n_a n_b$ is the induced spatial metric on Σ_t .

$$\sqrt{-g} = N \sqrt{h} .$$

$$(h_{ab} , \underset{\pounds_t h_{ab}}{\overset{\shortparallel}{\dot{h}_{ab}}}) \;\longrightarrow\; (h_{ab} , \pi^{ab})$$

$$I = \int_{\mathcal{M}} d^4x \, \underbrace{\sqrt{-g} \, R}_{= \mathcal{L}_G}$$

$$= \int d^4x \, \sqrt{h} \, N \left({}^{(3)}R + K_{ab}K^{ab} - K^2 \right)$$

$$\pi^{ab} \equiv \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h} \left(K^{ab} - K h^{ab} \right) .$$

where $K_{ab} = \frac{1}{2} \pounds_n h_{ab} = \frac{1}{2N} \left[\dot{h}_{ab} - D_a N_b - D_b N_a \right]$

" Extrinsic curvature of Σ_t .

• Hamiltonian density : $\mathcal{H}_G \equiv \pi^{ab}\dot{h}_{ab} - \mathcal{L}_G$.

$$= \sqrt{h}\,(N\mathcal{H} + N_a \mathcal{H}^a).$$

$$= \mathcal{H}_G[h_{ab}, \pi^{ab}, N, N^a].$$

Note that $\mathcal{H} = \mathcal{H}^a = 0$ for solutions.

• Hamiltonian : $H_G \equiv \int_{\Sigma_t} d^3x \sqrt{h}\,\mathcal{H}_G = 0$ in "on-shell"

$$\delta \int d^4x\, \mathcal{L}_G = \int d^4x\, \delta(\sqrt{-g}\, R_{ab} g^{ab})$$

$$= \int d^4x \sqrt{-g}\left[\left(R_{ab} - \tfrac{1}{2} g_{ab} R\right)\delta g^{ab} + \underbrace{\nabla^a v_a}_{\text{Boundary term}}\right].$$

$$v_a = \nabla^b \delta g_{ab} - g^{cd}\nabla_a \delta g_{cd}$$

$$\int_{\mathcal{M}} d^4x \sqrt{-g}\, \nabla^a v_a = \oint_{\partial\mathcal{M}} \underbrace{v_a n^a}\, d\Sigma$$

$$n^a v_a = n^a \underbrace{g^{bc}}_{= h^{bc} \mp n^b n^c} (\nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}) = n^a \underbrace{h^{bc}\nabla_c \delta g_{ab}}_{= 0} - h^{bc} \underbrace{n\cdot\nabla \delta g_{bc}}_{\neq 0 \text{ in general}}.$$

provided $\delta g_{ab} = 0$ on $\partial\mathcal{M}$.

Accurately speaking, the action principle is NOT well-defined with $\delta g_{ab} = 0$ only on the boundary.

Note that $\delta K = \frac{1}{2} \underline{h^{bc} n \cdot \nabla \delta g_{bc}}$.

Thus, this problem can be remedied easily as follows:

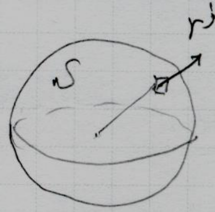
$$I = \int_{\mathcal{M}} d^4x\, \mathcal{L}_G \longrightarrow \underline{\underline{I' = \int_{\mathcal{M}} d^4x\, \mathcal{L}_G + 2 \oint_{\partial\mathcal{M}} K}} .$$

(Regge & Teitelboim '74).

Consequently,

$$H_G \longrightarrow H_G' = H_G + \alpha ,$$

where $\alpha = \frac{1}{16\pi G} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_S \left(\partial_i h_{ji} - \partial_j h_{ii}\right) r^j$



ADM energy:

$$\boxed{E = \frac{1}{16\pi G} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_S \left(\partial_i h_{ji} - \partial_j h_{ii}\right) r^j\, dS \quad \text{where } r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 .}$$

Ex) $ds^2 = -\left(1 - \frac{c}{r}\right) dt^2 + \frac{dr^2}{1 - c/r} + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

$$E = \frac{c}{2G} \equiv M . \left(= -\frac{1}{8\pi G} \oint_S \epsilon_{abcd} \nabla^c \xi^d \right) \qquad \xi^d = (\partial_t)^d .$$

3. Angular momentum

• Total spatial momentum: $P_a = (0, P_i)$

$$P_i = \frac{1}{8\pi G} \lim_{r\to\infty} \sum_{j=1}^{3} \int_S \left(K_{ji}\, r^j - K^j_{\ j}\, r_i \right) dS$$

ADM energy-momentum four-vector: $\mathcal{P}_a = -E n_a + P_a$.

• ADM angular momentum:

$$J \equiv \mathcal{P}_a \psi^a = P_a \psi^a, \quad \left(= \frac{1}{16\pi} \oint_S \epsilon_{abcd} \nabla^c \psi^d \right)$$

where ψ^a is a closed spacelike Killing vector field such as $\psi^a = \left(\frac{\partial}{\partial\varphi}\right)^a$.

4. Charges

$$Q = \frac{1}{8\pi} \oint_S {}^*\mathbb{F}.$$

Prob. 2-1

i) A black p-brane solution of the action

$$I = \frac{1}{2\kappa^2}\int d^D x \sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - e^{-\alpha\phi}\frac{1}{(d+1)!}F_{d+1}^2 \right]$$

may be given by

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 C(r)\,d\Omega_{D-d-1}^2 + D(r)\,\delta_{ij}\,dx^i dx^j .$$

$$\downarrow$$

$$SO(D-d)$$

$i = 1, \cdots, p$ with $p = d-1$.

Show that the ADM mass per unit volume is

$$M = \#\left[(\tilde{d}+1) r^{\tilde{d}+1}\partial_r C + (d-1) r^{\tilde{d}+1}\partial_r D - (\tilde{d}+1) r^{\tilde{d}}(B-C) \right]_{r\to\infty} .$$

Here $\tilde{d} = D-d-2$.

ii) Show that the total angular momentum of the Kerr black hole (p. 313 in Wald) is given by

$$J = Ma .$$