/

Bosonic closed string theory in flat space

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau \sqrt{-\gamma}\, \gamma^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$\alpha' = l^2/2$ $\quad l$ string length

Symmetry (local) reparametrization invariances

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$
$$\delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}$$
$$\delta(\sqrt{h}) = \partial_\alpha(\xi^\alpha \sqrt{h})$$

Weyl scaling $\quad \delta h^{\alpha\beta} = \Lambda h^{\alpha\beta}$

Global Symmetry Poincare invariance

$$\delta X^\mu = a^\mu_{\ \nu} X^\nu + b^\mu \qquad a_{\mu\nu} = -a_{\nu\mu}$$
$$\delta h^{\alpha\beta} = 0$$

index raised or lowered by $\eta_{\alpha\beta}$

Using repara invariances we set $\gamma_{ab} = \eta_{ab}$

Static gauge $X^1 = \sigma \quad X^0 = \tau$

a string moving in d-dim flat space

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau\, \partial_a X^\mu \partial^a X_\mu$$
$$= \frac{1}{\pi\alpha'}\int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu$$

$\sigma^+ = \tau - \sigma$

$\sigma^- = \tau + \sigma$

eq. of motion for X^μ $\quad \partial_+\partial_- X^\mu = 0$ (free wave equation)

With the periodic boundary condition

$$X^\mu(\sigma) = X^\mu(\sigma + 2\pi)$$

$$X^\mu = x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n}\left(\alpha^\mu_n e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}\right)$$

center of mass momentum

Impose the canonical commutation relation

$$[\dot{X}^\mu(\sigma,\tau),\ X^\nu(\sigma',\tau)] = 2\pi\alpha' i \eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\Rightarrow \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}$$

$$[\alpha^\mu_n, \alpha^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0} \quad \sim \text{harmonic oscillator}$$

$$(\alpha^\mu_n)^+ = \alpha^\mu_{-n} \qquad (\tilde{\alpha}^\mu_n)^+ = \tilde{\alpha}^\mu_{-n}$$

energy momentum tensor $\quad T_{ab} \sim \frac{\delta S}{\delta \gamma_{ab}}$

(eq. of motion $T_{ab} = 0$)

Conformal invariance $\to$ $\gamma^{ab} T_{ab} = 0$

$T_{+-} = 0$ automatically satisfied

$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + (X')^2)$

$$T_{++} = \frac{1}{\alpha'} (\partial_+ X^\mu)(\partial_+ X_\mu)$$

$$= \sum_n L_n e^{-in(\tau-\sigma)}$$

$$T_{--} = \frac{1}{\alpha'} (\partial_- X^\mu)(\partial_- X_\mu) = \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}$$

where

$$L_n = \sum_m \frac{1}{2} \alpha_{n-m} \cdot \alpha_m \qquad \tilde{L}_n = \sum_m \frac{1}{2} \tilde{\alpha}_{n-m} \tilde{\alpha}_m$$

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu$$

Hamiltonian $H = L_0 + \tilde{L}_0$

diffeomorphism invariance $\to$ constraints $T_{++} = T_{--} = 0$

as a quantum operator L_n, $\tilde{L}_n$ are well defined while L_0 and $\tilde{L}_0$ are subject to the normal ordering ambiguity.

Oscillator ground state $\alpha_m |0\rangle = 0$

one can have $|0 ; p^\mu\rangle$

One problem is that for time component

$[\alpha^0_n , \alpha^{0+}_n] = -n$ so we have negative norm states

e.g. $\langle 0| \alpha^0_n \alpha^{0+}_n |0\rangle = -n$

We need additional restrictions for the Hilbert space to obtain a sensible theory.

For a physical state, we impose

$$(L_n - a\delta_{n,0})|phys\rangle = 0 \qquad (\tilde{L}_n - a\delta_{n,0})|phys\rangle = 0 \qquad n \geq 0$$

$\to$ $(L_0 - \tilde{L}_0)|phys\rangle = 0$ level matching condition

(due to the invariance $\sigma \to \sigma + a$)

$\sim$

Noether charge for $\delta\sigma^1 = \text{const}$

$$\int T_{10} = \int \partial_1 X^\mu \partial_0 X_\mu = \int - T_{++} + T_{--} \sim L_0 - \tilde{L}_0$$

(This is similar to the Gupta-Bleuer quantization for the electromagnetic field. If we impose the Lorentz gauge $\partial \cdot A = 0$ the Maxwell eq becomes free eq for A^μ. But the usual quantization leads to negative norm states and we impose $\partial_\mu A^{\mu -} |\text{phys}\rangle$ where $-$ denote the negative frequency part)

One have to figure out the constant a in $(L_0 - a)|\text{phys}\rangle = 0$

This involves: $\frac{1}{2}\sum_n \alpha_{-n}\alpha_n = \frac{1}{2}\sum :\alpha_{-n}\alpha_n: + \frac{1}{2}\sum_{n=1}^{\infty} n$, for each direction of X^μ. The last term diverges and must be regularized. Here one uses the zeta function realization to obtain

$$\sum_{n=1}^{\infty} n \cong \zeta(-1) = -1/12 \qquad \text{with } \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and using the analytic continuation of Riemann zeta function

For D-dimension

$$\frac{1}{2}\sum_n \alpha^\mu_{-n}\alpha_{n\mu} = \frac{1}{2}\sum :\alpha^\mu_{-n}\alpha_{n\mu}: + \frac{D-2}{2}\sum_{n=1}^{\infty} n$$
$$= \frac{1}{2}\sum :\alpha^\mu_{-n}\alpha_{n\mu}: - \frac{D-2}{24}$$

Here naively the zero point energy of X^0 cancels the contribution from one X^i with X^i being a space like direction. This counting will be justified using the light cone gauge to be explained shortly.

For $D=26$ we have

$$(L_0 - 1)|\text{phys}\rangle = (\bar{L}_0 - 1)|\text{phys}\rangle = 0$$

$D=26$ is required for the decoupling of the negative norm states in the conformal gauge.

In doing that the Virasoro algebra

$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)$ plays an important role in figuring this out.

quantum part
(normal ordering effect)

Here we just accept that and look for the spectrum in $D=26$.

Define $N \equiv L_0 - \frac{1}{2}(\alpha_0)^2 = \sum_{n>0} \alpha_{-n}\cdot\alpha_n = L_0 - \frac{\alpha'}{4}p^2$

$p^2 = -m^2$ mass of the state

$N - a = \underbrace{L_0 - a}_{=0} + \frac{\alpha'}{4}m^2$

$m^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)$

for $D=26$ $\quad m^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-1)$

$N = \tilde{N} = 0 \quad |0\rangle \quad m^2 = -4/\alpha' \quad$ tachyon

$N = \tilde{N} = 1 \quad \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0\rangle \quad m^2 = 0 \quad$ massless modes

space-time two rank tensors $\to G^{\mu\nu}, B^{\mu\nu}, \phi$

graviton!

Thus we learn that only in $D=26$,

we can have the massless modes in the string spectrum

one of which can be interpreted as a graviton.

It's interesting that $D=26$ is picked up as a quantum consistency condition i.e., decoupling of the negative norm states in the conformal gauge.

Now let's comment on the light cone gauge. The light cone gauge is the gauge where we can just the physical degrees of freedom. Using the residual gauge invariance after the conformal gauge fixing, $\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$

One can choose the light cone gauge $X^+(\sigma,\tau) = x^+ + \alpha' p^+ \tau$

where $X^+ \equiv (X^0 + X^{D-1})/\sqrt{2} \quad X^- \equiv (X^0 - X^{D-1})/\sqrt{2}$.

The Virasoro constraint equations $(\dot{X} \pm X')^2 = 0$ become

$(\dot{X}^- \pm X'^-) = (\dot{X}^i \pm X'^i)^2 / 2\alpha' p^+ \qquad (V\cdot W = V^i W^i - V^+W^- - V^-W^+)$

This equation can be solved for X^- in terms of X^i so that in the light cone gauge both X^+ and X^- can be eliminated leaving only the transverse oscillators X^i.

For $X^- = x^- + \alpha' p^- \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n^- e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^- e^{-in(\tau+\sigma)} \right)$

One obtains

$$\alpha_n^- = \sqrt{\frac{\alpha'}{2}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha^i_{n-m} \alpha^i_m$$

$$\tilde{\alpha}_n^- = \sqrt{\frac{\alpha'}{2}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \tilde{\alpha}^i_{n-m} \tilde{\alpha}^i_m$$

In particular for $n=0$

$$\alpha' p^+ p^- = \sum_i \sum_{m=-\infty}^{\infty} \alpha^i_{-m} \tilde{\alpha}^i_m = \frac{\alpha'}{2}(p^i)^2 + 2\left(\sum_i \sum_{n>0} \alpha^i_{-n} \alpha^i_n - a\right)$$

after considering normal ordering

$$M^2 = 2p^- p^+ - (p^i)^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)$$

which is the same expression we obtained before

As an next example, let us consider the compactification on S^1

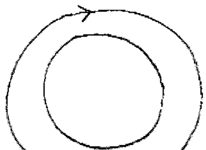
$X \sim X + 2\pi R$

Since the wave function should be periodic under $X \to X + 2\pi R$

$e^{ip\cdot(2\pi R)} = 1 \qquad p = n/R$

Furthermore string can wind around the circle

$X(2\pi R) = X(0) + 2\pi R m$. $\leftarrow$ this is called the winding mode



$$X = x + \alpha' p \tau + m R \sigma + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{i}{n} \left(\alpha_n e^{-in(\tau-\sigma)} + \tilde{\alpha}_n e^{-in(\tau+\sigma)} \right)$$

Note that the spectrum is invariant under $R \to \alpha'/R$. One can see that the string interaction is also invariant under such transformation. This symmetry of the string theory is called the T-duality. One sees the first example of the "stringy" geometry The geometry seen by a string could be quite different from the usual geometry seen by the field theory. In the string theory one cannot probe shorter than $\sqrt{\alpha'}$.

Partition function and the modular invariance for $\mathbb{R}^{25} \times S^1$

$$Z = \mathrm{Tr}\, q^{L_0 - 1} \bar{q}^{\bar{L}_0 - 1} \qquad q = e^{2\pi i \tau}$$

$$= \mathrm{Tr}\, e^{-2\pi \,\mathrm{Im}\,\tau \,(\underbrace{L_0 + \bar{L}_0 - 2}) + 2\pi i \,\mathrm{Re}\,\tau \,(L_0 - \bar{L}_0)}$$

Hamiltonian — level matching condition

$T_{++} = \frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu = \sum_n L_n e^{-in(\tau - \sigma)} + \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}$

$$L_0 = \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\tilde{p}^2 + \sum_{n>0} \alpha_{-n}\cdot\alpha_n$$

remaining 25 directions

$$\bar{L}_0 = \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \frac{\alpha'}{4}\tilde{p}^2 + \sum_{n>0} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

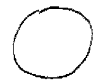
Note that $L_0 + \bar{L}_0 - 2 = \sum_\lambda \frac{\alpha'}{2}(p^2 + M_\lambda^2)$

all 26 directions

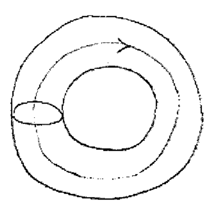
using $N - a = \frac{\alpha'}{4} m^2$

One loop vacuum amplitude

particle
$$\Lambda = \frac{1}{2} \ln \det (p^2 + m^2) = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \ln (p^2 + m^2)$$
$$= \int_0^\infty \frac{dt}{2t} \int \frac{d^D p}{(2\pi)^D} e^{-t(p^2+m^2)}$$



String
$$\int \frac{d\tau d\bar{\tau}}{2\,\mathrm{Im}\,\tau} \mathrm{Tr}\, e^{-2\pi \,\mathrm{Im}\,\tau\,(L_0 + \bar{L}_0 - 2) + 2\pi i \,\mathrm{Re}\,\tau\,(L_0 - \bar{L}_0)}$$
$$\| $$
$$\int \frac{d^D p}{(2\pi)^D} \sum_{M_\lambda} e^{-\alpha' \pi \,\mathrm{Im}\,\tau\,(p^2 + M_\lambda^2)}$$



~ torus amplitude

Oscillator parts (concentrate one direction)

$$\mathrm{Tr}\, q^{L_0} \sim \sum e^{2\pi i \tau \sum_{n>0} \alpha_{-n}\cdot\alpha_n}$$
$$= \prod_{n=1}^{\infty} \sum e^{2\pi i \tau\, n a_n^\dagger a_n}$$
$$'' \; 1 + e^{2\pi i \tau} + e^{4\pi i \tau} + \cdots$$
$$= \prod_{n=1}^{\infty} \frac{1}{1 - e^{2\pi i \tau n}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}$$

$$Z = \int \frac{d\tau d\bar{\tau}}{2\,\mathrm{Im}\,\tau} \mathrm{Tr}\, q^{L_0 - 1} \bar{q}^{\bar{L}_0 - 1}$$
$$= \int \frac{d\tau d\bar{\tau}}{2\,\mathrm{Im}\,\tau} \int \frac{d^{25}p}{(2\pi)^{25}} \mathrm{Tr}\, e^{-2\pi \,\mathrm{Im}\,\tau\,(L_0 + \bar{L}_0 - 2) + 2\pi i \,\mathrm{Re}\,\tau\,(L_0 - \bar{L}_0)}$$
$$= \int \frac{d\tau d\bar{\tau}}{2\,\mathrm{Im}\,\tau} \frac{1}{(\alpha' \,\mathrm{Im}\,\tau)^{25/2}} \frac{\sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2}}{|\eta(\tau)^{24}|^2}$$

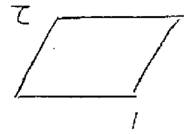
Here we define $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$

and we set $\alpha' = 2$ for the convenience

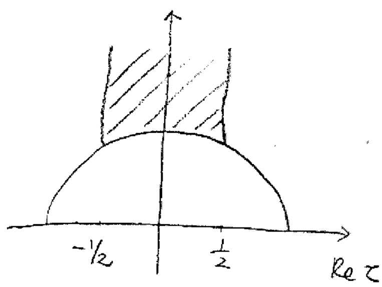
Note that the momentum integration is done over $26-d$ while the oscillator term integration is done over $24-d$ just the transverse directions. Recall that in the light cone gauge the zero mode momentums are still there in the light cone directions while the oscillators are expressed in terms of those in the transverse directions.

Modular invariance

torus $z \sim z+1 \sim z+\tau$



One can characterize the shape of a torus by a complex parameter τ



fundamental domain

The operations $T: \tau \to \tau+1$

$S: \tau \to -1/\tau$

generate large diffeomorphisms of the torus $\sim SL(2,\mathbb{Z})$

(i.e diffeomorphisms not connected with the identity)

$$\tau \longrightarrow \tau' = \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Thus the partition function should be invariant under $SL(2,\mathbb{Z})$ or T/S

It's known that
$$\eta(\tau+1) = \exp\frac{i\pi}{12}\,\eta(\tau)$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\;\eta(\tau)$$

Under T $\quad \tau \to \tau+1$

$$\frac{\sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2}}{\left| q^{1/24} \prod_n (1-q^n) \right|^2} \qquad e^{\pi i \frac{2n}{R}\cdot mR}$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$$

$$e^{2\pi i mn} = 1$$

winding modes form the dual lattice to those of momentum modes

The last term indicates that their inner product is always an even integer

Under S, using the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}^p} \exp -\pi(n+x)\cdot A\cdot(n+x) = (\det A)^{-1/2} \sum_{m\in\mathbb{Z}^p} \exp(-\pi m A^{-1} m + 2\pi i m\cdot x)$$

positive definite

With $x=0$, $A = \begin{pmatrix} 2\frac{\mathrm{Im}\,\tau}{R^2} & i\,\mathrm{Re}\,\tau \\ i\,\mathrm{Re}\,\tau & 2(\frac{R}{2})^2 \end{pmatrix}$

and $A^{-1} = \begin{pmatrix} 2(\frac{R}{2})^2\,\mathrm{Im}(-1/\tau) & i\,\mathrm{Re}\,{-1/\tau} \\ i\,\mathrm{Re}\,{-1/\tau} & 2\frac{1}{R^2}\,\mathrm{Im}(-1/\tau) \end{pmatrix}$ $\left(\text{as } \tau\to -1/\tau,\ R\to 2/R\right)$

One can see that the expression is modular invariant

This is a special example of self-dual lattice

winding and momentum modes are interchanged

dual lattice

winding + momentum $\rightarrow$ self-dual lattice

One sees the simplest example of an <u>even self-dual lattice,</u> which is crucial for the modular invariance.

Using the above results, one can directly see that

Z is indeed invariant under S and T.

Strings in general backgrounds and the low energy effective action

We saw that the massless modes of the bosonic string theory contains $G_{\mu\nu}$. $B_{\mu\nu}$. ϕ

graviton

Since the graviton is the fluctuation of the space time geometry we can consider a string theory action in a general background $g_{\mu\nu}$

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \cancel{\eta_{\mu\nu}}\, g_{\mu\nu}(X)$$

If one writes down the most general action for $X^\mu(\sigma, \tau)$ that is invariant under reparametrization of the string world sheet and renormalizable by power counting

with up to two derivatives of X

$$S_1 = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, g_{\mu\nu}(X^\rho)$$

$$S_2 = \frac{1}{2\pi\alpha'} \int d^2\sigma \cdot \epsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, B_{\mu\nu}(X^\rho)$$

$S_1 + S_2$ describes a string theory in a background with $g_{\mu\nu}$ and $B_{\mu\nu}$.

S_2 should be compared with the coupling of the charged particle to the electromagnetic field $\int A_\tau d\tau$

$$\int A_\tau d\tau = \int \frac{dX^\mu}{d\tau} A_\mu \cdot d\tau$$

Thus we see that the string is charged under $B_{\mu\nu}$.
It's rather subtle to describe a coupling of the dilaton, but it turns out to be

$$S_3 = \frac{1}{4\pi} \int d^2\sigma \sqrt{h}\; \Phi(X)\, R^{(2)}$$

world-sheet curvature

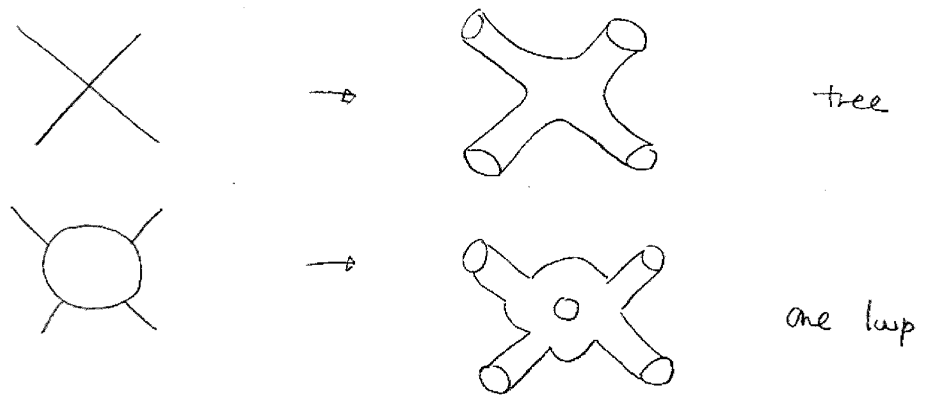
Note that $\frac{1}{4\pi}\int d^2\sigma\sqrt{h}\,R^{(2)}$ is a topological invariant

$= \chi = 2(1-g)$ ← genus of a Riemann surface

In the quantum effective action, this induces a term

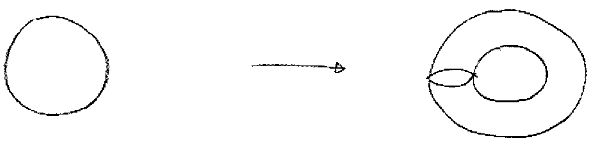
$e^{-S_3} \sim e^{-\phi\, 2(1-g)} = e^{\phi(-2+2g)}$ for a constant ϕ

Now e^{ϕ} is the coupling constant for the string perturbation theory



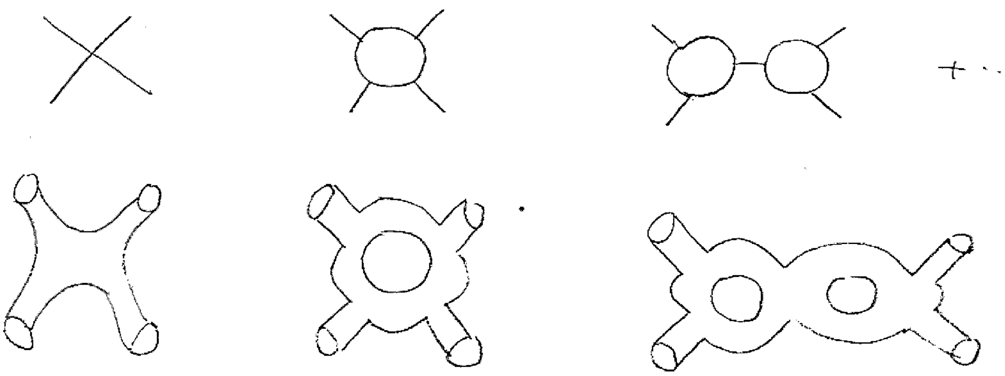
(this kind of thickning of the field theory diagram.

we have seen in the calculation of the vacuum amplitude



)

One can see that string diagrams involve all the Riemann surfaces and if we have many handles, the corresponding diagrams describe higher loops.



One can derive the equation of motion for $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ.
The basic idea is that the string theory has the conformal invariance and from $S = S_1 + S_2 + S_3$
(← action at page 1 reparametrization invariances + Weyl scaling)
One can calculate the beta function for the space time dependent coupling $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$.
The result is

$$R_{\mu\nu} + \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi = 0$$

$$D_\lambda H^\lambda{}_{\mu\nu} - 2 (D_\lambda \Phi) H^\lambda{}_{\mu\nu} = 0$$

$$4 (D_\mu \Phi)^2 - 4 D_\mu D^\mu \Phi + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$

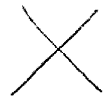
and D_μ is covariant derivative in D-dimension
This can be derived from the 26-D action

$$S_{26} = -\frac{1}{2k^2} \int d^{26}x \sqrt{-g}\, e^{-2\Phi} \left(R + 4 D_\mu \Phi D^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right)$$

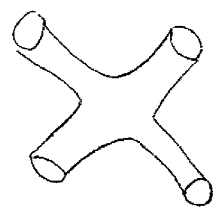
(convention $-++\!+$ $\quad R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \cdots \quad G_{\mu\nu} = 8\pi G T_{\mu\nu}$)

Recipe for the string amplitude

When we calculate the scattering amplitude, it involves a Feynman diagram like



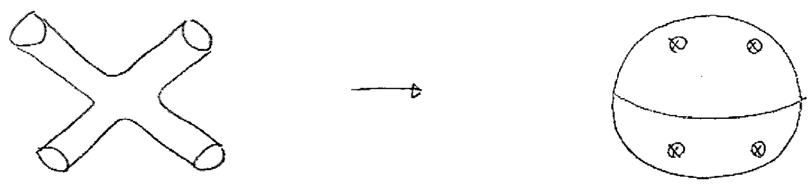
Corresponding string diagram is



It looks complicated to calculate such diagram.

What saves us is (again) conformal invariance so that we can carry out the conformal transformation.

We can shrink each leg at the above diagram to a point



In this figure, a scattering state is represented as a local operator insertion at a point. This operator is called a vertex operator.

A vertex operator taking momentum k has the form

$$\text{Oscillator part} \otimes e^{ik\cdot X}$$

↑ dependent on the corresponding state

e.g. tachyon $: e^{ik\cdot X} :$

graviton $: \zeta_{\mu\nu}\, \partial_\alpha X^\mu \partial^\alpha X^\nu e^{ik\cdot X} :$

antisymmetric field $: b_{\mu\nu}\, \varepsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu e^{ik\cdot X} :$

($\zeta_{\mu\nu}$ $b_{\mu\nu}$ suitable polarization tensor

where the momentum k should satisfy the on-shell condition

Actual string amplitude involves the evaluation

$$\int \mathcal{D}X(\sigma)\, \mathcal{D}h_{\alpha\beta}\, e^{-(S_1+S_2+S_3)} \int d^2\sigma \sqrt{h}\, e^{ik\cdot X} \int d^2\sigma \sqrt{h}\, \partial_\alpha X^\mu \partial^\alpha X^\nu e^{ik\cdot X}$$

$\cdots$

Open string theory

$$S = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \int d\tau \sqrt{-\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$\underset{\gamma^{ab}=\eta^{ab}}{=} \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\int d\tau\,(-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu)$$

eq. of motion $(-\partial_\tau^2 + \partial_\sigma^2) X^\mu = 0$

boundary condition $\delta X^\mu \partial_\sigma X_\mu \big|_{0,\pi} = 0$

$X^\mu =$ constant   Dirichlet

$\partial_\sigma X^\mu = 0$ Neumann

Choose the Neumann B.C.

$$X^\mu(\sigma,\tau) = x^\mu + \underset{\uparrow}{2\alpha' p^\mu \tau} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

identified with the total momentum $= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \frac{dX^\mu(\sigma)}{d\tau}$

$$L_m = \frac{1}{2\pi\alpha'}\int_0^\pi (e^{im\sigma}T_{++} + e^{-im\sigma}T_{--})\,d\sigma = \frac{1}{2}\sum \alpha_{m-n}\cdot\alpha_n$$

$H = L_0 = \frac{1}{2}\sum \alpha_{-n}\cdot\alpha_n$ with $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$

conditions on the physical state

$L_m|\text{phys}\rangle = 0$ $(L_0 - \underset{\uparrow}{a})|\text{phys}\rangle = 0$

1/24 per physical boson

$$L_0 = \alpha' p^2 + \underbrace{\sum_{n>0}\alpha_{-n}\cdot\alpha_n}_{N}$$

$m^2 = \frac{1}{\alpha'}(N-a) = \frac{1}{\alpha'}(N-1)$ in $D=26$

Spectrum

$N=0$ $|0\rangle$ $m^2 = -1/\alpha'$ tachyon $V = \exp ik\cdot X$

$N=1$ $a^\mu_{-1}|0\rangle$ $m^2 = 0$ gauge boson   # of physical state $D-2$

~ gauge symmetry

$V = \underset{(\zeta^\mu)}{\hat{\partial}_\tau X^\mu} \exp ik\cdot X$

For imposing Dirichlet condition at both ends

with $X^\mu|_0 = x_1^\mu \qquad X^\mu|_\pi = x_2^\mu$

$$X^\mu(\sigma, \tau) = x_1^\mu + \frac{x_2^\mu - x_1^\mu}{\pi}\sigma + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin n\sigma$$

(no momentum p)

Contribution for H and L_0 from X^μ

$$H = L_0 \leftarrow \frac{1}{4\pi\alpha'}(x_2^\mu - x_1^\mu)^2 + \sum_{n>0} \alpha_{-n}^\mu \alpha_n^\mu$$

generally (i Dirichlet direction μ: Neumann direction).

$$H = L_0 = (\alpha' p^\mu)^2 + \frac{1}{4\pi\alpha'}(x_2^i - x_1^i)^2 + \sum_{n>0} \alpha_{-n}^\mu \alpha_n^\mu + \sum_{n>0} \alpha_{-n}^i \alpha_n^i$$

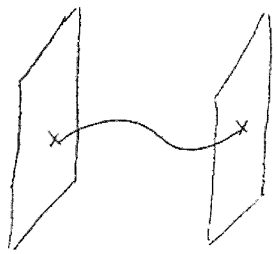
intersecting brane spectrum ?

the meaning of boundary conditions ?

vertex op $\partial_\sigma X^\mu \exp ik\cdot X$

→ clear in the context of D-branes

D-branes : hyperplane on which open string can end



D3-brane world volume

x^0, x^1, x^2, x^3 ← impose Neumann

transverse location eg. $x^4 = x^5 = \cdots = 0$

$x^4 = L \quad x^5 = x^6 = \cdots = 0$ impose Dirichlet

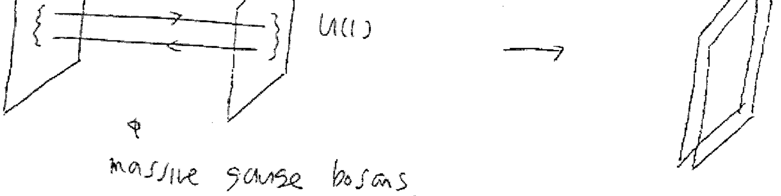
general open string states are described by

~~$|k, \lambda_{ij}\rangle$~~ $|k, \lambda\rangle \lambda_{ji}$ $\quad \lambda$ is hermitian (reality condition of the string ? field)

↑ world sheet fields

↑ Chan Paton factors (carry gauge group)

$\lambda, 1$: D-brane label



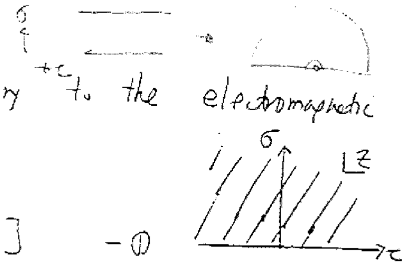
U(1) U(1) → U(2) gauge theory

↑ massive gauge bosons.

Coincident n D-branes ~ $U(n)$ gauge theory

What is the eqn of motion for an abelian gauge field coupled to the Bosonic open string?

conformal transf.



$z = \tau + i\sigma$

The action for the string coupled on the boundary to the electromagnetic field is (M^2 upper half plane)

$$S = \frac{1}{2\pi\alpha'}\left[\frac{1}{2}\int_{M^2} d^2z\, \partial^a X_\mu \partial_a X^\mu + i\int_{\partial M} d\tau A_\mu \partial_\tau X^\mu\right] \quad -①$$

where A_μ has been rescaled to contain a factor $2\pi\alpha'$.

In this calculation we will use the background field approach. This is the standard calculation in string theory. The basic idea is that the string theory has the conformal invariance. Thus if we compute the beta function β^A for the electromagnetic field A_μ, we should have $\beta^A = 0$

(different scale factor at different points

If we expand the action (2.1) around arbitrary background $\bar{X}$

$$X^\mu(\tau,\sigma) = \bar{X}^\mu(\tau,\sigma) + \xi^\mu(\tau,\sigma)$$

we get

$$A^\mu = A^\mu_0 + \partial_\nu A_\mu \xi^\nu + \frac{\partial_\alpha\partial_\beta A_\mu}{2!}\xi^\alpha\xi^\beta + \cdots$$

$$S[\bar{X}+\xi] = S[\bar{X}]$$

(cf) $\frac{1}{g^2}\int F^2$ g does not run if the theory is scale invariant

$$+\frac{1}{2\pi\alpha'}\Big[\int_{M^2} d^2z\,(\partial^a\bar{X}_\mu \partial_a\xi^\mu + \frac{1}{2}\partial^a\xi_\mu\partial_a\xi^\mu)$$

$$+ i\int_{\partial M} d\tau\,(F_{\mu\nu}\xi^\mu\partial_\tau\bar{X}^\nu + \frac{1}{2}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_\tau\bar{X}^\mu$$

$$+\frac{1}{2}F_{\mu\nu}\xi^\nu\partial_\tau\xi^\mu + \frac{1}{3}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_\tau\xi^\mu + \cdots)\Big]$$

where $\nabla_\nu = \frac{\partial}{\partial X^\nu}$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

assume that the fields vary slowly.

→ neglect terms with more than one deriv of F

On shell, the terms linear in ξ disappear since $\bar{X}$ satisfies the eq of motion

$$\Box\bar{X}^\mu = 0 \qquad (\Box = \partial_\tau^2 + \partial_\sigma^2)$$

mode expansion?

$$\partial_\sigma\bar{X}^\mu + iF_\nu{}^\mu\partial_\tau\bar{X}^\nu\big|_{\partial M} = 0$$

The on-shell action is

$$S[\bar{X}+\xi] = S[\bar{X}] + \frac{1}{2\pi\alpha'}\int_{M^2} d^2z \, \frac{1}{2}\, \partial^a \xi_\mu \, \partial_a \xi^\mu$$

$$+ \frac{i}{2\pi\alpha'}\int_{\partial M} d\tau \Big(\frac{1}{2} \nabla_\nu F_{\mu\lambda}\, \xi^\nu \xi^\lambda \, \partial_\tau \bar{X}^\mu + \frac{1}{2} F_{\mu\nu}\, \xi^\nu \partial_\tau \xi^\mu$$

$$+ \frac{1}{3} \nabla_\nu F_{\mu\lambda}\, \xi^\nu \xi^\lambda \, \partial_\tau \xi^\mu + \cdots \Big)$$

Now compute the field-theory 1-loop counter term to the gauge coupling term in ①, namely

$$\Delta S_I(\bar{X}) = \frac{i}{2\pi}\int_{\partial M} d\tau \, \Gamma_\mu \, \partial_\tau \bar{X}^\mu$$

~~From~~ The Neumann propagator in the upper half plane

$$\frac{1}{2\pi\alpha'} \Box G(z, z') = -\delta(z-z')$$

$$\partial_\sigma G(z, z')\Big|_{\sigma=0} = 0 \qquad \text{with} \quad z = \tau + i\sigma$$

The solution can be easily obtained by the image method to

$$G(z, z') = -\alpha' \left(\ln|z-z'| + \ln|z-\bar{z}'| \right)$$

We can compute the counter term with this propagator and sum up all 1-loop graphs with an external $\partial_\tau \bar{X}$ and all possible insertions of the vertex $F_{\mu\nu}\xi^\nu \partial_\tau \xi^\mu$.

More straightforward method is to compute the exact propagator in the presence of the gauge field F.

The B.C. is changed,

$$\partial_\sigma G(z, z')_{\mu\nu} + i F_\mu{}^\lambda \, \partial_\tau G_{\lambda\nu}(z, z') \Big|_{\sigma=0} = 0$$

The solution is

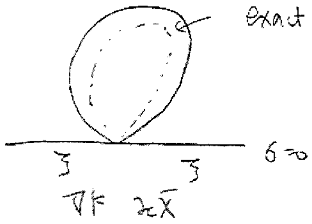
$\leftarrow$ meaning of this?

$$G_{\mu\nu}(z, z') = -\alpha' \Big[\delta_{\mu\nu} \ln|z-z'| + \frac{1}{2}\left(\frac{1-F}{1+F}\right)_{\mu\nu} \ln(z-\bar{z}')$$

$$+ \frac{1}{2}\left(\frac{1+F}{1-F}\right)_{\mu\nu} \ln(\bar{z}-z') \Big]$$

(symmetric under $z \leftrightarrow z'$, $\mu \leftrightarrow \nu$)

The only counterterm to S_Σ is



$$\Delta S_\Sigma = \frac{-\lambda}{2\pi\alpha'}\int d\tau\, \frac{1}{2}\nabla_\nu F_{\mu\lambda}\, \partial_\tau \bar{X}^\mu\, G^{\nu\lambda}(\tau\tau')\Big|_{\tau\to\tau'}$$

↑ propagator on the boundary ($\sigma=\sigma'=0$)

In the limit $\tau\to\tau'$

$$G_{\nu\lambda}(\tau\to\tau') = -\alpha'\left[1+\frac{1}{2}\frac{1-F}{1+F}+\frac{1}{2}\frac{1+F}{1-F}\right]_{\nu\lambda}\ln\Lambda$$

$$= -2\alpha'\ln\Lambda\,(1-F^2)^{-1}_{\nu\lambda}$$

Λ : short distance cutoff

$$\therefore\ \beta_\mu{}^A = \Lambda\frac{\partial}{\partial\Lambda}\Gamma_\mu = \nabla^\nu F_\mu{}^\lambda\,(1-F^2)^{-1}_{\lambda\nu}$$

eqn of motion $\nabla^\nu F_\mu{}^\lambda (1-F^2)^{-1}_{\lambda\nu} = 0$ — ②

positive definite

(F is actually $2\pi\alpha' F$ → ② contains all orders in α')

At the leading order in α' → Maxwell's equation

The beta function is not derivable from any action

but $\chi_{\mu\nu}\beta^\nu{}_A$ is for a suitable $\chi_{\mu\nu}$

$$(1-F^2)^{-1}_{\mu\nu}\,\beta^\nu{}_A = \nabla^\nu\left(\frac{F}{1-F^2}\right)_{\mu\nu} - \left(\frac{F}{1-F^2}\right)_{\mu\lambda}\nabla^\nu F^{\lambda\rho}\left(\frac{F}{1-F^2}\right)_{\rho\nu}$$

Bianchi identity → $$= \nabla^\nu\left(\frac{F}{1-F^2}\right)_{\mu\nu} + \frac{1}{4}\left(\frac{F}{1-F^2}\right)_{\mu\nu}\nabla^\nu\,\mathrm{tr}\ln(1-F^2)$$

This is derivable from

$$\mathcal{L}_{\rm eff} = \exp\frac{1}{4}\mathrm{tr}\ln(1-F^2) = \exp\frac{1}{2}\mathrm{tr}\ln(1+F) = \sqrt{\det(1+F)}$$

whose Euler-Lagrange eqs are

$$\sqrt{\det(1+F)}\,(1-F^2)^{-1}_{\mu\nu}\,\beta^\nu{}_A = 0$$

In the leading order in α', $\mathcal{L}_{\rm eff}$ is reduced to the action for the Maxwell theory.

Combined with

$$S_2 = \frac{1}{2\pi\alpha'} \int d^2\sigma \, \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)$$

with $\int_{\partial M} A = \int_{\partial M} \frac{dx^\mu}{d\tau} A_\mu$

we have $\frac{1}{2\pi\alpha'} \int B + \int_{\partial M} A$ (using the differential form notation)

This is invariant under the vector gauge transformation

$\delta A = d\Lambda$ ($\delta A^\mu = \partial^\mu \Lambda$) $\leftarrow$ ($\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$)

but the two-form gauge transf. $\delta B = d\xi$ gives a surface term cancelled by assigning a transformation $\delta A = -\xi/2\pi\alpha'$ ($\delta A^\mu = -\frac{\xi^\mu}{2\pi\alpha'}$)

Thus the combination $B + 2\pi\alpha' F$ is invariant under both transformation

This implies that the Born-Infeld action we obtain is modified

$$S = \int \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}$$

$G_{ab} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$ (pull back)

$B_{ab} = \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}$

(cf) Nambu-Goto action $\int \sqrt{-\det G_{ab}}$ is classically equivalent to string action we use $\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$

The dilaton coupling is simply

$$S = \int d^{p+1}x \, e^{-\phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}$$

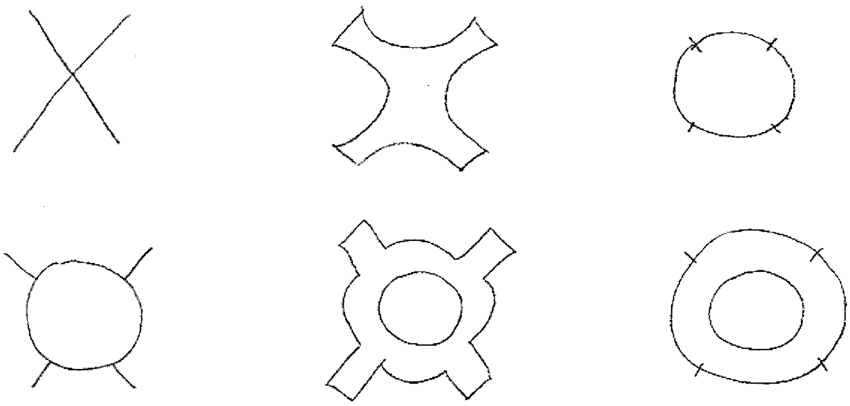
Since this is an open string tree level action

Previously we saw that closed string perturbation has all the Riemann surfaces.

In the case of oriented open string, we should consider Riemann surfaces with boundary. The expansion parameter is

$$e^{\phi(-2 + \underbrace{2h}_{\#\text{ of handles}} + \overbrace{b}^{\#\text{ of boundary}})}$$

The lowest diagram contains $e^{-\phi}$ $\to$ open string tree level



1 baundary

2 baundaries

Super strings

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \partial^\alpha X_\mu \partial_\alpha X^\mu$$

add Majorana fermions $\psi^\mu_A(\sigma, \tau)$
~ real but transforming in the vector representation of $SO(D-1, 1)$
~ internal group quantum number from the 2-D point of view

$S =$

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\frac{1}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \quad -②$$

ρ^α 2-D Dirac matrices $\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$

pure imaginary
Dirac operator $i\rho^\alpha\partial_\alpha$ real ~ ψ^μ real

$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$

$\bar{\chi} = \chi^+ \rho^0 = \bar{\chi}\rho^0$

World sheet supersymmetry

$\delta X^\mu = \alpha' \varepsilon \psi^\mu$

$\delta \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \varepsilon$ (supersymmetry on shell)

(conserved) Super current $J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu$

energy momentum tensor

$$T_{\alpha\beta} = \frac{1}{\alpha'}\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta X_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha X_\mu - \text{trace}$$

Classical equation of motion

$$\partial_+ \partial_- X^\mu = 0$$

2-D Dirac equation $\partial_+ \psi^\mu_- = 0$ and $\partial_- \psi^\mu_+ = 0$

$\sigma^\pm \equiv \tau \pm \sigma$

$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

$$S_F = \frac{i}{\pi}\int d^2\sigma\,(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+)$$

left mover and right mover decouple for both bosons and fermions. It is convenient to write the world-sheet super current and energy momentum tensor in terms of these.

$$J_+ = \psi_+^\mu \partial_+ X_\mu \qquad 0 = \partial_- J_+ = \partial_+ J_-$$
$$J_- = \psi_-^\mu \partial_- X_\mu$$

$$T_{++} = \frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2} \psi_+^\mu \partial_+ \psi_{+\mu}$$
$$T_{--} = \frac{1}{\alpha'} \partial_- X^\mu \partial_- X_\mu + \frac{i}{2} \psi_-^\mu \partial_- \psi_{-\mu}$$

Constraints $\quad J_+ = J_- = T_{++} = T_{--} = 0$

(In order to obtain these conditions, we should find the action which is reduced to the action ② upon the gauge fixing. In fact such action is known but we just assume it and proceed.)

Boundary conditions and mode expansions

Vanishing of the surface term $\psi_+ \delta\psi_+ - \psi_- \delta\psi_-$

$\to \psi_+ = \pm \psi_-$ at the boundary

Set $\quad \psi_+^\mu (0, \tau) = \psi_-^\mu (0, \tau)$

$\psi_+^\mu (\pi, \tau) = \pm \psi_-^\mu (\pi, \tau) \qquad$ + R (Ramond) boundary

− NS (Neveu-Schwarz) condition

R $\quad \psi_-^\mu (\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau - \sigma)}$ $\quad$ ($\sim$ spin structure of S^1)

$\psi_+^\mu (\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau + \sigma)}$ $\quad$ open

NS $\quad \psi_-^\mu (\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau - \sigma)}$ $\quad$ string

$\psi_+^\mu (\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau + \sigma)}$

Closed string

R $\quad \psi_-^\mu = \sum d_n^\mu e^{-2in(\tau - \sigma)} \qquad$ R $\quad \psi_+^\mu = \sum \tilde{d}_n^\mu e^{-2in(\tau + \sigma)}$

NS $\quad \psi_-^\mu = \sum b_r^\mu e^{-2ir(\tau - \sigma)} \qquad$ NS $\quad \psi_+^\mu = \sum \tilde{b}_r^\mu e^{-2ir(\tau + \sigma)}$

NS−NS $\quad$ R−R $\quad$ space time bosons $\quad$ (to be shown later)

NS−R $\quad$ R−NS $\quad$ space time fermions super Virasoro operator ← mode expansion of $T_{\alpha\beta}$ and J_α

open one independent set closed two independent sets

$$L_m = \frac{1}{\pi}\int_0^\pi d\sigma\ (e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}) = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma\ e^{im\sigma}T_{++}$$

fermionic generator

R $$F_m = \frac{\sqrt{2}}{\pi}\int_0^\pi d\sigma\ (e^{im\sigma}J_+ + e^{-im\sigma}J_-) = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\ e^{im\sigma}J_+$$

NS $$G_r = \frac{\sqrt{2}}{\pi}\int_0^\pi d\sigma\ (e^{im\sigma}J_+ + e^{-im\sigma}J_-) = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\ e^{ir\sigma}J_+$$

$$\left(G_r = \sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot b_{r+n}\ \ (NS) \qquad F_m = \sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot d_{m+n} \right)$$

Quantization

$$[\dot{X}^\mu(\sigma,\tau),\ X^\nu(\sigma',\tau)] = 2\pi\alpha' i\ \delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\to [\alpha^\mu_m,\ \alpha^\nu_n] = m\,\delta_{m+n}\eta^{\mu\nu}$$ (Similar for $\tilde{\alpha}^\mu_m$ in case of the closed string).

$$\{\psi^\mu_A(\sigma,\tau),\ \psi^\nu_B(\sigma',\tau)\} = \pi\ \delta(\sigma-\sigma')\eta^{\mu\nu}\delta_{AB}$$

$$\to \{b^\mu_r,\ b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s}$$

$$\{d^\mu_m,\ d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n}$$

Zero frequency part of the Virasoro constraint

NS $$\alpha' M^2 = N + (D-2)(-1/24 - 1/48)$$ ← fermion with anti periodic B.C.

$$= N - \frac{D-2}{16}$$ for a physical state

with $N = N^\alpha + N^b$ $\qquad N^\alpha = \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m$

$$N^b = \sum_{r=1/2}^{\infty} r\,b_{-r}\cdot b_r$$

where the ground state $|0\rangle$ is

defined by $\alpha^\mu_m|0\rangle = b^\mu_r|0\rangle = 0 \qquad m, r > 0$

↑ unique ground state

with the physical state condition

$$L_n|phys\rangle = G_r|phys\rangle = 0 \qquad n, m > 0$$

$$(L_0 - \frac{D-2}{16})|phys\rangle = 0$$

We can have $|0 : p^\mu\rangle$

Massless modes $b^\mu_{-1/2}|0\rangle$ for $\underline{D=10}$ ← critical dimension !

~ vector bosons in 10-D 8 physical degrees of freedom

GSO projection $(-1)^F$

where F is the world sheet fermion number

ψ^μ odd and X^μ even under $(-1)^F$

Choose $(-1)^F = -1$ on $|0\rangle$

then $(-1)^F = +1$ for $b^\mu_{-1/2}|0\rangle$ and keep the states with $(-1)^F = 1$

· Under GSO projection we will have space time supersymmetry

(the spectrum of the closed string contains a massless spin 3/2 particle. It is believed that interacting theory with a massless spin 3/2 particle would be consistent only if the particle couples to a conserved current, which is identified with the supersymmetric current)

Here we have the first example of the subtle relation between the physics of the world-sheet and that of the space-time. And there will be many examples to be encountered.

(Also GSO projection is related to the choice of the spin structure. In torus, such spin structures are mixed with each other under S and T transformation. Again if we require the modular invariance, we need the GSO projection.)

R $\quad \alpha' M^2 = N + (D-2)(-1/24 + 1/24)$

$\uparrow$ fermion with periodic B.C.

$= N$ for a physical state

with $N = N^\alpha + N^d \qquad N^\alpha = \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m$

$N^d = \sum_{m=1}^{\infty} d_{-m}\cdot d_m$

with the physical state condition

$L_n |phys\rangle = F_m |phys\rangle = 0 \qquad n, m > 0$

$\therefore L_0 |phys\rangle = F_0 |phys\rangle = 0$

Peculiar character of the R sector : degenerate vacua

due to $\{ d_0^\mu, d_0^\nu \} = \eta^{\mu\nu}$ —① (d_0^μ commute with M^2 operator)

$\uparrow$

Clifford algebra

$d_0^\mu \sim$ Dirac matrices

$\{ \Gamma^\mu, \Gamma^\nu \} = -2\eta^{\mu\nu} \qquad \Gamma^\mu = i\sqrt{2}\, d_0^\mu$

Each mass level of string states is a representation of ①

and the ground state should be an irreducible representation

since no other zero modes cause any degeneracy.

Irreducible representation of ① $\sim$ spinor of $SO(1,9)$

$\to$ Spacetime fermions !

It is known that spinors of $SO(1,9)$ has $2^{10/2} = 32$ components.

GSO projection corresponds to keep the Γ^{11} eigenstates out of these.

We can either have $(\Gamma^{11})' = +1 \sim 16$

or $(\Gamma^{11})' = -1 \sim 16'$

($\Gamma^{11} \equiv \Gamma^0\Gamma^1 \cdots \Gamma^9$ and plays a similar role of $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

in 4-D.)

Once we chose the ground states for the GSO projection,

we just keep the massive states generated by these ground

states by applying even number of d_{-n}^μ with $n > 0$.

Under the physical state condition, we just keep the half of

16 ground states, which are isomorphic to 8_s or 8_c of

$SO(8)$ spinors.

(This will be clear if we use the light cone gauge.)

The SO(8) spinors are the representation of the clifford algebra

$$\{ \Gamma^i, \Gamma^j \} = -2\delta^{ij} \quad \text{for } i = 1 \text{ to } 8$$

This spinor has the $2^{8/2} = 16$ components.

Again we can define $\Gamma^9 = \Gamma^1\Gamma^2 \cdots \Gamma^8$ with $(\Gamma^9)^2 = +1$

8_s denotes the spinor components with $\Gamma^9 = +1$ (Chirality projection)

8_c with $\Gamma^9 = -1$

Thus the lowest states of the 10-D open super string

are $8_v \sim b^{\mu}_{-1/2}|0\rangle \sim A^{\mu}$ where v denotes the vector representation

8_s 4_a (we denote 8_c by $4_{\dot{a}}$)

and $(A^{\mu}, 4_{\alpha})$ form $N=1$ 10-D supermultiplet.

The low energy theory is described by $N=1$ 10-D supersymmetric $U(N)$ gauge theories if we have N space time D-9 branes.

Now let us fill in the details.

It's convenient to consider the combinations

$$d_i^{\pm} = \frac{1}{\sqrt{2}} \left(d_0^{2i} \pm i\, d_0^{2i+1} \right) \qquad i = 1 \cdots 4$$

$$d_0^{\pm} = \frac{1}{\sqrt{2}} \left(d_0^1 \mp d_0^0 \right)$$

In this basis, the Clifford algebra takes the form

$$\{ d_i^+, d_j^+ \} = \delta_{ij}$$

$d_i^{\pm}$ $i = 0 - 4$ act as raising and lowering operators, generating 32 Ramond ground states. Denote these states

$$|s_0, s_1, s_2, s_3, s_4\rangle = |\vec{s}\rangle$$

where each of $s_i = \pm 1/2$ and

$$d_i^- |-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0$$

while d_i^+ raises s_i from $-1/2$ to $1/2$.

The fermionic part of the 10-D Lorentz generators is

$$S^{\mu\nu} = -\frac{i}{2}\sum_{m=-\infty}^{\infty} d^{\mu}_{-m} d^{\nu}_{m} - d^{\nu}_{-m} d^{\mu}_{m} \quad \text{for the R sector.}$$

The states above are eigenstates of $S_0 = i S^{01}$, $S_a = S^{2a, 2a+1}$ with S_i being the corresponding eigenvalues.

Since the Lorentz generators flip an even number of S_a, 32 states are decomposed into

16 with an even number of S_i

16' with an odd number.

As a next step, consider $F_0 |phys\rangle = \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot d_n |phys\rangle = 0$

For the ground states, this physical state condition is reduced to

$p_\mu d^{\mu}_0 = 0$. In the frame $p^0 = p^1$ this implies $S_0 = \frac{1}{2}$, which gives 16 physical Ramond ground states.

This is a representation of $SO(8)$, which again decomposes into 8_s with an even number of $-\frac{1}{2}$'s and 8_c with an odd number.

Massless spectrum of the closed string

We have seen that the mode expansion of the closed string is decomposed into left mover and right mover.

We should consider the GSO projections separately for left mover and right mover and then we have 4 sectors NS-NS, R-R, NS-R and R-NS.

For the open string, two possibilities of the GSO projection on the Ramond sector are equivalent but for the closed string, we can have relative difference between the right mover and the left mover.

Thus we have two choices

$$\text{Type IIA} \quad \overset{L}{(8_v \oplus 8_s)} \otimes \overset{R}{(8_v \oplus 8_c)}$$

$$\underset{}{}\quad NS \quad R$$

$$\text{Type IIB} \quad (8_v \oplus 8_s) \otimes (8_v \oplus 8_s)$$

For the NS-NS sector, we have for both IIA and IIB

$$8_v \otimes 8_v \sim b^\mu_{-1/2}\tilde{b}^\nu_{-1/2}|0\rangle$$

Again we have two rank tensors, which are decomposed into $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ.

For the NS-R and R-NS sectors

$$8_v \otimes 8_c = 8_s \oplus 56_c \qquad A^\mu \otimes \psi_{\dot{a}} \sim \underset{8_c}{\gamma^\mu_{a\dot{a}}\psi^a} \oplus \underset{8_s}{\psi^\mu_{\dot{a}}}$$

$$8_v \otimes 8_s = 8_c \oplus 56_s \qquad A^\mu \otimes \psi_a \sim \gamma^\mu_{a\dot{a}}\psi^{\dot{a}} \oplus \psi^\mu_a$$

We obtain the massless spin 3/2 particle, gravitino. eigenvalue of $\Gamma^9 = \Gamma^1 \cdots \Gamma^8$

For type IIA we have two gravitini with the opposite chirality i.e. 56_c, 56_s and for IIB we have two gravitini with the same chirality.

For the RR sector

IIA $\quad 8_s \otimes 8_c = [1] \oplus [3] = 8_v \oplus 56_t$

$$\psi_a \cdot \tilde{\psi}_{\dot{a}} \sim \bar{\psi}_a \Gamma^\mu \tilde{\psi}_{\dot{a}}, \quad \bar{\psi}_a \Gamma^{\mu\nu\lambda}\tilde{\psi}_{\dot{a}} \sim A^{\mu\nu\lambda}$$

IIB $\quad 8_s \otimes 8_s = [0] \oplus [2] \oplus [4]_+ = 1 \oplus 28 \oplus 35_+$

$$\psi_a \cdot \psi_a \sim \bar{\psi}_a\psi_a, \quad \bar{\psi}_a \Gamma^{\mu\nu}\psi_a, \quad \bar{\psi}_a \Gamma^{\mu\nu\lambda\tau}\psi_a - A^{\mu\nu\lambda\tau}$$

Here $[n]$ denotes the n-times antisymmetrized representation of $SO(8)$ with $[4]_+$ being self-dual. The representation $[n]$ and $[8-n]$ are the same being related by contraction with the 8-dimensional ϵ-tensor.

In summary, the massless spectrum is given

		NS-NS	R-R
for type IIA	bosonic	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ	A_μ, $A_{\mu\nu\lambda}$
	fermionic	ψ^μ_a, $\psi^\mu_{\dot{a}}$, ψ_a, $\psi_{\dot{a}}$	
type IIB	bosonic	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ	χ, $B^{(2)}_{\mu\nu}$, $B^+_{\mu\nu\lambda\sigma}$
	fermionic	$2(\psi^\mu_a, \psi_{\dot{a}})$	

The above matter contents form the supermultiplet of type IIA and type IIB supergravity, respectively.

The classical equation of motion for the RR sector is given by the physical state condition $F_0 \sim p^\mu \cdot d_{0\mu}$ for the ground states. p^μ acts as a differentiation and $d_{0\mu}$ as a multiplication by Γ^μ.

$$\Gamma^\nu \Gamma^{[\mu_1} \cdots \Gamma^{\mu_n]} = \Gamma^{[\nu\mu_1 \cdots \mu_n]} + \left(\delta^{\nu\mu}\Gamma^{[\mu_2} \cdots \Gamma^{\mu_n]} + \text{permutations}\right)$$

and similarly for right multiplication.

The physical state conditions become

$$dG = 0 \quad \text{and} \quad d^* G = 0 .$$

– Light cone gauge

Using the residual reparametrization invariances preserving the covariant gauge, one can set

$$X^+(\sigma, \tau) = x^+ + \alpha' p^+ \tau$$

And using the freedom of applying local supersymmetry transformation to set $\psi^+ = 0$ (we assume it.)

As a consistency check we note that under a global supersymmetry transformation

$$\delta X^+ = \bar{\varepsilon}\psi^+ = 0 \quad \text{so that } X^+ \text{ gauge choice is not altered.}$$

The constraints implied by the vanishing of J_α and $T_{\alpha\beta}$ takes the form

$$\psi \cdot \partial_+ X = 0$$

$$\text{and} \quad \frac{1}{\alpha'}(\partial_+ X)^2 + \frac{i}{2}\psi \cdot \partial_+ \psi = 0$$

From these, one obtains

$$\partial_+ X^- = \frac{1}{2p^+}\left(\frac{1}{\alpha'}\partial_+ X^i \partial_+ X^i + \frac{i}{2}\psi^i \partial_+ \psi^i\right)$$

$$\psi^- = \frac{2}{p^+ \alpha'}\psi^i \partial_+ X^i .$$

Again the mode expansion of X^- and ψ^- can be expressed in terms of that of X^i and ψ^i.

– Calculation of the partition function

aequatio identica satis abstrusa

let's calculate the partition function of the open string

$$Z = \operatorname*{Tr}_{NS-R} \underbrace{\left(\frac{1+(-1)^F}{2}\right)}_{\text{GSO projection}} q^{L_0 - a} \qquad \begin{array}{ll} a = 1/2 & \text{for NS} \\ 0 & \text{for R} \end{array}$$

$$L_0 = \alpha' p^2 + \sum_{n>0} \alpha_{-n}\cdot\alpha_n + \begin{cases} \sum_{r=1/2}^{\infty} r\, b_{-r}\cdot b_r & \text{NS} \\ \sum_{m>0} m\, d_{-m}\cdot d_m & \text{R} \end{cases}$$

Oscillator sum

NS
$$q^{L_0 - 1/2} \sim \frac{1}{\sqrt{q}} \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})^8}{(1-q^n)^8}$$

$$(-1)^F q^{L_0 - 1/2} \sim \frac{1}{\sqrt{q}} \prod_{n=1}^{\infty} \frac{(1-q^{n-\frac{1}{2}})^8}{(1-q^n)^8}$$

($\operatorname{Tr} q^{r b_{-r}\cdot b_r} = 1 + q^r$ for a particular fermionic oscillator)

R
$$q^{L_0} \sim 16 \prod_{n=1}^{\infty} \frac{(1+q^n)^8}{(1-q^n)^8}$$

space-time fermion (−); 16: degeneracy of the physical Ramond ground states

$$(-1)^F q^{L_0} = 0 \otimes \prod_{n=1}^{\infty} \frac{(1-q^n)^8}{(1-q^n)^8} = 0$$

half of the Ramond ground states have eigenvalue +1 and the other half −1

Thus without the momentum integration, we have

$$Z_{NS-R}(q) = \frac{1}{2\sqrt{q}} \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})^8}{(1-q^n)^8} + \frac{1}{2\sqrt{q}} \prod_{n=1}^{\infty} \frac{(1-q^{n-\frac{1}{2}})^8}{(1-q^n)^8} - 8 \prod_{n=1}^{\infty} \frac{(1+q^n)^8}{(1-q^n)^8}$$

$= 0$ by aequatio identica satis abstrusa (Jacobi)

For the closed string

$$Z = \operatorname{Tr} \left(\frac{1+(-1)^{F_L}}{2}\right)\left(\frac{1+(-1)^{F_R}}{2}\right) q^{L_0 - a}\, \bar{q}^{\bar{L}_0 - a}$$

$$\sim Z_{NS-R}(q)\, Z_{NS-R}(\bar{q}) = |Z_{NS-R}(q)|^2 = 0$$

→ evidence of the spacetime supersymmetry