February 26, 2004

Brief Survey of Maximally Supersymmetric Supergravity[*]

Hyeonjoon Shin

BK 21 Physics Research Division and Institute of Basic Science
Sungkyunkwan University, Suwon 440-746, Korea

hshin@newton.skku.ac.kr

Abstract

Possible types of spinors in a given space-time dimension are first investigated. We then give a brief survey of maximally supersymmetric supergravity theories in ten and eleven dimensions.

[*]Lecture given at the 8th Winter School of APCTP/KIAS on String Theory, February 2004.

1 Introduction

In this note, we give a brief survey of supergravity theories in ten and eleven dimensions. The focus is on the theories which have the maximal number of supersymmetries, that is, 32 supersymmetries.

Before going into the supergravity theory, we first investigate the possible types of spinors in various dimensions. This lead us to have information as to what the minimal spinor is for describing the supersymmetry and the field contents of a given supergravity theory.

In the eleven dimensional theory, the notion of Killing spinor is introduced. We then obtain the ten dimensional type IIA supergravity by compactifying the eleven dimensional theory on a circle. This gives the basic concept of Kaluza-Klein compactification. As a theory not obtainable directly from the eleven dimensional theory, the ten dimensional type IIB supergravity is finally considered. We present an interesting structure of it, which is the $SL(2,\mathbf{R})$ symmetry.

Much detail for the theories considered here and the other many important aspects of them may be found in the following nice review articles: Townsend [hep-th/9712004]; West [hep-th/9811101]; de Wit [hep-th/0212245], Bilal and Metzger [hep-th/0307152].

We would like to note that the purpose of this note is not the construction of supergravity theories themselves. Actually, almost all the interesting theories have been constructed and even the collection of important papers has been already given by Salam and Sezgin [Supergravities in Diverse Dimensions, (1989)].

2 Spinors in various dimensions

We consider spinors in arbitrary d space-time dimension with d_- time-like and d_+ space-like directions, that is, $d = d_- + d_+$. Since the properties of spinors are determined by those of Dirac gamma matrices, we first present the necessary facts about Dirac gamma matrices.

The references for this section are as follows: Kugo and Townsend [Nucl. Phys. B221 (1983) 357]; Sohnius [Phys. Rep. 128 (1985) 39]; West [hep-th/9811101]; Van Proeyen [hep-th/9910030]; Polchinski's book.

2.1 Gamma matrices

The Dirac gamma matrices Γ^a $(a = 1, \ldots, d)$ are defined to be irreducible representations of the Clifford algebra,

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \ . \tag{2.1}$$

The metric η^{ab} is the flat metric in $\mathbf{R}^{d_-,d_+}$, where $d_-(d_+)$ is the number of time-like (space-like) directions.

$$
\begin{aligned}
\eta^{ab} &= \mathrm{diag}(\overbrace{-\cdots-}^{d_-}\overbrace{+\cdots+}^{d_+}) \ , \\
d &= d_+ + d_- \ , \quad \Delta = d_+ - d_- \ ,
\end{aligned}
\tag{2.2}
$$

where Δ is defined for later convenience.

It follows from (2.1) that the matrices

$$
\Sigma^{ab} = -\frac{i}{2}\Gamma^{ab}
\tag{2.3}
$$

represent the algebra of $SO(d_-, d_+)$

$$
\begin{aligned}
\Sigma^{ab} =& \Sigma(J^{ab}) \\
[J^{ab}, J^{cd}] =& - i(\eta^{bc}J^{ad} + \eta^{ad}J^{bc} - \eta^{ac}J^{bd} - \eta^{ad}J^{bc}) \ .
\end{aligned}
\tag{2.4}
$$

To investigate the properties of Dirac gamma matrices, we begin with a possible representation of the Clifford algebra for the Euclidean space ($d_- = 0$) in terms of Pauli σ matrices.

$$
\begin{aligned}
\Gamma^1 =& \sigma^1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \\
\Gamma^2 =& \sigma^2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \\
\Gamma^3 =& \sigma^3 \otimes \sigma^1 \otimes \mathbf{1} \otimes \ldots \\
\Gamma^4 =& \sigma^3 \otimes \sigma^2 \otimes \mathbf{1} \otimes \ldots \\
\Gamma^5 =& \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \ldots \\
\vdots&
\end{aligned}
\tag{2.5}
$$

The dimension of a representation is $2^{d/2}$ for even d. For odd d, it is $2^{(d-1)/2}$, one example of which is the case of $d = 5$, where the last σ^1 in Γ^5 is unnecessary.

All the gamma matrices above are Hermitian, since σ matrices are so. If $d_- \neq 0$, we should multiply the first d_- matrices by i. For example, $\Gamma^1 = i\sigma^1 \otimes \mathbf{1}...$ for the Minkowskian case. This process makes the gamma matrices in the time-like directions to be anti-Hermitian. The Hermiticity property of the above gamma matrices is then expressed as

$$
\Gamma^{a\dagger} = \Gamma_a \ .
\tag{2.6}
$$

One important fact is that the gamma matrices can be used to make a complete set of even dimensional matrices. To be more precise, let us consider even d dimension and define anti-symmetric products of gamma matrices $\Gamma^{(n)}$ with $n = 0, 1, \ldots, d$;

$$\Gamma^{(n)} = \Gamma^{a_1 a_2 \cdots a_n} \equiv \Gamma^{[a_1} \Gamma^{a_2} \cdots \Gamma^{a_n]} \ , \tag{2.7}$$

which are orthogonal in the sense that $\mathrm{Tr}(\Gamma^{(n)} \Gamma^{(m)}) \propto \delta^{nm}$. Then $\{\Gamma^{(n)}\}$ provides a complete set of $2^{d/2} \times 2^{d/2}$ matrices. The last element $\Gamma^{(d)}$ which is used in considering Weyl spinors is denoted here simply as $\bar{\Gamma}$:

$$\bar{\Gamma} \equiv \Gamma^{(d)} = \Gamma^1 \Gamma^2 \cdots \Gamma^d \ , \quad \bar{\Gamma}^2 = (-1)^{-\Delta/2} \ , \quad \{\bar{\Gamma}, \Gamma^a\} = 0 \ . \tag{2.8}$$

An interesting fact is that, in even dimensions, the matrices $\Gamma^{(n)}$ and $\Gamma^{(d-n)}$ are related by $\bar{\Gamma}$ as

$$\Gamma_{a_1 \cdots a_n} = \frac{1}{(d-n)!} \epsilon_{a_1 \cdots a_n a_{n+1} \cdots a_d} (-1)^{d-} \bar{\Gamma} \Gamma^{a_d \cdots a_{n+1}} \ , \tag{2.9}$$

where $\epsilon_{12 \cdots d} = 1$.

Since $\{\bar{\Gamma}, \Gamma_a\} = 0$, $\bar{\Gamma}$ in even d dimension can be used as Γ^{d+1} in odd $d+1$ dimension by setting $\Gamma^{d+1} = i^{\Delta/2} \bar{\Gamma}$. Then the matrix $\bar{\Gamma}$ in odd dimension given by product of all gamma matrices is just the identity matrix multiplied by ± 1 or $\pm i$. This leads to the fact that, in odd d dimension, a set $\{\Gamma^{(n)}\}$ with $n = 0, 1, \ldots, (d-1)/2$ forms the basis of $2^{(d-1)/2} \times 2^{(d-1)/2}$ matrices.

2.2 Equivalence relations

The Dirac gamma matrices such as Eq. (2.5) is not the unique representation of the Clifford algebra (2.1). One can obtain another set of gamma matrices satisfying the Clifford algebra by means of similarity transformation,

$$\Gamma'^a = S \Gamma^a S^{-1} \ , \tag{2.10}$$

where S is a non-singular matrix. In this case, two sets of gamma matrices, $\{\Gamma^a\}$ and $\{\Gamma'^a\}$, are said to be in the same equivalence class. For the equivalence relations, there are two useful theorems, which are given below without proofs.

- For a given even d dimension with a given signature of the metric, all irreducible representations of Clifford algebra are $2^{d/2} \times 2^{d/2}$ dimensional and are equivalent to one another. That is, for any two representations $\{\Gamma^a\}$ and $\{\Gamma'^a\}$, there is a non-singular matrix S such that (2.10) holds.

- For a given odd d dimension with a given signature of the metric, there are two equivalence classes of irreducible representations in terms of $2^{(d-1)/2}\times 2^{(d-1)/2}$ matrices. In particular, if $\{\Gamma^a\}$ is in one equivalence class then $\{-\Gamma^a\}$ is in the other.

We note that, for a given representation $\Gamma\equiv\{\Gamma^a\}$, all the following are also representations of the Clifford algebra.

$$\Gamma, -\Gamma, \Gamma^\dagger, -\Gamma^\dagger, \Gamma^*, -\Gamma^*, \Gamma^T, -\Gamma^T \ . \tag{2.11}$$

According to the first theorem above, these are all equivalent in even dimensions. Thus there must exist interwiners between them. We let A, B, and C the interwiners connecting Γ with $\pm\Gamma^\dagger$, $\pm\Gamma^*$, and $\pm\Gamma^T$, respectively. The interwiner connecting Γ and $-\Gamma$ is simply $\bar{\Gamma}$. The second theorem implies that, in odd dimensions, the above eight representations are grouped into two equivalence classes, each of which contains four representations. For example, if A exists connecting Γ and $-\Gamma^\dagger$, then there is no non-singular matrix connecting Γ and $\Gamma^\dagger$.

In view of the similarity transformation between representations, the Hermiticiy property of the gamma matrices, (2.6), can be expressed as

$$\Gamma^{a\dagger} = (-1)^{d_-} A\Gamma^a A^{-1} \ , \quad A = \Gamma^1\Gamma^2\cdots\Gamma^{d_-} \ . \tag{2.12}$$

The interwiner in this case is for $\Gamma^\dagger$ or $-\Gamma^\dagger$. As an example, for $d_- = 0$, $A = \mathbf{1}$ is the interwiner connecting Γ and $\Gamma^\dagger$ both in even and odd dimensions. The interwiner for $-\Gamma^\dagger$ is given by $\bar{\Gamma}$ in even dimensions but it does not exist in odd dimensions.

The interwiner B, which is crucial in the discussion of Majorana spinor, will be considered later.

The matrix C, which is usually called the charge conjugation matrix, connects Γ and $\pm\Gamma^T$ as follows.

$$\Gamma^{aT} = \eta C\Gamma^a C^{-1} \ , \quad \eta = \pm 1 \ . \tag{2.13}$$

With the help of Schur's lemma, we get from this relation the information about the charge conjugation matrix as

$$C^T = \epsilon C \ , \quad \epsilon = \pm 1 \ . \tag{2.14}$$

We should determine which values of η and ϵ are allowed in a given d_+ and d_-. In order to do that, we use the completeness of $\{\Gamma^{(n)}\}$ matrices as alluded to in the last subsection and consider the symmetry property of the matrices $C\Gamma^{(n)}$. We first note that

$$\left(C\Gamma^{(n)}\right)^T = \epsilon\eta^n(-1)^{n(n-1)/2} C\Gamma^{(n)} \ . \tag{2.15}$$

d (mod 8)	S	A	ϵ	η
0	0,3	1,2	$+$	$-$
	0,1	2,3	$+$	$+$
1	0,1	2,3	$+$	$+$
2	0,1	2,3	$+$	$+$
	1,2	0,3	$-$	$-$
3	1,2	0,3	$-$	$-$
4	1,2	0,3	$-$	$-$
	2,3	0,1	$-$	$+$
5	2,3	0,1	$-$	$+$
6	2,3	0,1	$-$	$+$
	0,3	1,2	$+$	$-$
7	0,3	1,2	$+$	$-$

Table 1: Symmetric (S) and anti-symmetric (A) properties of $C\Gamma^{(n)}$ with n mod 4 in dimension d mod 8. The quantities ϵ and η are those appearing in $C^T = \epsilon C$ and $\Gamma^{aT} = \eta C \Gamma^a C^{-1}$.

This is periodic under the shift $n \to n+4$, and thus the number of symmetric or anti-symmetric matrices can be computed by using the following formulae.

$$\begin{aligned}
\binom{d}{0} + \binom{d}{4} + \cdots &= 2^{d-2} + 2^{d/2-1} \cos \frac{\pi d}{4} \ , \\
\binom{d}{1} + \binom{d}{5} + \cdots &= 2^{d-2} + 2^{d/2-1} \sin \frac{\pi d}{4} \ , \\
\binom{d}{2} + \binom{d}{6} + \cdots &= 2^{d-2} - 2^{d/2-1} \cos \frac{\pi d}{4} \ , \\
\binom{d}{3} + \binom{d}{7} + \cdots &= 2^{d-2} - 2^{d/2-1} \sin \frac{\pi d}{4} \ .
\end{aligned} \tag{2.16}$$

On the other hand, we know that there are

$$2^{[d/2]-1}(2^{[d/2]} \pm 1) \tag{2.17}$$

symmetric ($+$) and anti-symmetric ($-$) matrices, where $[n]$ denotes the integer part of n. Now, by equating the results from two counting methods for the number of symmetric or anti-symmetric matrices, we get possible values of η and ϵ, which are listed in table 1.

2.3 Irreducible spinors

The Dirac spinor is the one that transforms in the spinor representation of $SO(d_-, d_+)$, Eq. (2.4). Although the Dirac gamma matrices form an irreducible representation of the Clifford algebra, the spinor representation of $SO(d_-, d_+)$ which they define may be reducible in some situations. Thus the Dirac spinor is reducible in those cases. Basically, there are two ways of reducing the Dirac spinor while keeping manifest $SO(d_-, d_+)$ covariance. They are provided by the chirality projection, which is valid only in even dimensions, and the reality condition.

Weyl spinor: In even d dimensions, there exists the matrix $\bar{\Gamma}$, whose eigenvalues are from Eq. (2.8)

$$\pm\sqrt{\beta} \ , \tag{2.18}$$

where $\beta \equiv (-1)^{-\Delta/2} = \pm 1$. There are equal number of $+\sqrt{\beta}$ and $-\sqrt{\beta}$ eigenvalues, since $\mathrm{Tr}\bar{\Gamma} = 0$.[1] We now define the projection operators

$$P_\pm = \frac{1 \pm \bar{\Gamma}/\sqrt{\beta}}{2} \ . \tag{2.19}$$

They commute with the $SO(d_-, d_+)$ generators Σ^{ab} because $\{\bar{\Gamma}, \Gamma^a\} = 0$, and thus $\Sigma^{ab}_\pm \equiv P_\pm \Sigma^{ab}$ are also representations of the algebra (2.4), called chiral representations. Weyl or chiral spinors are the spinors transforming in chiral representations and are defined by

$$\psi_\pm = P_\pm \psi \ , \tag{2.20}$$

where ψ_+ (ψ_-) is said to have positive (negative) chirality.

As a remark, we note that, from the dependence of β on Δ, the eigenvalues of $\bar{\Gamma}$ are ± 1 in $\Delta = 0$ mod 4 and $\pm i$ in $\Delta = 2$ mod 4. This implies that chiral spinors in $\Delta = 2$ mod 4 change their chiralities under the complex conjugation, while those in $\Delta = 0$ mod 4 do not.

Majorana spinor: The components of a Dirac spinor are in general complex numbers. Even if we start with a component-wise real spinor, it becomes complex after $SO(d_-, d_+)$ transformation generated by Σ^{ab}.

In some dimensions, however, it is possible to impose a reality condition, which is stated as

$$\psi^* = X\psi \ , \tag{2.21}$$

for some matrix X. For the consistency $\psi^{**} = \psi$, X should satisfy

$$X^* X = 1 \ . \tag{2.22}$$

[1] According to the representation of Clifford algebra, (2.5), $\bar{\Gamma} = \sigma^3 \otimes \sigma^3 \otimes \cdots$, which is obviously traceless. Since the similarity transformation does not change this property, $\mathrm{Tr}\bar{\Gamma} = 0$ in any other gamma matrix representation.

The dimensions allowing the reality condition are determined by requiring that the condition is consistent under the $SO(d_-, d_+)$ transformations. We first consider the reality properties of the Dirac gamma matrices. Combining Eqs. (2.12), (2.13), and (2.14), it is given by

$$\Gamma^{a*} = \eta(-1)^{d_-} B\Gamma^a B^{-1} \ , \quad B = A^{-1T} C \ . \tag{2.23}$$

We note that if the Hermiticity of the gamma matrices, (2.6), is assumed then all of the interwiners A, B, and C have to be unitary. This leads us to obtain

$$B^* B = \epsilon\eta^{d_-}(-1)^{\frac{1}{2}d_-(d_- -1)} \ . \tag{2.24}$$

If the $SO(d_-, d_+)$ transformation is applied on both sides of the reality condition, (2.21), by using $\delta\psi = \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi$ with infinitesimal parameter ω_{ab}, we now obtain

$$B\Gamma^{ab}B^{-1}X = X\Gamma^{ab} \ . \tag{2.25}$$

This implies that

$$X = \alpha B \tag{2.26}$$

with some constant α. From Eqs. (2.22) and (2.24), we see that the consistency of the reality condition leads to $|\alpha| = 1$ and $B^* B = 1$.

The possible dimensions satisfying $B^* B = 1$ are determined from the table 1 and Eq. (2.24) as follows.

$$\begin{aligned} &\Delta = 0, 1, 7 \text{ mod } 8 \\ &\Delta = 2 \text{ mod } 8 \text{ with } \eta(-1)^{d/2} = -1 \\ &\Delta = 6 \text{ mod } 8 \text{ with } \eta(-1)^{d/2} = +1 \ . \end{aligned} \tag{2.27}$$

In these cases, there can be spinors satisfying the reality condition (2.21), and such spinors are called Majorana or real spinors. We note that the condition $|\alpha| = 1$ is actually irrelevant in the discussion of Majorana spinors and is used to make, for example, the fermion bilinears to be Hermitian in the action construction.

For the values of Δ other than those of (2.27), $B^* B = -1$. In this case, if we have even number of spinors ψ_i with $i = 1, \ldots, 2n$, there is actually an another possibility, the symplectic Majorana condition

$$\psi^{i*} = B\Omega_{ij}\psi^j \ . \tag{2.28}$$

The matrix Ω is antisymmetric and satisfies $\Omega\Omega^* = -1$. A typical form of Ω is given by $\begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & 0 \end{pmatrix}$.

Majorana-Weyl spinor: Finally, one may ask whether there are chiral spinors which also satisfy the reality condition, that is,

$$\psi_\pm^* = X\psi_\pm \ . \tag{2.29}$$

A bit of manipulation for the left hand side gives

$$\begin{aligned} \psi_\pm^* &= \frac{1}{2}(1 \pm \bar{\Gamma}^*/\sqrt{\beta}^*)\psi^* \\ &= \frac{1}{2}(1 \pm B\bar{\Gamma}B^{-1}/\sqrt{\beta}^*)X\psi \\ &= X\frac{1}{2}(1 \pm \bar{\Gamma}/\sqrt{\beta}^*)\psi \ . \end{aligned} \tag{2.30}$$

This show that the reality condition satisfies if $\sqrt{\beta}^* = \sqrt{\beta}$. The condition for β is thus

$$\beta = +1 \ , \tag{2.31}$$

which means $\Delta = 0$ mod 4. From the dimensions allowing Majorana spinors (2.27), we now see that the condition (2.29) is satisfied when

$$\Delta = 0 \text{ mod } 8 \ . \tag{2.32}$$

The spinors satisfying the condition (2.29) are called Majorana-Weyl spinors.

The other possibilities $\Delta = 4$ mod 8 for $\beta = +1$ correspond to the symplectic Majorana-Weyl spinors, which exist when

$$\Delta = 4 \text{ mod } 8 \ . \tag{2.33}$$

Until now, we have investigated the reductions of Dirac spinors which are consistent with the $SO(d_-, d_+)$ transformations. Based on the results, we give tables 2 and 3 which show the possible types of spinors in various space-time dimensions with Minkowkian and Euclidean signature respectively.

3 Local supersymmetry and gravity

In this section, we consider the local supersymmetry and consequences following it.

The theories that are invariant under the local supersymmetry transformation are called the supergravity theories, which contain the gravity. This seems to imply that the local

d	d_Γ	$d_{\min}$	W	M	MW
1	1	1		real	
2	2	1	self	real	real
3	2	2		real	
4	4	4	complex	real$^-$	
5	4	8		symplectic	
6	8	8	self	symplectic	symplectic
7	8	16		symplectic	
8	16	16	complex	real$^+$	
9	16	16		real	
10	32	16	self	real	real
11	32	32		real	
12	64	64	complex	real$^-$	

Table 2: Spinors in Minkowskian space-time. d: dimension of space. d_Γ: dimension of Dirac gamma matrix. $d_{\min}$: real dimension of minimal spinor. W: Weyl, M: Majorana, MW: Majorana-Weyl. The signs of real$^\pm$ are those of η in $\Gamma_a^T = \eta C \Gamma_a C^{-1}$. The table continues with d mod 8. One should notice that the symplectic case exists only when the number of spinors is $2n$ with $n \geq 1$.

d	d_Γ	$d_{\min}$	W	M	MW
1	1	1		real	
2	2	2	complex	real$^+$	
3	2	4		symplectic	
4	4	4	self	symplectic	symplectic
5	4	8		symplectic	
6	8	8	complex	real$^-$	
7	8	8		real	
8	16	8	self	real	real
9	16	16		real	
10	32	32	complex	real$^+$	
11	32	64		symplectic	
12	64	64	self	symplectic	symplectic

Table 3: Spinors in Euclidean space-time. The labelling of columns is the same with that of table 2.

supersymmetry requires the inclusion of gravity sector in a locally supersymmetric theory. To see this clearly, let us consider the schematic form of the supersymmetry algebra $\{Q, Q\} = P$, where Q is the supercharge and P the momentum. If we take the commutator of two successive supersymmetry transformations $[\delta_1, \delta_2]$ where $\delta_{1,2} = \epsilon_{1,2} Q$ with the transformation parameters $\epsilon_{1,2}$, the supersymmetry algebra tells us that the translation with an amount proportional to $\epsilon_1 \epsilon_2$ is generated. Since the ϵ's are space-time dependent in the case of local supersymmetry, the translation generated by two supersymmetry transformations is also space-time dependent. We note that this is nothing but the general coordinate transformation. Therefore we can conclude that the inclusion of gravity is an inevitable consequence of the local supersymmetry.

The gravity is the theory describing the dynamics of spin-2 particle, that is , the graviton. The graviton is in the bosonic sector of the so-called gravity multiplet of the supersymmetry algebra, which is the massless multiplet with maximal helicity 2, $|\lambda_{\max}| = 2$. Usually, there is no need to take into account the gravity multiplet if one does not consider the local supersymmetry. In some cases, however, it is contained in a theory automatically, and we should consider supergravity theories. This situation depends on the number of supersymmetry, the total number of real components of supercharges, which we let n_Q. By a simple arithmetic, we see that $|\lambda_{\max}| \geq \frac{n_Q}{16}$, for massless multiplets. Thus, if $n_Q > 16$, $|\lambda_{\max}| \geq \frac{3}{2}$, so that the theory should contain Rarita-Schwinger fields. By the way, an interacting supersymmetric theory of this type should contain gravity, that is, the graviton. Therefore, the gravity multiplet is included in a theory automatically if $n_Q > 16$. If $n_Q > 32$, the multiplets with the helicity $\lambda > 2$ begin to enter, and hence are beyond of our concern here. This implies that $n_Q = 32$ is the maximum number of supersymmetries that supergravity theories can have. If we look at the table 2, we see that the bound $n_Q = 32$ restricts the space-time dimension to eleven.

In this note, we are concerned about the maximally supersymmetric case, $n_Q = 32$. The eleven dimensional supergravity is first considered. After that, we go down to ten dimensions and consider two maximally supersymmetric supergravity theories.

4 Eleven dimensional supergravity

The $d = 11$ $N = 1$ supergravity action was constructed first by Cremmer, Julia, and Scherk [Phys. Lett. B76 (1978) 409]. The superspace formulation was given by Cremmer and Ferrara [Phys. Lett. B91 (1980) 61].

We begin with quoting the viewpoint about the $d = 11$ supergravity in mid 80's.

Eleven-dimensional supergravity remains an enigma. It is hard to believe that its existence is just an accident, but it is difficult at the present time to state a compelling conjecture for what its role may be in the scheme of things. ...

Green, Schwarz, and Witten

... before the appearance of M-theory. From mid 90's, the supergravity has been recognized as the low energy limit of the big theory, the M-theory. Then one may ask what the M-theory is. Unfortunately, at present, we do not yet have the complete answer. Story continues ....

Fist of all, we work out the massless representation of the supersymmetry algebra, that is, the graviton supermultiplet. Since $N = 1$, we have one supercharge which is the Majorana spinor with 32 real components as one finds in the table 2. The supersymmetry algebra is[2]

$$\{Q_\alpha, Q_\beta\} = -2(\Gamma^M C^{-1})_{\alpha\beta} P_M \ , \tag{4.1}$$

where C may be chosen as Γ^0. Here we have ignored the terms of central charges on the right hand side corresponding to the presence of membranes (M2-brane) and fivebranes (M5-brane), since they are not our concern.

If we simply take the momentum of the massless state to be $P_M = (-1, -1, 0, \dots, 0)$, the algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2(1 + \Gamma^{01})_{\alpha\beta} \ . \tag{4.2}$$

Since Γ^{01} squares to the identity, its eigenvalues are ± 1, and, since it is traceless, precisely half are $+1$ and half -1. Then, after the diagonalization, we see that 16 components of supercharge give non-vanishing anti-commutators. These 16 components can be split into 8 creation operators and 8 annihilation operators. Therefore, by acting creation operators on the state which is annihilated by all annihilation operators, we get $2^8 = 256$ states, of which 128 are bosons and 128 fermions:

$$\mathbf{256} = \mathbf{128}_B + \mathbf{128}_F \ . \tag{4.3}$$

This is the number of on-shell *physical* states in the graviton supermultiplet of $d = 11$ supergravity. The contents of the graviton supermultiplet are the graviton G_{MN}, 3-from gauge field C_{MNP}, and gravitino Ψ^A_M. The number of *physical* states for each field gives the above numbers as

$$\begin{aligned} \mathbf{128}_B &= \mathbf{44}\ (G_{MN}) + \mathbf{84}\ (C_{MNP}) \ , \\ \mathbf{128}_F &= \mathbf{128}\ (\Psi^A_M) \ . \end{aligned} \tag{4.4}$$

[2] The notation for the indices are as follows: $M, N, \dots$ $(A, B, \dots)$ are the curved (flat tangent) space-time indices taking values of $0, 1, \dots, 9, 11$, while $\alpha, \beta, \dots$ are the spinor indices taking values of $1, 2, \dots, 32$.

The dynamics of the fields in the graviton supermultiplet is described by the $d = 11$ $N = 1$ supergravity action,

$$S = S_B + S_F \ , \tag{4.5}$$

where S_B (S_F) is the bosonic (fermionic) part. The bosonic part is given by

$$S_B = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-G} \left(R - \frac{1}{2 \cdot 4!} G_{(4)}^2 + \frac{1}{(12)^4} \varepsilon^{M_1 \ldots M_{11}} C_{M_1 M_2 M_3} G_{M_4 \ldots M_7} G_{M_8 \ldots M_{11}} \ , \right) \tag{4.6}$$

where R is the Ricci scalar, $G_{MNPQ} = 4\partial_{[M} C_{NPQ]}$, $G_{(4)}^2 = G_{MNPQ} G^{MNPQ}$, and $2\kappa_{11}^2$ is the 11-dimensional gravitational constant defined by using the 11-dimensional Planck length l_p as

$$2\kappa_{11}^2 \equiv (2\pi)^8 l_p^9 \ . \tag{4.7}$$

We note that the Chern-Simons term in the action is required for the invariance of the full action S under the local supersymmetry transformation.

The introduction to some conventions and notations adopted in this note is in order. For p-form gauge field $A_{(p)}$, its field strength $F_{(p+1)}$ is

$$F_{N_1 N_2 \ldots N_{p+1}} = (p+1)\partial_{[N_1} A_{N_2 \ldots N_{p+1}]} \ , \tag{4.8}$$

the squaring of which is given by $F_{(p+1)}^2 = F_{N_1 N_2 \ldots N_{p+1}} F^{N_1 N_2 \ldots N_{p+1}}$. The ε-tensor in d-dimensional Minkowskian space-time is defined as

$$\varepsilon^{01 \cdots (d-1)} = \frac{1}{\sqrt{-G}} \epsilon^{01 \cdots (d-1)} \ , \tag{4.9}$$

where G is the determinant of metric tensor and the Levi-Civita ϵ-symbol is taken to have its value as $\epsilon_{01 \cdots (d-1)} = -\epsilon^{01 \cdots (d-1)} = 1$.

The fermionic part of the action S is

$$\begin{aligned} S_F = & \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-G} \Bigg[-\bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P \\ & - \frac{1}{96} \left(\bar{\Psi}_R \Gamma^{MNPQRS} \Psi_S + 12 \bar{\Psi}^M \Gamma^{NP} \Psi^Q \right) G_{MNPQ} + \cdots \Bigg] \ , \end{aligned} \tag{4.10}$$

where dots denote the four fermion terms and the covariant derivative is given by

$$D_M = \partial_M + \frac{1}{4} \Omega_M{}^{AB} \Gamma_{AB} \ , \tag{4.11}$$

with the spin connection $\Omega_M{}^{AB}$. The four fermion terms, which are of course required for the local supersymmety invariance, lead to the contact interactions. Because of the nature of contact interaction and the non-renormalizability of the four fermion terms in the usual sense of quantum field theory, they are activated at ultra high energy. However, since the supergravity is at present regarded as the low energy effective theory of some mother theory, that is, M or string theory, we should not consider the supergravity at high energy where the four fermion interactions begin to give significant effects.

The 11-dimensional supergravity action given by Eqs. (4.5), (4.6), and (4.10) is invariant under the following local supersymmetry transformations.

$$\begin{aligned} \delta E_M^A &= \frac{1}{2}\bar{\epsilon}\Gamma^A\Psi_M \ , \\ \delta C_{MNP} &= -\frac{3}{2}\bar{\epsilon}\Gamma_{[MN}\Psi_{P]} \ , \\ \delta\Psi_M &= D_M\epsilon + \frac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})\epsilon G_{NPQR} + \cdots \ , \end{aligned} \tag{4.12}$$

where $\epsilon(X^M)$ is the infinitesimal local supersymmetry parameter and dots in third line denote the terms quadratic in gravitino field.

Let us now discuss about the supersymmetric background or vacuum by using the above supersymmetry transformation rules with the parameter ϵ. Firstly, let $|\Omega\rangle$ be the vacuum state of the theory. The expectation values of bosonic fields for the vacuum give the background field configuration. If it is annihilated by some fraction of supercharge Q, e.g. half of 32 components, it is said to be supersymmetric and such fraction of supercharge, generate unbroken supersymmetries. How do we find such unbroken supersymmetries? The unbroken supercharge Q annihilating $|\Omega\rangle$ implies that $\langle\Omega|\{Q,\phi\}|\Omega\rangle = 0$ for all field operators ϕ. This will be so if ϕ is a bosonic operator, since $\{Q,\phi\}$ is fermionic. On the other hand, if ϕ is fermionic, we have non-trivial equation $\langle\Omega|\delta\phi|\Omega\rangle = 0$ for the supersymmetry parameter ϵ, where $\delta\phi = \{Q,\phi\}$ has been used. At the classical level, $\langle\Omega|\delta\phi|\Omega\rangle$ coincides with $\delta\phi$. Therefore, finding a supersymmetry transformation such that $\delta\phi = 0$ for fermionic fields ϕ leads to the finding of a unbroken supersymmetry. In the present case, we have $\delta\Psi_M = 0$, that is,

$$\left(D_M + \frac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})G_{NPQR}\right)\epsilon = 0 \ . \tag{4.13}$$

This is called the Killing spinor equation. The simplest example is the vacuum given by the 11-dimensional Minkowskian space-time; $G_{MN} = \eta_{MN}$, $C_{(3)} = 0$, and $\Psi_M = 0$. It is easy to see that all the 32 supersymmetries are unbroken and hence the Minkowskian space-time is maximally supersymmetric.

5 Type IIA supergravity

The $d = 10$ IIA supergravity has nonchiral $N = (1,1)$ supersymmetry, whose algebra is given by

$$\{Q^+_\alpha, Q^+_\beta\} = -2(P_+\Gamma^\mu C^{-1})_{\alpha\beta}P_\mu \ , \quad \{Q^-_\alpha, Q^-_\beta\} = -2(P_-\Gamma^\mu C^{-1})_{\alpha\beta}P_\mu \ . \tag{5.1}$$

$$Q^+_\alpha \in \mathbf{16} \text{ of } SO(1,9) \ , \quad Q^-_\alpha \in \mathbf{16}' \text{ of } SO(1,9)$$

Although α ranges from 1 to 32, the number of independent components are 16. For a proper choice of momentum as that of Eq. (4.2), the massless multiplets for the superalgebras are obtained as

$$\begin{aligned} \{Q^+, Q^+\} &\longrightarrow \mathbf{8_v} + \mathbf{8_c} \ , \\ \{Q^-, Q^-\} &\longrightarrow \mathbf{8_v} + \mathbf{8_s} \ , \end{aligned} \tag{5.2}$$

where $\mathbf{8_v}$, $\mathbf{8_s}$, and $\mathbf{8_c}$ are the eight dimensional representations of the transverse $SO(8)$. Then the total number of states in the IIA supergravity multiplet is $(8+8)\times(8+8) = 256 = 128_B + 128_F$, which is decomposed as follows.

$$\begin{array}{ccccccccccc} \mathbf{128}_B & = & \mathbf{1} & + & \mathbf{35_v} & + & \mathbf{28} & + & \mathbf{8_v} & + & \mathbf{56_v} \\ & & \phi & & g_{\mu\nu} & & B_{(2)} & & A_{(1)} & & A_{(3)} \\ \\ \mathbf{128}_F & = & \mathbf{8_s} & + & \mathbf{8_c} & + & \mathbf{56_s} & + & \mathbf{56_c} & & \\ & & \lambda^+ & & \lambda^- & & \psi^+_\mu & & \psi^-_\mu & & \end{array} \tag{5.3}$$

where the type IIA field corresponding to each representation has been shown and the signs of superscripts for the fermionic fields denote the $SO(1,9)$ chirality.

The action describing the dynamics of type IIA supergravity fields have been constructed by Huq and Namazie [Class. Quantum Grav. 2 (1985) 293, 597 (E)], Giani and Pernici [Phys. Rev. D 30 (1984) 325], and Campbell and West [Nucl. Phys. B243 (1984) 112].

From now on, we obtain the $d = 10$ type IIA supergravity action from the Kaluza-Klein circle compactification of the $d = 11$ supergravity described in Eqs. (4.5), (4.6), and (4.10). We only consider the bosonic part.

Let us assume that x^{11} is the isometry direction, which is compactified on a circle of coordinate radius R_{11}, and $d = 11$ supergravity fields are independent on that direction. Then, after introducing convention for indices

$$M = (\mu, \tilde{11}) \ , \quad A = (a, 11) \ ,$$

$$\mu \ (a) \ : \quad d = 10 \text{ curved (flat tangent) space-time index} \ ,$$

the $d = 11$ metric may be written as

$$\begin{aligned} ds_{11}^2 =& G_{MN} dx^M dx^N \\ =& \tilde{G}_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma} (dx^{11} + A_\mu dx^\mu)^2 \ , \end{aligned} \tag{5.4}$$

where

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - e^{2\sigma} A_\mu \ .$$

We note that A_μ is actually an Abelian gauge field whose transformation law originates from the diffeomorphism $\delta x^{11} = \xi^{11}$. By taking $\xi \equiv \xi^{11}$, the gauge transformation is

$$\delta A_\mu = -\partial_\mu \xi \ . \tag{5.5}$$

As for the 3-form gauge field, it is related to the $d = 10$ quantities as

$$\begin{aligned} & C_{MNP} = (C_{\mu\nu\rho}, C_{\mu\nu\tilde{1}1}) = (A_{\mu\nu\rho}, B_{\mu\nu}) \\ \Rightarrow \quad & G_{MNPQ} = (G_{\mu\nu\rho\sigma}, G_{\mu\nu\rho\tilde{1}1}) = (F_{\mu\nu\rho\sigma}, H_{\mu\nu\rho}) \ . \end{aligned} \tag{5.6}$$

If we plug Eqs. (5.4) and (5.6) into the bosonic part of the $d = 11$ supergravity action, we get

$$S_B = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\tilde{G}} \left[e^\sigma \tilde{R} - \frac{1}{2 \cdot 3!} e^{-\sigma} H_{(3)}^2 + \cdots \right] \tag{5.7}$$

where $\kappa_{10}^2 = \kappa_{11}^2 / 2\pi R_{11}$ and we have kept the most important terms while the remaining terms are denoted as dots. This is not of the usual canonical form.

There are two canonical forms for IIA supergravity. One is the form in the string frame and another in the Einstein frame. In order to have the canonical forms, we perform the Weyl transformation,

$$\begin{aligned} \tilde{G}_{\mu\nu} &= e^{2\Lambda} g_{\mu\nu} \ , \\ \tilde{R} &= e^{-2\Lambda} \left[R - 2(d-1) \nabla^\mu \partial_\mu \Lambda - (d-1)(d-2) g^{\mu\nu} \partial_\mu \Lambda \partial_\nu \Lambda \right] \ , \end{aligned} \tag{5.8}$$

where Λ is taken as

$$\Lambda = a\sigma \tag{5.9}$$

with some constant a. Under this transformation, S_B becomes

$$\begin{aligned} S_B = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \Big[& e^{(4a+1)\sigma} \left(R - 9a \nabla^\mu \partial_\mu \sigma - 18a^2 g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right) \\ & - \frac{1}{2 \cdot 3!} e^{(2a-1)\sigma} H_{(3)}^2 + \cdots \Big] \ . \end{aligned} \tag{5.10}$$

5.1 String frame

In the string frame, there is an overall factor $e^{-2\phi}$ in the Lagrangian for the NS-NS sector fields, that is, dilaton, graviton, and 2-form gauge field. It turns our that the choice $a = -1$ with the identification

$$\sigma = \frac{2}{3}\phi \tag{5.11}$$

corresponds to the IIA supergravity in the string frame.

Following the terminology from string theory, the bosonic part of IIA supergravity action in the string frame is given by

$$S^B_{\text{IIA}} = S_{\text{NS-NS}} + S_{\text{R-R}} + S_{CS} \ , \tag{5.12}$$

where each term on the right hand side is

$$\begin{aligned} S_{\text{NS-NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\cdot 3!}H_{(3)}^2\right) \ , \\ S_{\text{R-R}} &= -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(\frac{1}{2\cdot 2!}F_{(2)}^2 + \frac{1}{2\cdot 4!}\widetilde{F}_{(4)}^2\right), \\ S_{\text{CS}} &= -\frac{1}{4\kappa_{10}^2}\int B_{(2)}\wedge F_{(4)}\wedge F_{(4)} \ . \end{aligned} \tag{5.13}$$

The field strength $\widetilde{F}_{(4)}$ is defined by

$$\widetilde{F}_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]} - 4A_{[\mu}H_{\nu\rho\sigma]} \tag{5.14}$$

which is invariant under the $U(1)$ transformation (5.5).

5.2 Einstein frame

If we choose $a = -1/4$ with the identification (5.11), we get the IIA supergravity action in the Einstein frame. On the other hand, we can go to the Einstein frame from the string frame via the relation

$$g_{\mu\nu}\ (\text{string})\ = e^{\phi/2}g_{\mu\nu}\ (\text{Einstein})\ . \tag{5.15}$$

Then the resulting action is the sum of the following actions.

$$\begin{aligned} S_{\text{NS-NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\cdot 3!}e^{-\phi}H_{(3)}^2\right) \ , \\ S_{\text{R-R}} &= -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(\frac{1}{2\cdot 2!}e^{\frac{3}{2}\phi}F_{(2)}^2 + \frac{1}{2\cdot 4!}e^{\frac{1}{2}\phi}\widetilde{F}_{(4)}^2\right), \\ S_{\text{CS}} &= -\frac{1}{4\kappa_{10}^2}\int B_{(2)}\wedge F_{(4)}\wedge F_{(4)} \ . \end{aligned} \tag{5.16}$$

6 Type IIB supergravity

The $d = 10$ IIB supergravity has chiral $N = (2,0)$ supersymmetry, whose algebra is

$$\{Q^A_\alpha, Q^B_\beta\} = -2\delta^{AB}(P_+\Gamma^\mu C^{-1})_{\alpha\beta}P_\mu \ . \tag{6.1}$$

The two supercharges Q^A_α with $A = 1,2$ have the same chirality (positive chirality).

As in the case of IIA supergravity, the number of states in the massless IIB supergravity multiplet is $256 = 128_B + 128_F$. However, the decomposition of the states is different:

$$\begin{array}{ccccccccccccc} \mathbf{128}_B & = & \mathbf{1} & + & \mathbf{35_v} & + & \mathbf{28} & + & \mathbf{1} & + & \mathbf{28} & + & \mathbf{35_c} \\ & & \phi & & g_{\mu\nu} & & B_{(2)} & & A_{(0)} & & A_{(2)} & & A^+_{(4)} \\ \\ \mathbf{128}_F & = & \mathbf{8_c} & + & \mathbf{8_c} & + & \mathbf{56_s} & + & \mathbf{56_s} & & & & \\ & & \lambda^1 & & \lambda^2 & & \psi^1_\mu & & \psi^2_\mu & & & & \end{array} \tag{6.2}$$

where we attach the superscripts $1,2$ to the fermionic fields to distinguish the fermions with the same chirality and $A^+_{(4)}$ means that its field strength is self-dual.

Important progresses in the study of IIB supergravity were made by Schwarz [Nucl. Phys. B226 (1983) 269], Howe and West [Nucl. Phys. B238 (1984) 181], and Schwarz and West [Phys. Lett. B126 (1983) 301]. Due to the presence of self-dual 5-form field strength, it is hard to construct the convariant action for such a field. However, one can formulate the IIB supergravity on-shell, that is, the covariant equations of motion.

One interesting property of IIB supergravity is the presence of $SL(2,\mathbf{R})$ symmetry, in which we are interested here. Although there is no covariant action, we may ignore the problem of self-dual field strength in the investigation of $SL(2,\mathbf{R})$ and use the action. In the string frame, the bosonic part of the IIB supergravity action is

$$S^B_{\rm IIB} = S_{\text{NS-NS}} + S_{\text{R-R}} + S_{CS} \ , \tag{6.3}$$

where the terms on the right hand side are given by

$$\begin{aligned} S_{\text{NS-NS}} &= \frac{1}{2\kappa^2_{10}}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\cdot 3!}H^2_{(3)}\right) \ , \\ S_{\text{R-R}} &= \frac{1}{2\kappa^2_{10}}\int d^{10}x\sqrt{-g}\left(-\frac{1}{2}F^2_{(1)} - \frac{1}{2\cdot 3!}\widetilde{F}^2_{(3)} - \frac{1}{4\cdot 5!}\widetilde{F}^2_{(5)}\right) \ , \\ S_{\rm CS} &= -\frac{1}{4\kappa^2_{10}}\int A^+_{(4)}\wedge H_{(3)}\wedge F_{(3)} \ . \end{aligned} \tag{6.4}$$

The field strengths with tilde are defined as

$$\begin{aligned} \widetilde{F}_{(3)} &= F_{(3)} - A_{(0)}\wedge H_{(3)} \ , \\ \widetilde{F}_{(5)} &= F_{(5)} - \frac{1}{2}A_{(2)}\wedge H_{(3)} + \frac{1}{2}B_{(2)}\wedge F_{(3)} \ . \end{aligned} \tag{6.5}$$

To see the $SL(2,\mathbf{R})$ symmetry clearly, it is better to go to the Einstein frame. By using (5.15), we can get the IIB supergravity action in the Einstein frame as follows:

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa^2_{(10}} \int d^{10}x\sqrt{-g}\bigg(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial A)^2 - \frac{1}{2\cdot 3!}e^{-\phi}H^2_{(3)} \\ & - \frac{1}{2\cdot 3!}e^{\phi}\widetilde{F}^2_{(3)} - \frac{1}{4\cdot 5!}\widetilde{F}^2_{(5)}\bigg) - \frac{1}{4\kappa^2_{(10)}}\int A^+_{(4)}\wedge H_{(3)}\wedge F_{(3)} \ , \end{aligned} \tag{6.6}$$

where we set

$$A \equiv A_{(0)} \ .$$

Now we see that there seems to be some relation between ϕ and A (also between $H_{(3)}$ and $\widetilde{F}_{(3)}$). Indeed it is so. To see this, we define following quantities.

$$\begin{aligned} &\tau = A + ie^{-\phi} \ , \\ &\mathcal{M}_{ij} = \frac{1}{\mathrm{Im}\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{pmatrix} \ , \quad F^i_{(3)} = \begin{pmatrix} H_{(3)} \\ F_{(3)} \end{pmatrix} \end{aligned} \tag{6.7}$$

The action then becomes

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\bigg(R - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\mathrm{Im}\tau)^2} - \frac{1}{2\cdot 3!}\mathcal{M}_{ij}F^i_{(3)}\cdot F^i_{(3)} - \frac{1}{4\cdot 5!}\widetilde{F}^2_{(5)}\bigg) \\ & - \frac{1}{8\kappa^2}\epsilon_{ij}\int A^+_{(4)}\wedge F^i_{(3)}\wedge F^j_{(3)} \ . \end{aligned} \tag{6.8}$$

We consider the following $SL(2,\mathbf{R})$ transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{6.9}$$

where a, b, c, and d are real numbers and satisfy $ad - bc = 1$. This leads to the transformation rule for the matrix $\mathcal{M}$ as

$$\mathcal{M}' = (\Lambda^{-1})^T\mathcal{M}\Lambda^{-1} \ , \quad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \ . \tag{6.10}$$

Then the IIB supergravity is invariant under the following tranformation rules in addition to that of τ.

$$\begin{aligned} &F'^i_{(3)} = \Lambda^i{}_j F^j_{(3)} \ , \\ &g'_{\mu\nu} = g_{\mu\nu} \ , \quad \widetilde{F}'_{(5)} = \widetilde{F}_{(5)} \ . \end{aligned} \tag{6.11}$$

As a final remark, we note that the $SL(2,\mathbf{R})$ symmetry is crucial in the study of non-perturbative dynamics of type IIB string theory, which is unfortunately beyond of the scope of this note.

Bon voyage!