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Exercises for Perturbative String Theory.

2004. Feb. @ Kias - APCTP. Winterschool.

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[1] Show: $S=\sum_{n=1}^{\infty}(n-\theta) = \frac{1}{24}-\frac{1}{8}(2\theta-1)^2$.

let $f(t) = \sum_{n=1}^{\infty} e^{-(n-\theta)t}$

Then $f(t) = e^{\theta t}\cdot\frac{e^{-t}}{1-e^{-t}} = \frac{e^{\theta t}}{e^{t}-1}$.

$f'(t) = -\sum e^{-(n-\theta)t}\cdot(n-\theta)$. So we want: $\underline{\underline{-f'(0)}}$.

On the other hand, near $t\simeq 0$.

$$f(t) = \frac{(1+\theta t+\frac{1}{2}\theta^2t^2+\cdots)}{(t+\frac{t^2}{2}+\frac{t^3}{6}+\cdots)} = \frac{1}{t}\left(1+\theta t+\frac{1}{2}\theta^2t^2+\cdots\right)\left(1-\frac{t}{2}-\frac{t^2}{6}+\cdots\right.$$

$$= \frac{1}{t}\left(1+(\theta-\tfrac{1}{2})t+(\tfrac{1}{2}\theta^2-\tfrac{\theta}{2}+\tfrac{1}{4}-\tfrac{1}{6})t^2+\cdots\right)$$

$$f'(t) = -\frac{1}{t^2}+\cancel{(\theta-\tfrac{1}{2})}^{\to 0}+(\tfrac{1}{2}\theta^2-\tfrac{\theta}{2}+\tfrac{1}{12})\cdot 1+O(t).$$

as $t\to\epsilon$: $S = -\frac{1}{\epsilon^2}+\frac{1}{24}-\frac{1}{8}(2\theta-1)^2 = -f'(\epsilon)$

If you want further justification: look zeta function regularization method.

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[2] $L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha^{\mu}_{-n}\alpha^{\nu}_{n}\eta_{\mu\nu} = \sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha^{\nu}_{n}\eta_{\mu\nu} + \frac{1}{2}\sum\eta_{\mu\nu}\eta^{\mu\nu}n.$

$$= \; :L_0: + \frac{1}{2}D\left(-\frac{1}{12}\right).$$

not $\underline{\underline{D-2}}$.

* ~~Longitudinal modes~~ Negative normed ~~states~~ oscillators does not cancel transverse mode's in zero-point Energy Calculation.

But for mass formula, it is D-2 { Either X + ghost in covariant Qz. or $X^{\perp}$ in light cone Qz (D-2)

$$\frac{\alpha'}{4}M^2 = N - \frac{D-2}{24}$$

[3] ③ Show that

$$S = \frac{1}{4\pi\alpha'}\int \sqrt{-h}\, \underbrace{h^{\alpha\beta}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu G_{\mu\nu}}_{A} + \frac{1}{2\pi\alpha'}\int \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) = \frac{1}{2\pi\alpha'}\int \sqrt{-\det(\partial_a X\cdot \partial_b X)} + \frac{1}{2\pi\alpha'}\int_{B} \text{E-term}$$

$$\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \frac{1}{2\pi\alpha'}\left(\partial_\alpha X\cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta}\, \partial X\cdot \partial X\right) + 0.$$

So: $\det(\partial_\alpha X\cdot\partial_\beta X) = \left(\frac{1}{2}\partial X\cdot \partial X\right)^2 \det h_{\alpha\beta}.$

Taking square root:

$$\sqrt{-h}\cdot A = \sqrt{-\det \partial_\alpha X \partial_\beta X}\cdot \frac{A}{\frac{1}{2}\cancel{\partial X\cdot\partial X}} = 2\sqrt{-G}. \qquad \boxed{\text{QED}}$$

One should notice that

$$h^{\alpha\beta}\delta h_{\alpha\beta} = \delta \operatorname{Tr} \ln (h_{\alpha\beta}) = \delta \ln(\det(h))$$

$$\delta\sqrt{-h} = \frac{1}{\sqrt{-h}}\delta\sqrt{-h} = \delta \ln\sqrt{-h} = \frac{1}{2}\delta \operatorname{Tr} \ln h_{\alpha\beta}$$

$$= \frac{1}{2} g^{-1}\delta g.$$

$$= -\frac{1}{2} g\cdot \delta g^{-1}$$

$$= -\frac{1}{2} g_{\alpha\beta}\,\delta g^{\alpha\beta}.$$

(So Even before we consider the Eq. of M for X, we get the Equivalence.)

II. Mode Expansion.

II-1

Closed String. ($X(\sigma+l) = X(\sigma)$)

$$X = x + ap\tau + b \sum_{n\neq 0} \left[\frac{\alpha_n}{n} e^{-i\frac{2\pi n}{l}(\sigma+\tau)} + \frac{\tilde{\alpha}_n}{n} e^{+i\frac{2\pi n}{l}(\sigma-\tau)} \right]$$

$$\dot{X} = ap - i\frac{2\pi}{l} b \sum_{n\neq 0} \left[\alpha_n e^{-i\frac{2\pi n}{l}(\sigma+\tau)} + \tilde{\alpha}_n e^{i\frac{2\pi n}{l}(\sigma-\tau)} \right]$$

$$[X(\sigma), \dot{X}(\sigma')] = ai - i\frac{2\pi}{l} b^2 \sum_{n,m} \left\{ \left[\frac{\alpha_n}{n}, \alpha_m\right] e^{-i\frac{2\pi}{l}(n\sigma+n'\sigma')} + \left[\frac{\tilde{\alpha}_n}{n}, \tilde{\alpha}_m\right] e^{i\frac{2\pi}{l}(n\sigma+m\sigma')} \right\} \times e^{-i\frac{2\pi n}{l}\tau}$$

So, set $\boxed{\frac{a}{2} = -\frac{2\pi}{l} b^2}$ $\quad b = i\sqrt{\frac{al}{4\pi}}$

Then $[X(\sigma), \dot{X}(\sigma')] = ai \left(1 + \sum e^{-i\frac{2\pi}{l} n(\sigma-\sigma')} \right)$

$$= 2\pi a i \, \delta\left(\frac{2\pi}{l}(\sigma-\sigma')\right)$$

$$= i a l \, \delta(\sigma-\sigma').$$

Then: $\boxed{al = 2\pi\alpha'}$ since $P_X = \frac{\dot{X}}{2\pi\alpha'}$

Then. $b = i\sqrt{\frac{\alpha'}{2}}$ regardless of l.

$$\boxed{a = \frac{2\pi\alpha'}{l}}$$

[This is consistent with

$p = \int \frac{\dot{X}}{2\pi\alpha'} d\sigma = \frac{apl}{2\pi\alpha'}$ So: $\boxed{\frac{al}{2\pi\alpha'} = 1}$]

$$\cancel{\sum_n e^{-i(n+\frac{1}{2})(\sigma-\sigma')} = e^{-i\frac{1}{2}(\sigma-\sigma')} \sum e^{-in(\sigma-\sigma')} \stackrel{?}{=} e^{-i\frac{1}{2}(\sigma-\sigma')} [2\pi\delta(\sigma-\sigma') - 1]}$$

$$\sum \cos(n+\tfrac{1}{2})\sigma \cos(n+\tfrac{1}{2})\sigma' = \pi\,\delta(\sigma-\sigma')$$

b

$\sum a_i = 1$
$\sum b_i = 1$
$\boxed{\sum\left(\frac{a_i+b_i}{2}\right) = 1}$

$$1 + \sum_{n\neq 0} e^{in(\sigma-\sigma')} = 2\pi\,\delta(\sigma-\sigma')$$

Open string

$$X = x + 2\alpha' p\tau + i\sqrt{\frac{\alpha'}{2}}\beta \sum \frac{\alpha_n}{n}\cos n\sigma\, e^{-in\tau}$$

$$\dot X = 2\alpha' p + \sqrt{\frac{\alpha'}{2}}\beta \sum \alpha_n \cos n\sigma\, e^{-in\tau}$$

$$[X, \dot X] = 2\alpha'[x, p] + i\frac{\alpha'}{2}\beta^2\sum_n\left[\frac{\alpha_n}{n}, \alpha_m\right]\cos n\sigma \cos m\sigma'\, e^{-i(n+m)\tau}$$

$$= 2\alpha' i\left[1 + \frac{\beta^2}{4}\sum \cos n\sigma \cos n\sigma'\right] = \pi\alpha' i\,\delta(\sigma-\sigma'), \text{ with } \beta = 2$$

$-\frac{l\alpha}{\pi}$

$$\left[\because \sum \cos n\sigma \cos n\sigma' = \sum \frac{1}{4}\left(e^{in\sigma} + e^{-in\sigma}\right)\left(e^{in\sigma'} + e^{-in\sigma'}\right)\right.$$

$$= \sum_{n\neq 0}\left[\frac{1}{4}\left(e^{in(\sigma+\sigma')} + e^{-in(\sigma+\sigma')}\right) + \frac{1}{4}\left(e^{in(\sigma-\sigma')} + e^{-in(\sigma-\sigma')}\right)\right]$$

$$= \frac{1}{4}\left((2\pi\,\delta_{\sigma+\sigma',0} - 1)\times 2 + (2\pi(\delta_{\sigma-\sigma',0}) - 1)\times 2\right)$$

$$= -1 + \frac{\pi}{2}\left(\delta_{\sigma,\sigma'} + \cancel{\delta_{\sigma,-\sigma'}}\right) = \quad \delta(|\sigma| - |\sigma'|)$$

So take $\underline{\beta = 2}$ $\quad \boxed{0 \leq \sigma \leq \pi.}$

then, $[X, \dot X] = 2\alpha' i\left(\cancel{1} + \frac{\pi}{2}\delta_{\sigma,\sigma'} \cancel{-1}\right)$

$= \pi\alpha' i.$ $\quad$ So,

If $p = \frac{\dot X}{2\pi\alpha'}$,

So:

$$X = x + ap\tau + b\sum a_n \cos\frac{n\sigma\pi}{l}\, e^{-\frac{n\pi i}{l}\tau}$$

$$\dot X = ap + b\sum_n \frac{(-i)n a_n \pi}{l}\cos\left(\frac{n\pi\sigma}{l}\right) e^{-\frac{n\pi i}{l}\tau}.$$

$$[X, \dot X] = ai + b^2\left(\frac{-i\pi}{l}\right)\sum\left[a_n, n' a_{n'}\right]\cos\frac{n\pi\sigma}{l}\cos\frac{n'\pi\sigma'}{l}$$

$$1 + \sum \cos\frac{n\pi\sigma}{l}\cos\frac{n'\pi}{l}\sigma' = \frac{\pi}{2}\delta\left(\frac{\pi}{l}(\sigma-\sigma')\right)$$

$$= \frac{l}{2}\delta(\sigma-\sigma')$$

$$[a_n, n' a_{n'}] = \delta_{n+n',0}$$

$$[a_n, a_{-n}] = -\frac{1}{n},$$

l

$[\alpha_n, \alpha_n] = n \cdot$ $\qquad [a_n, a_n] = -\frac{1}{n}$

$\alpha_n = n a_n$ $\qquad$ So: $\boxed{[a_n, -n a_n] = 1}$

$[n a_n, -n a_n] = n$

$\boxed{-b^2 \frac{\pi}{l} = +a}$ $\qquad b = i\sqrt{\frac{al}{\pi}} =$

Then

$[X, \dot{X}] = \cdot i a \cdot \frac{l}{\cancel{2}} \delta(\sigma-\sigma') \cdot = 2\pi\alpha' \delta(\sigma-\sigma')$

So: $\quad a = \frac{2\pi\alpha'}{l}$ $\qquad$ if $l = \pi$; $\boxed{a = 2\alpha'}$

$p = \int_0^l \frac{1}{2\pi\alpha'} a p \, d\sigma = \cdot \frac{apl}{2\pi\alpha'}$ $\quad \therefore$ $a = \frac{2\pi\alpha'}{l}$

Something wrong!

So: $\quad X = x + \frac{2\pi\alpha'}{l} p\tau + i\sqrt{\frac{al}{\pi}} \sum (\cdots)$

$b = i\sqrt{\frac{la}{\pi}}$

$l = \pi,$

$b = \sqrt{2\alpha'}$

It seems that for consistency, we need to set

$$\boxed{1 + \sum_{n \neq 0} \cos n\sigma \cos n\sigma' = \pi \delta(\sigma - \sigma')}$$

Then; let $p = \frac{1}{2\pi\alpha'} \dot{X}$,

$[X(\sigma) P(\sigma')] = i\delta(\sigma-\sigma')$,

$[X(\sigma), \dot{X}(\sigma')] = ia \cdot l \, \delta(\sigma-\sigma') = i 2\pi\alpha' \delta$

So: $\boxed{a l = 2\pi\alpha'}$ ①

So $\quad a = \frac{2\pi\alpha'}{l}$

Consistency

note: $p = \int_0^l \frac{1}{2\pi\alpha'} (ap) = \frac{al}{2\pi\alpha'} p$.

So: $\boxed{al = 2\pi\alpha'}$ ②

$$\boxed{b = i\sqrt{\frac{l}{\pi} \frac{2\pi\alpha'}{l}} = i\sqrt{2\alpha'} \text{ regardless of } l}$$

① and ② is consistent

for open string set $l = \pi$, then $\boxed{a = 2\alpha'}$

$b = i\sqrt{\frac{al}{\pi}}$, $\quad l = \pi \Rightarrow \boxed{b = i\sqrt{2\alpha'}}$

Summary: $X = x + \frac{2\pi\alpha'}{\ell} p\tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n}{n} \cos\frac{n\pi}{\ell}\sigma \; e^{-\frac{n\pi}{\ell} i\tau}$

open string:

$n\sigma + m\sigma'$

· Closed string: $X = x + \frac{2\pi\alpha'}{\ell} p\tau + i\sqrt{\frac{\alpha'}{2}} \sum \left[\frac{\alpha_n}{n} e^{-\frac{2\pi i}{\ell} n(\sigma+\tau)} + \frac{\tilde{\alpha}_n}{n} e^{+\frac{2\pi i}{\ell} n(\sigma-\tau)} \right]$

So, Except the $\sqrt{\frac{\alpha'}{2}}$ v.s $\sqrt{2\alpha'}$

Notice that if one put $\boxed{\alpha_n = \tilde{\alpha}_n}$

closed string Expansion reduces to open string

if one set $\ell \to 2\ell$

Suppose $X(\sigma+\ell) = -X(\sigma)$: Then

The zero mode must be zero. $x = p = 0$.

$$X = b \sum \left[\frac{\alpha_n}{n+\frac{1}{2}} e^{-i\frac{2\pi}{\ell}(n+\frac{1}{2})(\sigma+\tau)} + \frac{\tilde{\alpha}_n}{n+\frac{1}{2}} e^{i\frac{2\pi}{\ell}(n+\frac{1}{2})(\sigma-\tau)} \right]$$

$$[X(\sigma), \dot{X}(\sigma')] = +b\left(-i\frac{2\pi}{\ell}\right) b \sum \left\{ \left[\frac{\alpha_n}{n+\frac{1}{2}}, \alpha_{n'} \right] e^{-i\frac{2\pi}{\ell}[(n+\frac{1}{2})\sigma + (n'+\frac{1}{2})\sigma']} + \left[\frac{\tilde{\alpha}_n}{n+\frac{1}{2}}, \tilde{\alpha}_{n'} \right] e^{+i\frac{2\pi}{\ell}[(n+\frac{1}{2})\sigma + (n'+\frac{1}{2})\sigma']} \right\} e^{-i\frac{2\pi}{\ell}(n+\frac{1}{2}\dots}$$

let $\boxed{[\alpha_n, \alpha_{-n}] = (n+\frac{1}{2})}$. $\quad n+\frac{1}{2} = -(n'+\frac{1}{2})$. $\quad \boxed{n+n'+1=0}$

1

$$= b^2\left(-i\frac{2\pi}{\ell}\right) \sum_n \left[e^{-i\frac{2\pi}{\ell}(n+\frac{1}{2})(\sigma-\sigma')} + e^{+i\frac{2\pi}{\ell}(n+\frac{1}{2})(\sigma-\sigma')} \right] \cancel{e^{-i\frac{2\pi}{\ell}(n+n'+1)\tau}}$$

$$\cancel{= b^2\left(-i\frac{2\pi}{\ell}\right) \cdot \left(\frac{\ell}{2\pi} 2\pi\delta(\sigma-\sigma') - e^{-i\frac{\pi}{\ell}(\sigma-\sigma')} + 2\pi\delta(\sigma-\sigma') - e^{+\frac{\pi i}{\ell}(\sigma-\sigma')}\right)}$$

$\sigma') - 1]$

$b = \sqrt{\alpha'/2}$

$$= b^2(-i2\pi) \cdot 2\delta - \cancel{b^2\left(-i\frac{2\pi}{\ell}\right) 2\cos\left[\frac{\pi}{\ell}(\sigma-\sigma')\right]}$$

$$= -4\pi i b^2 \left[\delta_{\sigma,\sigma'} - \cos\frac{\pi}{\ell}(\sigma-\sigma') \right]$$

$$= i2\pi\alpha' \left[\delta_{\sigma,\sigma'} - \cos\frac{\pi}{\ell}(\sigma-\sigma') \right]$$

III ◎ Calculation of β-function.

$$I_B(\phi) = \frac{1}{2}\int d^2x\, g_{ij}(\phi)\, \partial_\mu\phi^i \partial_\mu\phi^j \qquad \text{let } \pi = \delta\phi. \qquad \phi \xrightarrow{\delta\phi} \phi'$$

$$I_B(\phi+\pi) = I_B(\phi) + \frac{\delta I_B}{\delta\phi}\cdot\pi + \frac{1}{2}\pi\cdot\frac{\delta^2 I}{\delta\phi\delta\phi}\cdot\pi + (\text{higher}).$$

(* In this expansion π is not covariant.)

Consider

$$\Omega_B[\phi] = \int[d\pi]\, e^{\frac{i}{\hbar}\left[I_B[\phi+\pi] - I_B[\phi] - \frac{\delta I_B}{\delta\phi}\cdot\pi\right]}$$

It generates all diagram with at least one loop with external trees amputated.

$\Omega_B[\phi]$ is reparametrization invariant.

The problem of non-covariance in the π-expansion

$\Rightarrow$ Express π^i in terms of ξ which is a covariant tensor.

$$\lambda(0) = \phi \xrightarrow[\lambda(t)]{} \phi+\pi = \lambda(1)$$

$$g_{ij}\xi^i\xi^j = s^2, \qquad s = \int_\phi^{\phi+\pi} dt\, \sqrt{g_{ij}\dot\lambda_i\dot\lambda_j}$$

$\underline{\underline{\xi^i}}$ is a contravariant vector under reparametrization.

$\therefore$ Expansion of Tensor in ξ is covariant (explicitly)

$$T_{k_1\cdots k_m}(\phi+\pi) = \sum \frac{1}{n!}\frac{\partial}{\partial\xi_{i_1}}\cdots\frac{\partial}{\partial\xi_{i_n}} T_{k_1\cdots k_m}(\phi)\,\xi^{i_1}\cdots\xi^{i_n}$$

Use Riemann <u>Normal Coordinate</u>

<u>geodesic = straight line</u> $\qquad \dot\xi(t) = \dot{x}t.$

iv)

Method of

$\boxed{\text{Normal co-ordinate Expansion.}}$

$\phi \xrightarrow[\lambda(t)]{} \phi' = \phi + \pi \qquad g_{ij}\xi^i\xi^j = s^2$

III - 2

$\lambda(t) = \lambda_0 + t\lambda_1 + \frac{t^2}{2}\lambda_2 + \frac{t^3}{3!}\lambda_3 + \cdots \qquad \lambda_1 = \xi \qquad \lambda_2$

$\ddot{\lambda}^i + \Gamma^i_{jk}(\lambda)\dot{\lambda}^j\dot{\lambda}^k = 0 \ : \ (\ddot{\lambda} = \lambda_2 + \frac{t}{1}\lambda_3 + \frac{1}{2}\lambda_4 t^2 + \cdots$

$+ \Gamma^i_{jk} + t\xi^l \partial_l \Gamma^i_{jk} \ (\lambda_1^j + t\lambda_2^j + \frac{t^2}{2!}\lambda_3^j + \cdots)(\lambda_1^k + t\lambda_2^k + \frac{t^2}{2}\lambda_3 + \cdots) = 0.$

$$\begin{cases} \lambda_2^i + \Gamma^i_{jk}\lambda_1^j\lambda_1^k = 0 \qquad \therefore \lambda_2^i = -\Gamma^i_{jk}\xi^j\xi^k \\ \lambda_3^i + \Gamma^i_{jk}(\lambda_1^j\lambda_2^k + \lambda_2^j\lambda_1^k) + \lambda_1^l \partial_l\Gamma^i_{jk}\lambda_1^j\lambda_1^k = 0. \\ \lambda_3^i = +\Gamma^i_{jk}(\Gamma^k_{lm}\xi^l\xi^m\xi^j + \Gamma^j_{lm}\xi^l\xi^m\xi^k) - \partial_l\Gamma^i_{jk}\xi^l\xi^j\xi^k \\ \quad = -\tilde{\nabla}_l\Gamma^i_{jk}\xi^l\xi^j\xi^k \end{cases}$$

$\boxed{\text{geodesic lines in Normal co-ordinate System.}}$

$\overset{s'}{\underset{x}{0}} \ \overset{s}{\underset{x'}{}} \qquad \xi = \dot{\lambda}(0). \qquad \|\xi\| = s$

$\lambda(t) = t\xi$. So $\Gamma^i_{jk}\xi^j\xi^k = 0$, e.t.c. & $\partial_l\Gamma_{jk}) = 0$ etc.

In normal coordinate,

$\bar{R}^i_{jkl} = \partial_k\bar{\Gamma}^i_{jl} - \partial_l\bar{\Gamma}^i_{jk}. \qquad \boxed{\partial_l\Gamma^i_{jk} + \partial_j\Gamma^i_{lk} + \partial_k\Gamma^i_{lj} = 0}$

$\bar{R}^i_{lkj} = \partial_k\Gamma^i_{lj} - \partial_j\Gamma^i_{lk}$

$\bar{R}^i_{jkl} + \bar{R}^i_{lkj} = 2\partial_k\Gamma^i_{jl} + (\partial_k\Gamma^i_{jl}) = 3\partial_k\Gamma^i_{jl}$

So: $\partial_k\bar{\Gamma}^i_{jl} = \frac{1}{3}(\bar{R}^i_{jkl} + \bar{R}^i_{lkj})$

$\frac{\partial}{\partial\xi^i}T_{k_1\cdots k_n}(\phi) = D_i\bar{T}_{k_1\cdots k_n} = \frac{\partial}{\partial\xi^i}T - \Gamma^j_{ik_p}T_{k_1\cdots \underset{p\text{th}}{j} \cdots k_n}$

$\partial_{\xi_{i_1}}\partial_{\xi_{i_2}}T_{k_1\cdots k_m} = D_{(i_1}D_{i_2)}\bar{T}_{k_1\cdots k_m}$

$D_{i'}D_iT = \frac{\partial}{\partial\xi^{i'}}\frac{\partial}{\partial\xi^i}T - \partial_{i'}\Gamma^j_{ik_p}T_{k_1\cdots j\cdots k_n}$ in normal coordinate.

$\hat{\partial} X^{\nu}$

Therefore,

$$g_{ij}(\varphi+\pi) = g_{ij}(\varphi) + \frac{1}{3} R_{i k_1 j k_2}(\varphi)\, \xi^{k_1}\xi^{k_2} + \cdots$$

$$\phi+\pi = \phi + \xi - \frac{1}{2}\Gamma^{i}_{j_1 j_2}\xi^{j_1}\xi^{j_2} - \frac{1}{3!}\Gamma^{i}_{j_1 j_2 j_3}\xi^{j_1}\xi^{j_2}\xi^{j_3} - \cdots$$

So $\partial_\mu(\varphi+\pi) = \partial_\mu\phi + \partial_\mu\xi - \frac{1}{2}(\partial_j \Gamma^{i}_{j_1 j_2})\xi^{j_1}\xi^{j_2}\partial_\mu\varphi^j + \cdots$

$$= \partial_\mu\phi^i + D_\mu\xi^i - \frac{1}{2}\frac{1}{3}\left(\bar{R}^i_{j_1 j j_2} + \bar{R}^i_{j_2 j j_1}\right)\xi^{j_1}\xi^{j_2}\partial_\mu\varphi^j + \cdots$$

$$= \partial_\mu\phi^i + D_\mu\xi^i - \frac{1}{3}R^i_{j_1 j j_2}\xi^{j_1}\xi^{j_2}\partial_\mu\varphi^j$$

$$= \partial_\mu\phi^i + D_\mu\xi^i + \frac{1}{3}R^i_{j_1 j_2 j}\xi^{j_1}\xi^{j_2}\partial_\mu\varphi^j$$

$$I_B(\varphi+\pi) - I_B(\varphi) = \int d^2x \left[g_{ij}\,\partial_\mu\varphi^i D_\mu\xi^i + \left(\frac{1}{2}+1\right)\frac{1}{3}R^i_{j_1 j_2 j}\,\xi^{j_1}\xi^{j_2}\partial_\mu\varphi^j\partial^\mu\varphi_i + \frac{1}{2}D_\mu\xi D^\mu\xi + \cdots \right]$$

Require $\frac{\delta I_B(\varphi)}{\delta\xi} = 0.$

So $\langle I_B(\varphi+\pi)\rangle = I_B(\varphi) + \frac{1}{2}\int d\sigma^2\, \overbrace{\underbrace{R_{iabj}\,\partial_\mu\varphi^i\partial_\mu\varphi^j}\langle\xi^a\xi^b\rangle}^{C_{aa}}$

$$= I_B(\varphi) + \frac{\mu^{n-2}}{2\pi\epsilon}C_{aa}$$

So $g_{ij} \to g_{ij} + R_{iabj}\,\delta^{ab}\frac{\ln\Lambda}{2\pi} = g_{ij} + \delta g_{ij}$

and $\beta = \Lambda\frac{\partial}{\partial\Lambda}g_{ij} = \underline{R_i{}^a{}_{aj}} = R_{ij}$

$$\boxed{\beta_{g_{ij}} = R_{ij}}$$

$$\frac{k^{n-1}}{(k+\mu)(k-\mu)} \sim \frac{\mu^{n-1}}{2\mu} = \left(\frac{\mu^{n-2}}{2}\right) \quad k^{n-3}$$

ρ^{ab}

$$i\Delta_F(0) = \frac{1}{(2\pi)^2}\int d^n k\, \frac{1}{k^2+\mu^2}$$

$$\frac{1}{2\pi}\int\frac{k^{n-1}dk}{k^2} = \frac{dk}{k^{3-n}} = \frac{1}{n-2}\frac{1}{k^{2-n}}$$

or $\frac{2\pi}{(2\pi)^2}\int^{\Lambda}\frac{k\,dk}{k^2+\mu^2} \sim \frac{1}{2\pi}\ln\Lambda$

IV. Toroidal compactification in CFT. & Poisson Resummation.

Set $G_d = 1$.

Effect of Cptz. : $e^{2\pi i R p} = 1$: $p = \frac{n}{R}$. $n \in \mathbb{Z}$.

$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w$: $w \in \mathbb{Z}$.

$$\partial X = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{+\infty} \frac{\alpha_m}{z^{m+1}}, \quad \bar\partial X = -i\sqrt{\frac{\alpha'}{2}} \sum \frac{\tilde\alpha_m}{\bar z^{m+1}}.$$

$$\boxed{2\pi R W = \int_0^{2\pi} d\sigma\, \partial X}$$

$$\boxed{z = e^{(\tau + i\sigma)}} \quad \partial_\sigma = \frac{\partial z}{\partial \sigma}\partial_z + \frac{\partial \bar z}{\partial \sigma}\partial_{\bar z} = i(\partial_z - \partial_{\bar z})$$

$$d\sigma = \frac{1}{zi}(dz - d\bar z)$$

$$\int d\bar z(\partial_z X) = \oint d\bar z \cdot \frac{\alpha_m}{z^{m+1}} = 0$$

The most basic but Important exercise is the mode Expansion.

$$X(\tau,\sigma) = x^i + \alpha' p \tau + i\sqrt{\frac{\alpha'}{2}} \sum\left[\frac{\alpha_n}{n} e^{-\frac{2\pi i n(\sigma+\tau)}{\ell}} + \frac{\tilde\alpha_n}{n} e^{+\frac{2\pi i n}{\ell}(\sigma-\tau)}\right]$$

$+ \sigma W R$

Take. $\boxed{\tau \to i\tau}$, $\boxed{z = e^{+\frac{2\pi}{\ell}(\tau + i\sigma)}}$ $\quad \boxed{i\sigma - \tau}$

$i(i\tau + \sigma) = -(\tau - i\sigma)$.

$\boxed{it = t_E}$

$p \cdot \tau = \frac{dx}{d\tau} \cdot \tau \to (ip_E) \cdot (-i\tau_E)$

$$z = \exp\left[+\frac{2\pi}{\ell}(\tau + i\sigma)\right], \quad \bar z = \exp\left[+\frac{2\pi}{\ell}(\tau - i\sigma)\right]$$

$\boxed{\sigma W R}$ $\quad \boxed{\sigma = 0, \ell}$

$$X(\tau,\sigma) = x^i + \alpha' p \cdot \tau + \sigma W R + i\sqrt{\frac{\alpha'}{2}} \sum\left(\frac{\alpha_n}{n} \cdot \frac{1}{z^n} + \frac{\tilde\alpha_n}{n} \frac{1}{\bar z^n}\right).$$

take $\ell = 2\pi$

$$\Rightarrow \frac{x^i}{2} + i\frac{p}{2}(\ln z + \ln \bar z) + 2\pi$$

∂

Set $\boxed{\frac{p}{2} = p_L}$ $\quad \boxed{z = e^{\tau + i\sigma}}$ $\quad \ell = R$

$$\boxed{\sigma = \frac{\ln z - \ln \bar z}{2i}}$$

$\frac{n}{R} \tau + w R \sigma$

$$\ln z = \frac{2\pi}{\ell}(\tau + i\sigma) = \boxed{\tau + i\sigma}$$

$$\partial X = \frac{\alpha' p}{2}\left(\frac{-i}{z}\right) + i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n \left(\frac{-1}{z^{n+1}}\right) =$$

$$\frac{\partial \sigma}{\partial z} = \frac{1}{2i}\frac{1}{z} \qquad \tau = \frac{1}{2}(\ln z + \ln \bar z), \quad \tau = \frac{1}{2}\ln z$$

$$\frac{\partial \tau}{\partial z} = \frac{1}{2}\frac{1}{z}$$

$$= -i\sqrt{\frac{\alpha'}{2}} \sum \frac{\alpha_n}{z^{n+1}} \qquad \left(\quad\right) = \sqrt{\frac{\alpha'}{2}}\alpha_0 = \frac{p\alpha'}{}+ \frac{WR}{} \quad p$$

$\partial_z X = -i\sqrt{\frac{\alpha'}{2}}\sum \frac{\alpha_n}{z^{n+1}}$. $p = \frac{\dot X}{2\pi\alpha'}$

So: $p = \int^{2\pi} \frac{\dot X}{2\pi\alpha'} d\sigma = .$

So $\dot X = p\alpha'$

$$\boxed{p = \int_0^{2\pi} d\sigma \left(\frac{\dot X}{2\pi\alpha'}\right) = \frac{p\alpha'}{2\pi\alpha'}\cdot 2\pi = p}$$

$$\boxed{z = e^{\tau + i\sigma}}$$

So: $X = x + \alpha' p \tau + w\sigma R + i\sqrt{\frac{\alpha'}{2}}\sum_n \left[\frac{\alpha_n}{n} e^{-in(\sigma+\tau)} + \frac{\tilde\alpha_n}{n} e^{+in(\sigma-\tau)}\right]$

$X_E = x - i\alpha' p\tau + w\sigma R + i\sqrt{\frac{\alpha'}{2}}\sum\left[\frac{\alpha_n}{n}\frac{1}{z^n} + \frac{\tilde\alpha_n}{n}\frac{1}{\bar z^n}\right]$.

$$\left[\begin{array}{l} \tau = \frac12(\ln z + \ln\bar z) \\ \sigma = \frac{1}{2i}(\ln z - \ln \bar z) \end{array}\right.$$

$\partial_z X_E = -\frac{i}{2}(\alpha' p + wR) - i\frac{\sqrt{\alpha'}}{\sqrt 2}\sum_{n\neq 0}\alpha_n \frac{1}{z^{n+1}} = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n/z^{n+1}$.

$\partial_{\bar z} X_E = -\frac{i}{2}(\alpha' p - wR) - i\sqrt{\frac{\alpha'}{2}}\sum\tilde\alpha_n \frac{1}{\bar z^{n+1}}$.

So: $\frac12(\alpha'p + wR) = \sqrt{\frac{\alpha'}{2}}\alpha_0 = \frac{\alpha'}{2}P_L$ or $P_L = \frac{p}{R} + \frac{wR}{\alpha'}$

$\frac12(\alpha'p - wR) = \sqrt{\frac{\alpha'}{2}}\tilde\alpha_0$: $P_R = \frac{p}{R} - \frac{wR}{\alpha'}$

$p = \sqrt{\frac{1}{2\alpha'}}(\alpha_0+\tilde\alpha_0) = \frac12(P_L+P_R)$.

$L_0 = \frac14\alpha' P_L^2 + \sum_{n=1}^\infty \alpha_{-n}\alpha_n$, $\bar L_0 = \frac{\alpha'}{4}P_R^2 + \sum\tilde\alpha_{-n}\tilde\alpha_n$

(proof): $T_{zz} = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$.

$= \frac12\sum_{n,m} :\frac{\alpha_n}{z^{n+1}}\cdot\frac{\alpha_m}{z^{m+1}}: = \sum_{l=-\infty}^{\infty}\frac{L_l}{z^{l+2}}$.

$L_l = \frac12\sum_{n=-\infty}^{\infty} :\alpha_n\alpha_{l-n}:$

$$\boxed{L_0 = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n}$$

⊙ Partition function for X : $\quad Z = (q\bar{q})^{-\frac{c}{24}} \, Tr \, q^{L_0} \bar{q}^{\bar{L}_0}$, $\underline{c=1.}$

- oscillator part gives $\frac{1}{\eta\bar{\eta}}$.
- Zero-mode (momentum + Winding) gives a factor.

$$Z_0 = \sum_{n,w} q^{\alpha' P_L^2/4} \, \bar{q}^{\alpha' P_R^2/4} \qquad q = e^{2\pi i \tau} = e^{2\pi i (\tau_1) - 2\pi \tau_2}.$$

$$= \sum_{n,w} e^{2\pi i \alpha' [\frac{P_L^2}{4} - \frac{P_R^2}{4}]\tau_1 - 2\pi \tau_2 \alpha' (\frac{P_L^2}{4} + \frac{P_R^2}{4})}, \quad \begin{cases} P_L = \frac{n}{R} + \frac{wR}{\alpha'} . \\ P_R = \frac{n}{R} - \frac{wR}{\alpha'} . \end{cases}$$

$$= \sum_{n,w} e^{2\pi i \cancel{\alpha'} \cdot \frac{\tau_1}{\cancel{4}} (\cancel{4} nw / \cancel{\alpha'}) - 2\pi\tau_2 \alpha' \frac{1}{2} [(\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2]}$$

$$= \sum_{n,w} e^{2\pi i \tau_1 \cdot nw - \pi \tau_2 \alpha' [(\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2]} .$$

Poisson Resummation formula:

$$\sum e^{-\pi a n^2 + 2\pi i b n} = \frac{1}{\sqrt{a}} \cdot \sum_m e^{-\frac{\pi}{a}(m-b)^2}, \qquad \text{with } a = \frac{\alpha' \tau_2}{R^2} . \quad b = \tau_1 w$$

$$\Rightarrow Z_0 = \sum \frac{R}{\sqrt{\tau_2 \alpha'}} e^{-\frac{\pi}{\alpha' \tau_2} R^2 (m - \tau_1 w)^2 - \frac{\pi}{\alpha'} \tau_2 (wR)^2}$$

$$= \sum \frac{R}{\sqrt{\tau_2 \alpha'}} e^{-\frac{\pi}{\alpha'} \frac{R^2}{\tau_2} (|m - \tau w|^2)}$$

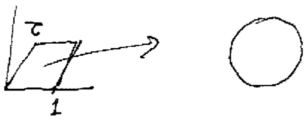
So: $\quad Z_X = \frac{1}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} \cdot \sum e^{-\pi R^2 |m - \tau w|^2 / \alpha' \tau_2}$

Poisson Resummation formular.

let $\quad \tilde{f}(y) = \int_{-\infty}^{\infty} dx \, e^{2\pi i x y} f(x).$

$\Rightarrow \quad \sum_n f(n) = \sum_m \tilde{f}(m).$

Ex : Perform the Path Integral.



with. $\quad X(\sigma^1 + 2\pi, \sigma^2) = X + (2\pi w R)$

$\quad X(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) = X(\sigma^1, \sigma^2) + 2\pi m R.$

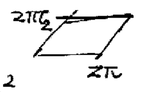
$\Rightarrow \quad X_{cl} = \sigma^1 w R + \sigma^2 (m - w\tau_1) R/\tau_2 .$

Poisson Resummation formula and Modular Inv. (3)

$$\cancel{\sum \int_{-\frac{1}{2}+k}^{+\frac{1}{2}+k} e^{2\pi i n x} f(x)\,dx = \tilde{f}(n)}. \quad \text{or} \quad \boxed{\sum_n e^{2\pi i n\cdot x} = \sum_n \delta(x-n)}$$

$$\sum_n \tilde{f}(n) = \sum_n \int_{-\infty}^{\infty} e^{2\pi i n x} f(x)\,dx$$

$$= \int dx \sum_n \delta(x-n)\, f(x) = \sum_n f(n).$$



$$\partial_1 X_{cl} = wR, \quad \partial_2 X_{cl} = \frac{R}{\tau_2}(m - w\tau_1).$$

$$S_E = \int (\partial_1 X_{cl})^2 + (\partial_2 X_{cl})^2 = \tau_2 \left[(wR)^2 + \left(\frac{R}{\tau_2}(m - w\tau_1)\right)^2 \right] = \tau_2 \left(\frac{R}{\tau_2}\right)^2 (2\pi)^2 \left| m - w\tau_1 - i\tau_2 \right|^2$$

$$= \left(\frac{R^2}{\tau_2}\right)(2\pi)^2 \left| m - w\tau \right|^2$$

$\frac{1}{4\pi\alpha'}$.

$$\text{So}: \quad S_E = \frac{1}{4\pi\alpha'} \cdot \frac{R^2}{\tau_2} \cdot (2\pi)^2 \, |m - w\tau|^2$$

$$= \frac{\pi R^2}{\tau_2} |m - \tau w|^2.$$

$$\int e^{-S_E(X)}\, dX = \sum_{m,w} e^{-S(m,w)} \cdot Z_{\text{fluctuation}}.$$

$\odot$ Fluctuation Integral :

$$\cancel{S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[(\partial_1 X)^2 + (\partial_2 X)^2 \right]}$$

$$\cancel{= \frac{1}{\pi\alpha'} \int \left[(\partial_z X)^2 + (\partial_{\bar z} X)^2 \right]}$$

$$\cancel{X = \text{zero mode} + \sum i\sqrt{\frac{\alpha'}{2}} \frac{\alpha_n}{n} \frac{1}{z^n}.}$$

$$\cancel{\partial X = -i\sqrt{\frac{\alpha'}{2}} \sum \frac{\alpha_n}{z^{n+1}}.}$$

$$\cancel{\int \partial X \cdot \partial X = -\frac{\alpha'}{2} \sum \alpha_n \alpha_{-n}}$$

Once we expressed $\mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$

in terms of Path Integral, the modular inv. is guaranteed since modular tr. is a Reparametrization.

But one can prove this more explicitly using Poisson Resum.

For fermions, $\psi = \sum_{n\in\mathbb{Z}} \psi_n e^{-in(\tau+\sigma)} + h.c.$ for Ramond sector.

$$\psi = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r e^{-ir(\tau+\sigma)} + h.c. \quad \text{for NS Sector.}$$

$$L_0 = (\cdots \text{from } X) + \sum_{r\in\mathbb{Z}+\frac{1}{2}} r\, \psi_{-r}\psi_r - a.$$

Zero point Energy.

NS sector: $8\cdot(-\frac{1}{24}) + 8(-\frac{1}{48}) = -\frac{1}{2}$.

R sector: $8(-\frac{1}{24}) + 8(\frac{1}{24}) = 0$.

Here we used. $\frac{1}{2}(1+2+3+\cdots) = -\frac{1}{12}\cdot\frac{1}{2} = -\frac{1}{24}$

$$\frac{1}{2}(\tfrac{1}{2}+\tfrac{3}{2}+\cdots) = \frac{1}{24}\cdot\frac{1}{2} = +\frac{1}{48}.$$

<u>Fermion zero point Energy</u> $= \frac{1}{2}\sum_{r\in-(\mathbb{Z}+\frac{1}{2})} r = \frac{1}{2}(-\frac{1}{2}-\frac{3}{2}-\cdots)$

- Dirichlet Boundary Condition and T-duality.

$$X(\sigma,\tau) = x_0 + 2\alpha' p\tau + i\sqrt{2\alpha'}\sum \frac{\alpha_n}{n}\cos n\sigma\, e^{-in\tau}$$

Exchange of $\tau\leftrightarrow\sigma$, $p\leftrightarrow w$. $R\to\frac{\alpha'}{R}$ etc.

$$X = X_L + \bar{X},$$

$$X = \tfrac{1}{2}x + \alpha' p(\tau+\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum\frac{\alpha_n}{n}e^{-in(\tau+\sigma)}$$

$$\bar{X} = \tfrac{1}{2}x + \alpha' p(\tau-\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum\frac{\alpha_n}{n}e^{-in(\tau-\sigma)}$$

Then. T-duality can be obtained from

$$\bar{X}\to-\bar{X}. \qquad \partial_\sigma X\big|_{\sigma=\pi}=0 \Rightarrow \partial_\tau X = 0.$$

$$X \overset{T}{\longmapsto} X' = X - \bar{X} = 2\alpha'\frac{n}{R}\sigma + i\sqrt{2\alpha'}\sum\frac{\alpha_n}{n}\sin n\sigma\, e^{-in\tau}$$

$$= 2nR'\sigma + \quad ''$$

$$X'(\sigma=\pi) = X'(\sigma=0) = 0.$$

$$\boxed{\partial_\sigma X = 0 \leftrightarrow \partial_\tau X = 0.}$$

V. (String frame $\longrightarrow$ Einstein frame) // Intersecting Dbrane, V-①

$\boxed{\odot \text{ Metric \& Curvature Convention : Landau - Lifshitz.}}$

$$R^i{}_{klm} = \partial_l \Gamma^i_{km} - \partial_m \Gamma^i_{kl} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \quad \cdots ①$$

$[l,m]$

$$\Gamma^i_{km} = \frac{1}{2} g^{in} (\partial_k g_{nm} + \partial_m g_{nk} - \partial_n g_{km}). \quad \cdots ②$$

Then, under $g_{ij} \doteq e^{2\phi}\tilde g_{ij}$ $\cdots ③$

$$\Gamma^i_{km} \doteq \tilde\Gamma^i_{km} + \partial_k\phi\,\delta^i_m + \partial_m\phi\,\delta^i_k - \partial_n\phi\cdot\tilde g^{in}\tilde g_{km} \quad \cdots ④$$

Then, $R_{km} = R^l{}_{klm} = \partial_l \Gamma^l_{km} - \partial_m\Gamma^l_{kl} + \Gamma^l_{nl}\Gamma^n_{km} - \Gamma^l_{nm}\Gamma^n_{kl}$ $\cdots ⑤$

under ③,

$$⑤ \Rightarrow R_{km} \doteq \tilde R_{km} + \partial_l(\partial_k\phi\,\delta^l_m + \partial_m\phi\,\delta^l_k - \partial_n\phi\,\tilde g^{ln}\tilde g_{km})$$
$$- \partial_m(\partial_k\phi\,\delta^l_l + \partial_l\phi\,\delta^l_k - \partial_n\phi\,\tilde g^{ln}\tilde g_{kl})$$
$$+ (\tilde\Gamma^l_{nl} + \partial_n\phi\,\delta^l_l + \partial_l\phi\,\delta^l_n - \partial_j\phi\,\tilde g^{lj}\tilde g_{nl}) \times$$
$$\times(\tilde\Gamma^n_{km} + \partial_k\phi\,\delta^n_m + \partial_m\phi\,\delta^n_k - \partial_j\phi\,\tilde g^{nj}\tilde g_{km}) - \tilde\Gamma^l_{nl}\tilde\Gamma^n_{km}$$
$$-(\tilde\Gamma^l_{nm} + \partial_n\phi\cdot\delta^l_m + \partial_m\phi\,\delta^l_n - \partial_j\phi\,\tilde g^{lj}\tilde g_{nm})\times + \tilde\Gamma^i_{nm}\tilde\Gamma^n_{kl}$$
$$\times(\tilde\Gamma^n_{kl} + \partial_k\phi\,\delta^n_l + \partial_l\phi\,\delta^n_k - \partial_j\phi\,\tilde g^{nj}\tilde g_{kl})$$

$i \to n$
$n \to l$

$$= \tilde R_{km} + \cancel{\partial_m\partial_k\phi} + \cancel{\partial_k\partial_m\phi} - \partial_l[\partial_n\phi\, g^{ln} g_{km}]$$
$$- \cancel{\partial_k\partial_m\phi} - \cancel{\partial_m\partial_k\phi} + \partial_m[\partial_n g^{ln} g_{kl}]$$
$$+ \tilde\Gamma^l_{nl}(\partial_k\phi\,\delta^n_m + \partial_m\phi\,\delta^n_k - \partial_j\phi\cdot\tilde g^{nj}\tilde g_{km})$$
$$+ \tilde\Gamma^n_{km}(D\cdot\partial_n\phi + \cancel{\partial_n\phi} - \partial_j\phi\cdot\cancel{\tilde g^{lj}\tilde g_{nl}}\,\delta^j_n)$$
$$+ (\partial_k\phi\,\delta^n_m + \partial_m\phi\,\delta^n_k - \partial_j\phi\,\tilde g^{nj}\tilde g_{km})\cdot(D\cdot\partial_n\phi)$$
$$- \tilde\Gamma^l_{nm}(\partial_k\phi\,\delta^n_l + \partial_l\phi\,\delta^n_k - \partial_j\phi\,\tilde g^{nj}\tilde g_{kl})$$
$$- \tilde\Gamma^n_{kl}(\partial_n\phi\,\delta^l_m + \partial_m\phi\,\delta^l_n - \partial_j\phi\,\tilde g^{lj}\tilde g_{nm})$$
$$- (\partial_n\phi\,\delta^l_m + \partial_m\phi\,\delta^l_n - \partial_j\phi\, g^{lj} g_{nm})\cdot(\partial_k\phi\,\delta^n_l + \partial_l\phi\,\delta^n_k - \partial_j\phi\,\tilde g^{nj}\tilde g_{kl})$$

$$[\partial_m\phi\,\partial_k\phi + \partial_k\phi\,\partial_m\phi - \partial_n\phi\,\partial_j\phi\,\tilde g^{nj}\tilde g_{km}$$
$$+ \partial_m\phi\,\partial_k\phi\cdot D + \partial_m\phi\,\partial_k\phi - \partial_m\phi\,\partial_j\phi\,\tilde g^{lj}\tilde g_{kl}$$
$$- \partial_j\phi\,\partial_k\phi\, g^{lj} g_{lm} - \partial_j\phi\,\partial_l\phi\, g^{lj} g_{km} + \partial_j\phi\, g^{lj}\underbrace{g_{nm}\,\partial_i\phi\, g^{ni}}_{\delta^i_m} g_{kl}]$$

δ^j_k

$+g_{ab})$

$\frac{-4}{D-2}$

$\partial^2 = \partial^\alpha \partial_\alpha$ $\quad \partial_\alpha \theta_\beta = -\varepsilon_{\alpha\beta}$, $\partial^\alpha = -\varepsilon^{\alpha\beta}\partial_\beta$, $\quad \int d\theta\, d\theta\ \theta\theta = 1$

V- 2

$\partial^\alpha \theta^\beta = -\varepsilon^{\alpha\beta}$ $\qquad d^2\theta = \frac{1}{2} d\theta^1 d\theta^2 = -\frac{1}{4}\ d\theta^\alpha d\theta^\beta\, \varepsilon_{\alpha\beta}$

$\partial^2(\theta\theta) = 4$ $\qquad d^2\bar\theta = -\frac{1}{4}\, d\bar\theta_\alpha d\bar\theta_\beta\, \varepsilon^{\alpha\beta}$

So. $R = g^{km} R_{km}$

$$= e^{-2\phi}\tilde g^{km}\Big[\tilde R_{km} - \partial_\ell[\partial_n\phi\, \tilde g^{\ell n}\tilde g_{km}] + \partial_m[\partial_n\phi\, \tilde g^{\ell n}\tilde g_{k\ell}] +$$

$$\tilde\Gamma^\ell_{m\ell}\partial_k\phi + \tilde\Gamma^\ell_{k\ell}\partial_m\phi - \tilde\Gamma^\ell_{n\ell}\,\tilde g^{nj}\tilde g_{km} + \tilde\Gamma^n_{km}\, D\partial_n\phi$$

$$+ D\cdot(\partial_k\phi\partial_m\phi + \partial_m\phi\partial_k\phi - \partial_j\phi\partial_n\phi\cdot \tilde g^{nj} g_{km})$$

$$- \Gamma^\ell_{\ell m}\partial_k\phi - \Gamma^\ell_{\ell k}\partial_m\phi + \Gamma^\ell_{nm}\partial_j\phi\, g^{nj} g_{k\ell}$$

$$- \Gamma^n_{km}\partial_n\phi - \Gamma^\ell_{k\ell}\partial_m\phi + \Gamma^n_{k\ell}\partial_j\phi\, g^{\ell j} g_{nm}$$

$$- \Big[2\partial_m\phi\partial_k\phi - \partial_n\phi\partial^n\phi\, g_{km} + (D+1)\partial_m\phi\partial_k\phi - \partial_m\phi\partial_k\phi$$

$$\cancel{- \partial_m\phi\partial_k\phi - \partial_\ell\phi\partial^\ell\phi\, g_{km} + \partial_m\phi\partial_k\phi}\Big]\Big]$$

Take Normal coordinate.

$$= e^{-2\phi}\Big[\tilde R - D\,\partial_\ell\partial^\ell\phi + \partial^k(\partial_k\phi) + 0.$$

$$+ D\,(2\,\partial_k\phi\,\partial^k\phi - \partial\phi\cdot\partial\phi\cdot D) - 0.$$

$$- [2\,\partial\phi\cdot\partial\phi - 2D\,\partial\phi\cdot\partial\phi + (D+1-1)\partial\phi\cdot\partial\phi]\Big]$$

$$= e^{-2\phi}[\tilde R - (D-1)\nabla^2\phi - D(D-2)\partial\phi\cdot\partial\phi - (2-D)\partial\phi\cdot\partial\phi$$

$$= e^{-2\phi}[\tilde R - (D-1)\nabla^2\phi - (D-1)(D-2)\,\partial\phi\cdot\partial\phi\,]$$

This is a tensor expression valid in Normal coordinate, hence valid in all co-ordinate.

Now, $\sqrt{-g} = \sqrt{(e^{2\phi})(-\tilde g)} = e^{D\phi}\sqrt{\tilde g}$ in D-dim. ($\because$ $\det(e^{2\phi}g_{ab}) = e^{2D\phi}\cdot\det g_{ab}$)

So let $\phi = \alpha\Phi$

$$\sqrt{g}\, e^{-2\Phi} R = \sqrt{\tilde g}\, e^{(D\alpha - 2 - 2\alpha)\Phi}\Big[\tilde R - \alpha(D-1)\nabla^2\Phi - \alpha^2(D-1)(D-2)\partial\Phi\cdot\partial\Phi\Big].$$

$$= 1\cdot[\quad '' \quad].$$

if $(D-2)\alpha = 2$. or $\alpha = \frac{2}{D-2}$

$$\boxed{\frac{2D}{D-2} - 2 = \frac{4}{D-2}}$$

$$\sqrt{g}\, e^{-2\Phi}\, 4(\nabla\Phi)^2 = \sqrt{\tilde g}\, 4\, \nabla^2\Phi$$

$$\boxed{4 - \alpha^2(D-1)(D-2) = 4\frac{(D-2)-(D-1)}{(D-2)} = \frac{-4}{D-2}}$$

$$\sqrt{g}\, e^{-2\Phi} = \sqrt{\tilde g}\, e^{(D\alpha-2)\Phi} = \sqrt{\tilde g}\ e^{\frac{4}{D-2}\Phi}$$

$$\sqrt{-g}\, e^{-2\Phi}\, H_{\mu\nu\lambda}H_{\mu'\nu'\lambda'}\, g^{\mu\mu'}g^{\nu\nu'}g^{\lambda\lambda'} = 1\cdot e^{-4\phi}\sqrt{\tilde g}\,\tilde H^2 = e^{-\frac{8\Phi}{D-2}}\tilde H^2$$

Q.E.D

D.

Summary: $g^{(S)}_{\mu\nu} = e^{\frac{4}{D-2}\Phi} g^{(E)}_{\mu\nu}$. $g^{(S)}_{\mu\nu} = g_{\mu\nu}$, $g^{(E)}_{\mu\nu} = \tilde{g}_{\mu\nu}$.

$$S = \int \sqrt{-g}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 + \cdots \right) \quad \text{if ③} - \frac{2}{3\alpha'}(D-26)$$

$$= \int \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{4}{D-2}(\nabla\Phi)^2 - \frac{1}{12} e^{-\frac{8}{D-2}\Phi} H^2 + \cdots \right) - \frac{2}{3\alpha'}(D-26)\, e^{\frac{4}{D-2}\Phi}$$

We omitted a term $\sqrt{-\tilde{g}} \cdot \left(-\frac{2(D-1)}{D-2} \right) \nabla^2 \Phi$.
Since it is a Total derivative

$g_{\mu\nu} = e^{2\phi} \tilde{g}_{\mu\nu}$

$$R = e^{-2\phi} \left[\tilde{R} - (D-1)\nabla^2\phi - (D-1)(D-2)\partial\phi\cdot\partial\phi \right]$$

$$\sqrt{g} R = e^{(D-2)\phi} \left[\quad " \quad \right]$$

$\phi = \alpha\Phi$ (dilaton Φ). notice that there is no Einstein frame in $D = 2$. $[\, (D-2)\alpha\phi - 2\Phi = 0 \,]$
$\Rightarrow \alpha = \frac{2}{D-2}$

ction,

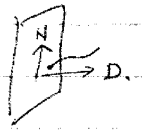
ely)

t.

Intersecting D-branes & SUSY

0	1	2	3	4	5	6	7	8	9
•	•	•	•						
•	•			•	•	•			

X^0, X^1 : Neunman-Neunman (NN) : $X = x_0 + 2\alpha' p\tau + i\sqrt{2\alpha'} \sum \frac{\alpha_n}{n} \cos n\sigma$



$X^{2,3}$: ND : $X = i\sqrt{2\alpha'} \sum \frac{\alpha_{n+\frac{1}{2}}}{n+\frac{1}{2}} \cos(n+\frac{1}{2})\sigma$

$X^{4,5,6}$: DN : $X = \sqrt{2\alpha'} \sum \frac{\alpha_{n+\frac{1}{2}}}{n+\frac{1}{2}} \sin(n+\frac{1}{2})\sigma.$

$X^{7,8,9}$: DD : $X = x_0 + WR + \sqrt{2\alpha'} \sum \frac{\alpha_n}{n} \sin n\sigma$

Contribution to the zero point Energy.

B.C	boson	fermion
NN	$\frac{1}{2}\sum_{1}^{\infty} n = -\frac{1}{24}$,	$\frac{1}{2}\sum_{0}^{\infty} -(n+\frac{1}{2}) = -\frac{1}{48}$
ND	$\frac{1}{2}\sum_{0}^{\infty}(n+\frac{1}{2}) = +\frac{1}{48}$,	$\frac{1}{2}\sum_{n=1}^{\infty}(-n) = \frac{1}{24}$ for NS
DN	〃 $= \frac{1}{48}$	〃 $= \frac{1}{24}$ 〃
DD	$\frac{1}{2}\sum n = -\frac{1}{24}$	$\frac{1}{2}\sum(-n+\frac{1}{2}) = -\frac{1}{48}$

(* for Ramond sector fermion has the same moding as ~~Ramond~~ Boson so always 0 z.p.e.)

If there are ν of ND boundary conditions.

then $z.p.e = (8-\nu)(-\frac{1}{24}-\frac{1}{48}) + \nu(\frac{1}{24}+\frac{1}{48})$

$= \frac{1}{16}(\nu+\nu-8).$

So for space-time SUSY. $\boxed{\nu = 4}$

$\Rightarrow$ Balance between $\begin{cases} \text{NS sector (Boson)} \\ \text{R} \quad 〃 \quad \text{(Sp-time Fermion).} \end{cases}$

Spectrum : $\alpha' M^2 = N - a$.

$$a = \begin{cases} -\frac{\nu}{8} + \frac{1}{2} & : \text{NS sector} \\ 0 & : \text{R sector} \end{cases}$$

Since the modes of $\psi_r^{(NS)}$ can change N by $\frac{1}{2}$ + integer, ν must be multiple of 4 as a necessary condition for Space-time SUSY.