Lectures on $N=2$ Supersymmetric Gauge Theories

① Duality in Maxwell's (U(1)) theory.
  i) Statement of duality
  ii) Dirac Monopole and quantization of electric charge

② Georgi-Glashaw model
  i) 't Hooft-Polyakov monopole
  ii) BPS Solution
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③ $N=2$ Supersymmetry algebra & representations
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⑤ $N=2$ Pure gauge theory
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⑥ Monodromies and invariant particle spectrum
  i) Analysis near singular points
  ii) Dual gauge theory near singular points
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⑦ U(1) gauge theory coupled to monopole multiplet.
  i) Explicit breaking of $N=2$ SUSY to $N=1$
  ii) Monopole condensation and Confinement

① Duality in Maxwell's Theory

Conventions: 3+1 dim. Minkowski space with $(+\,-\,-\,-)$ signature. $c=\hbar=1$

- Maxwell's equations in free space

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 0$$

- These equations are invariant under electric-magnetic duality $\vec{E}\to\vec{B}$ and $\vec{B}\to-\vec{E}$

- In covariant notation, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\qquad F_{0i} = E^i$, $\quad F_{ij} = -\epsilon_{ijk}B^k$

  Define $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

- Equations of motion $\quad \partial_\mu F^{\mu\nu} = 0 \;;\; \partial_\mu \tilde{F}^{\mu\nu} = 0$

- Duality transformation is $\quad F_{\mu\nu}\to\tilde{F}_{\mu\nu}$ and $\tilde{F}_{\mu\nu}\to -F_{\mu\nu}$.

- This symmetry is broken in presence of electric charges.

$$\partial_\mu F^{\mu\nu} = -J^\nu \;,\quad \partial_\mu\tilde{F}^{\mu\nu} = 0$$

  where J^ν is electric 4-current.

- However, we can restore this symmetry if we introduce magnetic charges as well.

- Presence of both electric and magnetic charges give

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad\text{and}\quad \partial_\mu\tilde{F}^{\mu\nu} = -K^\nu$$

  where K^ν is magnetic 4-current.

- These equations are invariant under

$$F_{\mu\nu}\to\tilde{F}_{\mu\nu}\,,\;\tilde{F}_{\mu\nu}\to -F_{\mu\nu}\;;\; J^\mu\to K^\mu,\; K^\mu\to -J^\mu$$

In fact, they are invariant under larger symmetry $SO(2)$ group which rotates

- We need to introduce a notion of magnetic (monopole) charge.
- Since electric and magnetic charges are on equal footing in a duality symmetric theory we can readily write down magnetic field produced by a magnetic monopole of strength 'g'.

$$\vec{B} = g\frac{\vec{r}}{4\pi r^3}$$

- In this magnetic field, an electrically charged particle of mass 'm' and charge 'e' experiences a force

$$m\ddot{\vec{r}} = e\dot{\vec{r}}\times\vec{B} = eg\frac{\dot{\vec{r}}\times\vec{r}}{4\pi r^3}$$

- The change in the orbital angular momentum of the charged particle is

$$\frac{d}{dt}(m\vec{r}\times\dot{\vec{r}}) = m\vec{r}\times\ddot{\vec{r}} = eg\frac{\vec{r}\times(\dot{\vec{r}}\times\vec{r})}{4\pi r^3} = \frac{d}{dt}\left(\frac{eg}{4\pi}\frac{\vec{r}}{r}\right)$$

- Thus the conserved angular momentum is given by

$$\vec{J} = m\vec{r}\times\dot{\vec{r}} - \frac{eg}{4\pi}\frac{\vec{r}}{r}$$

- If we assume that orbital angular momentum is quantized then

$$\frac{eg}{4\pi} = \frac{1}{2}n \Rightarrow eg = 2\pi n \qquad n\in\mathbb{Z}$$

Thus for a fixed magnetic charge 'g', electric charges are quantized. This is the Dirac charge quantization condition.

- Suppose we have particles carrying both electric and magnetic charge then the quantization condition becomes

$$e_1g_2 - e_2g_1 = 2\pi n \qquad n\in\mathbb{Z}$$

This is called the Dirac-Schwinger-Zwanziger condition.

- This condition is invariant under $SO(2)$ $[(e+ig)\to e^{i\phi}(e+ig)]$ transformation.

② Georgi - Glashaw model

- So far we have studied pointlike magnetic charges. In the Georgi - Glashaw model we will see that a magnetic monopole exists as a solution to the classical equations of motion.

- Georgi - Glashaw model is a Yang - Mills - Higgs system with SO(3) gauge group.

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}(D_\mu\phi)^a (D^\mu\phi)^a - V(\phi) \qquad a = 1,2,3$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\,\epsilon^{abc} A^b_\mu A^c_\nu$$

$$D_\mu \phi^a = \partial_\mu \phi^a - e\,\epsilon^{abc} A^b_\mu \phi^c$$

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2 \;; \qquad \phi^2 = \phi^a\phi^a$$

- The energy density is given by

$$T_{00} = \frac{1}{2}\left[(B^{ai})^2 + (E^{ai})^2 + (D^i\phi)^a(D^i\phi)^a + (D^0\phi)^a(D^0\phi)^a\right] + V(\phi)$$

$$E^{ai} = -F^{a0i} \;, \qquad B^{ai} = -\frac{1}{2}\epsilon^{ijk} F^a_{jk}$$

- $T_{00} \geq 0$, $\quad T_{00} = 0$ if $F^a_{\mu\nu} = 0$, $(D_\mu\phi)^a = 0$ and $V(\phi) = 0$

  $\Rightarrow$ Ground state of the system has SO(3) broken to U(1)

  $A^a_\mu = 0$, $\partial_\mu \phi^a = 0 \Rightarrow \phi^a$ constant and $\phi^a\phi^a = a^2$ $\boxed{\text{choose, } \phi^3 \neq 0}$

  This is the Higgs vacuum. A^3_μ is massless in this vacuum.

  $W_{\mu\pm} = \frac{1}{\sqrt{2}}(A^1_\mu \mp i A^2_\mu)$ are massive $(M_\pm = ae)$ and are charged with respect to A^3_μ (charge $\pm e$).

  ϕ^3 is a neutral scalar with mass $M_\phi = \sqrt{2\lambda}\,a$

- Static Solution

i.e. $E^{ai} = 0 \; , \; D_0 \phi^a = 0$

Mass or energy of the static solution is

$$M = \int d^3x \left[\frac{1}{2}(B^{ai})^2 + \frac{1}{2}((D^i\phi)^a)^2 + V(\phi) \right]$$

$$= \int d^3x \left[\frac{1}{2} \left[B^{ai} \pm (D^i\phi)^a \right]^2 \mp B^{ai}(D^i\phi)^a + V(\phi) \right]$$

- Since potential $V(\phi)$ is positive semidefinite

$$M \geqslant \mp \int d^3x \; B^{ai}(D^i\phi)^a = \mp \int_V d^3x \; \partial_i (B^{ai}\phi^a)$$ $\boxed{\because D_i B^{ai} = 0 \text{ due to Bianchi identity}}$

$$= \mp \int_{\underset{\text{Sphere at infinity}}{\partial V = S}} d^2s \; (B^{ai}\phi^a) = \mp a \underset{\text{magnetic charge}}{g}$$

- $M \geqslant \mp a g$ — equality holds only when $V(\phi) = 0$ and $B^{ai} = \mp (D^i\phi)^a$. <u>BPS limit</u>

Bogomol'nyi-Prasad-Sommerfield

- Magnetic charge 'g' is a topological charge

$$g = \frac{1}{a}\int d^3x \; B^{ai}(D^i\phi)^a = \frac{1}{a}\int d^3x \; \partial_i (B^{ai}\phi^a)$$

$$= -\frac{1}{2a}\int d^3x \; \epsilon^{ijk}\partial_i(\phi^a F^a_{jk})$$

- Integrand is zeroeth component of the current

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\partial_\nu(\phi^a F^a_{\rho\sigma})$$

which is conserved by construction $\partial_\mu K^\mu = 0$

- Thus K^μ is topological current and 'g' is topological charge.

- BPS solution is obtained by taking $\lambda \to 0$ in such a way that $\langle \phi^a\phi^a \rangle = a^2$ and solving

$$B^{ai} = (D^i\phi)^a \quad . \quad M = ag$$

- Ansatz

$$\phi^a = \frac{r^a}{er^2} H(aer) \ , \quad A_i^a = -\epsilon_{aij} \frac{r^j}{er^2}(1 - K(aer)) \ , \quad A_0^a = 0$$

$$aer = \xi$$

- Relevant expressions are

$$eB_i^a = -\left[\frac{\delta_{ai} - \hat{r}_a \hat{r}_i}{r} K' + \frac{\hat{r}_a \hat{r}_i}{r^2}(K^2 - 1) \right]$$

prime is derivative with respect to r.

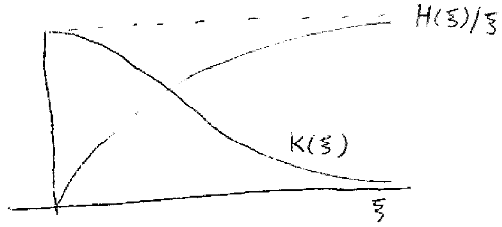
$$e(D_i\phi)^a = \left[\frac{\delta_{ai} - \hat{r}_a \hat{r}_i}{r^2} HK + \frac{\hat{r}_a \hat{r}_i}{r^2}(rH' - H) \right]$$

- $B^{ai} = (D^i\phi)^a$ gives

$$\xi \frac{dK}{d\xi} = -HK \ , \qquad \xi \frac{dH}{d\xi} - H = 1 - K^2$$

- Solution is

$$H_0(\xi) = \xi \coth \xi - 1 \ ,$$
$$K_0(\xi) = \xi / \sinh \xi$$



- For large r, $\quad B^a_i \longrightarrow \dfrac{\hat{r}^a \hat{r}_i}{er^2} \ , \quad \phi^a \longrightarrow a\hat{r}^a$

$$\Rightarrow \quad M_{mon} = 4\pi r^2 \frac{a}{er^2} = \frac{4\pi a}{e} \qquad \text{but} \quad M_{mon} = ag$$

$\Rightarrow \ g = \dfrac{4\pi}{e} \ \Rightarrow \ n = 2$ in the Dirac quantization condition.

- θ - parameter

If we allow CP violation in Georgi-Glashow model then we can add one more term to this action.

$$\mathcal{L}_\theta = \frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}$$

where $\tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$

- In absence of θ-term, the electric charge operator is

$$Q = \frac{1}{a}\int d^3x \; E^{ai}\cdot(D_i\phi)^a$$

- θ-term, being a total derivative, does not affect equations of motion. It, however, does modify electric charge operator to

$$eN = \frac{1}{a}\int d^3x \; E^{ai}\cdot(D_i\phi)^a - \frac{\theta e^2}{8\pi^2 a}\int d^3x \; B^{ai}\cdot(D_i\phi)^a$$

$$= Q - \frac{\cancel{8}\theta e^2}{8\pi^2} M$$

- Q and M are electric and magnetic charge operators with eigenvalues q and g respectively.

- Since N is quantized in integer units, electric charge in presence of θ-term becomes

$$g = \frac{4\pi}{e} m$$

$$q = ne + \frac{\theta e^2}{8\pi^2} g \qquad \Rightarrow \quad q = ne + \frac{\theta e}{2\pi} m$$

(Witten Effect)

- If we represent charges as points on the complex plane, then an arbitrary charge state can be written as

$$q + ig = e(n + m\tau) \qquad \text{where} \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

- e being the coupling constant, $\mathrm{Im}\,\tau$ should always be positive.

- n and m are integers $\Rightarrow$ All charged states lie on a two-dimensional lattice with lattice parameter τ.

- A General Statement of Duality:

Quantum Theory A at coupling e

$\equiv$ Quantum Theory B at coupling $1/e$

- Electrically charged particles of theory A get mapped on to magnetically charged particles of theory B

Consequences of Duality:

① Classical - quantum duality:

Strong-coupling results of theory A can be obtained by doing Semiclassical calculations in theory B.

$e^+e^- \to$ something in theory A at strong coupling [loop calculations + Nonperturbative effects]	=	$M^+M^- \to$ something in theory B at weak coupling [Classical approximation valid]

② Elementary - Composite duality

Elementary $\downarrow$ electrons, quarks

Composite $\downarrow$ Bound states: Proton, H-atom

Composite $\searrow$ Solitons: monopoles, Vortices

In duality symmetric theories distinction between elementary and composite disappears.

- Since duality is a statement for full quantum theory, S-matrices of two theories must be equal and as a result

  mass spectrum of theory A at coupling e
  = mass spectrum of theory B at coupling 1/e

- While duality is difficult to prove, it is possible to test duality.

- Supersymmetry comes handy for this purpose.

- BPS states give easiest check on duality conjecture.

③ N=2 SUSY algebra and representations

i) N=2 SUSY algebra without central charges $[Q^1_\alpha, Q^2_\alpha; \bar{Q}^1_{\dot\alpha}, \bar{Q}^2_{\dot\alpha}]$

(We will use 2-component notation for SUSY charges)

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^{ij} \qquad \sigma^\mu = (\mathbb{1}, \sigma^i)$$

$$\{Q^i_\alpha, Q^j_\beta\} = 0 \quad , \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0 \qquad i,j = 1,2$$

• <u>Representations</u>

• Massive states : choose rest frame, $P_\mu = (M, 0, 0, 0)$

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2M \delta_{\alpha\dot\beta} \delta^{ij} , \quad \{Q^i_\alpha, Q^j_\beta\} = 0 \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0$$

• Define $a^i_\alpha = \dfrac{Q^i_\alpha}{\sqrt{2M}}$, $a^{\dagger i}_\alpha = \dfrac{\bar{Q}^i_{\dot\alpha}}{\sqrt{2M}}$

Generators of fermionic harmonic oscillator

$$\{a^i_\alpha, a^{\dagger j}_\beta\} = \delta^{ij}\delta_{\alpha\beta} , \quad \{a^i_\alpha, a^j_\beta\} = 0 \quad \{a^{\dagger i}_\alpha, a^{\dagger j}_\beta\} = 0$$

• Notice we have 4 creation and 4 annihilation operators.
($\alpha = 1,2$ and $i = 1,2$)

• Consider a state $|\Omega\rangle$ such that

$$a^i_\alpha |\Omega\rangle = 0 \quad \forall i \text{ and } \alpha$$

• We can now write down the SUSY multiplet by using creation operators.

$\alpha, \beta, \cdots = 1,2$
$i, j, \cdots = 1,2$

$\lvert\Omega\rangle$	1	$a^{\dagger i}_\alpha \lvert\Omega\rangle$	4
$a^{\dagger i}_\alpha a^{\dagger j}_\beta \lvert\Omega\rangle$	6	$a^{\dagger i}_\alpha a^{\dagger j}_\beta a^{\dagger k}_\gamma \lvert\Omega\rangle$	4
$a^{\dagger i}_\alpha a^{\dagger j}_\beta a^{\dagger k}_\gamma a^{\dagger l}_\delta \lvert\Omega\rangle$	1	Fermionic (Bosonic)	
Bosonic (Fermionic)			

This is a 16 component multiplet with 8 bosonic and 8 fermionic degrees of freedom (This is called the long multiplet.)

- Massless states : choose a frame such that $P_\mu = E(1,0,0,1)$

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2(\sigma^0+\sigma^3)_{\alpha\dot\beta} E\,\delta^{ij} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{(\alpha\dot\beta)} \delta^{ij}$$

$\Rightarrow \{Q^i_1, \bar{Q}^j_{\dot 1}\} = 4E\,\delta^{ij}$ and rest all anticommutators vanish.

- Define $a^i = \dfrac{Q^i_1}{\sqrt{4E}}$, $a^{\dagger i} = \dfrac{\bar{Q}^i_{\dot 1}}{\sqrt{4E}}$

$$\{a^i, a^{\dagger j}\} = \delta^{ij} \quad , \quad \{a^i, a^j\} = 0 \qquad i,j = 1,2$$

- Choose $|\Omega\rangle$ such that $a^i|\Omega\rangle = 0 \;\; \forall i$ then the multiplet is given by

$|\Omega\rangle$ 1 $\qquad a^{\dagger i}|\Omega\rangle$ 2

$a^{\dagger i}a^{\dagger j}|\Omega\rangle$ 1

2 bosonic and 2 fermionic degrees of freedom
4 component multiplet (short multiplet)

ii) $N=2$ SUSY algebra with central charges

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta^{ij}$$
$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta} Z^{ij} \qquad Z^{ij} = -Z^{ji}$$
$$\{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 2\epsilon_{\dot\alpha\dot\beta} \bar{Z}^{ij}$$

$\boxed{\text{$Z^{ij}$ commutes with all generators}}$

$\Rightarrow N=2$ theory has only one central charge $Z = Z^{12}$

$\boxed{\text{Write $Z = |Z|e^{i\alpha}$ \& $\bar{Z} = |Z|e^{-i\alpha}$. Redefine Q and $\bar{Q}$ to soak up this phase. In the end, $\{Q,Q\}$ and $\{\bar{Q},\bar{Q}\}$ depends on $|Z|$ only.}}$

- Massive states : $P_\mu = (M,0,0,0)$

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2M\,\delta^{ij}\delta_{\alpha\dot\beta}$$

- Define new operators

$$a_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_{\alpha\beta}\bar{Q}^{2\beta}\right) \qquad a^\dagger_\alpha = \frac{1}{\sqrt{2}}\left(\bar{Q}^1_\alpha + \epsilon_{\alpha\beta} Q^{2\beta}\right)$$

$$b_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_{\alpha\beta}\bar{Q}^{2\beta}\right) \qquad b^\dagger_\alpha = \frac{1}{\sqrt{2}}\left(\bar{Q}^1_\alpha - \epsilon_{\alpha\beta} Q^{2\beta}\right)$$

- Anticommutation relations

$$\{a_\alpha, a_\beta^\dagger\} = 2(M+|Z|)\delta_{\alpha\beta}, \quad \{b_\alpha, b_\beta^\dagger\} = 2(M-|Z|)\delta_{\alpha\beta}$$

All other anticommutators vanish

- It is clear from the anti-commutation relations that right hand side should be positive semidefinite, i.e.,

$$M \geqslant |Z| \qquad \text{This is known as BPS bound.}$$

- Take $M = |Z|$ then $\{b_\alpha, b_\beta^\dagger\} = 0$ and

$$\{a_\alpha, a_\beta^\dagger\} = 4M\delta_{\alpha\beta}$$

Define $\hat{a}_\alpha = \frac{1}{\sqrt{4M}} a_\alpha \quad \Rightarrow \quad \{\hat{a}_\alpha, \hat{a}_\beta^\dagger\} = \delta_{\alpha\beta}$

- Choose $|\Omega\rangle$ such that $\hat{a}_\alpha|\Omega\rangle = 0$

State	multiplicity	J_z (spin)
$\lvert\Omega\rangle$	1	j
$\hat{a}_\alpha^\dagger\lvert\Omega\rangle$	2	$j \pm \frac{1}{2}$
$\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger\lvert\Omega\rangle$	1	j

- Thus we see that a multiplet satisfying BPS condition is 4 component (short multiplet)

- Those which do not saturate BPS bound involve $b^\dagger$ operators and therefore are 16 component (long multiplets).

- Notice, in absence of central charges massless representations were also 4 component.

- It is clear from BPS bound that massless representations do not carry any central charge.

- Masses of BPS multiplet are protected even after taking quantum effects into account. If quantum effects take $M=|Z|$ to $M>|Z|$ then corresponding multiplet should be 16 component.

  However, quantum mechanically we cannot create new degrees of freedom. Therefore, BPS multiplet should remain 4 component and this can happen iff $M=|Z|$.

- This is not to say that BPS masses do not have quantum corrections. In fact, they do get corrected but all those corrections occur through corrections to coupling ~~and~~ in such a way that they preserve $M=|Z|$ condition.

- It is instructive to check that every $N=2$ BPS state in 4 dimensions can be obtained from a massless representation of $N=1$ SUSY algebra in 6 dimensions.

- $N=2$ massless representation is, in fact, the lowest member of this class.

- Thus if we had not discovered Kaluza-Klein mechanism before Supersymmetry, we would have discovered it in the context of $N=2,4$ SUSY in 4 dimensions.

- While BPS multiplet is 4 component, $N=2$ supermultiplet contains 8 components. In other words it contains two BPS multiplets. This is because $N=2$ superfield should be invariant under CPT conjugation.

- To illustrate this point let us consider the state $|\Omega\rangle$

with j_z eigenvalue $1/2$.

The BPS multiplet we get has spin assignments

$|\Omega\rangle$ $\quad 1/2$

$\hat{a}^\dagger_\alpha |\Omega\rangle$ $\quad 1, 0$

$\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta |\Omega\rangle$ $\quad 1/2$

It does not contain states with spin projections $-1/2$ and -1. These projections exist in the CPT conjugate BPS multiplet.

The CPT conjugate multiplet is

$|\Omega\rangle$ $\quad -1/2$

$\hat{a}^\dagger_\alpha |\Omega\rangle$ $\quad -1, 0$

$\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta |\Omega\rangle$ $\quad -1/2$

We combine these two to write down ~~what~~ $N=2$ vector multiplet.

- $N=2$ Vector multiplet in component form contains a gauge field A^a_μ, a complex scalar ϕ^a and two Weyl fermions λ^a_1, λ^a_2, all in adjoint rep. of gauge group G.

- In $N=1$ superspace notation it is a combination of $N=1$ vector multiplet W^a_α and $N=1$ chiral multiplet ~~to~~ Φ^a in adjoint representation of gauge group G.

- $N=2$ hypermultiplet in component form contains two complex scalar fields in representation 'R' and two Weyl fermions in the same representation 'R' of the group G. $(q^i_j, \tilde{q}^{i\dagger}_j ; \chi^i_\alpha, \tilde{\chi}^i_\alpha)$

- In $N=1$ superspace notation, a hypermultiplet consists of two $N=1$ ~~chiral~~ multiplets $\underset{\text{(chiral)}}{Q^i}$ and $\underset{\text{(anti-chiral)}}{\tilde{Q}^{i\dagger}}$, both in representation R of the group G.

④ $N=2$ Supersymmetric Lagrangians

i) $N=2$ SUSY Gauge Theory

As stated earlier $N=2$ vector multiplet contains $N=1$ chiral multiplet and $N=1$ vector multiplet.

• In $N=1$ superspace formulation, Lagrangian for $N=1$ chiral multiplet coupled to $N=1$ vector multiplet is given by

$$\mathcal{L} = \frac{1}{8\pi} \operatorname{Im}\left(\tau \int d^2\theta \, W^{\alpha a} W^a_\alpha\right) + \int d^2\theta \, d^2\bar{\theta} \, \operatorname{Tr}\left(\Phi^\dagger e^{-2V} \Phi\right)$$

$$+ \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \overline{W}(\Phi^\dagger) \qquad \boxed{\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}}$$

• Since all the fields in $N=2$ vector multiplet belong to same group representation and have same mass, superpotential terms are inconsistent with the gauge invariance of the multiplet.

• Thus the Lagrangian which is invariant under $N=2$ SUSY is

$$\mathcal{L}_{N=2} = \frac{1}{8\pi} \operatorname{Im}\left[\tau\left\{\left(\operatorname{Tr}\int d^2\theta \, W^\alpha W_\alpha\right) + 2\int d^2\theta \, d^2\bar{\theta} \, \operatorname{Tr} \Phi^\dagger e^{-2V} \Phi\right\}\right]$$

where, we have rescaled $\Phi \to \Phi/e$ to conform to $N=2$ SUSY.

• Absence of superpotential makes F-term trivial. The D-term contribution to the bosonic potential is

$$\frac{1}{e^2} \operatorname{Tr}\left(\frac{1}{2} D^a D + D[\phi^\dagger, \phi]\right)$$

• Integrating out D gives

$$V = -\frac{1}{2e^2} \operatorname{Tr}\left([\phi^\dagger, \phi]^2\right)$$

- $N=2$ superspace formulation

Define $N=2$ chiral superfield Ψ with constraint

$\bar{D}_{\dot{\alpha}} \Psi = 0, \quad \bar{\tilde{D}}_{\dot{\alpha}} \Psi = 0$

- We can expand Ψ in terms of $N=1$ superfields as

$$\Psi = \Phi(\tilde{y},\theta) + \sqrt{2}\,\tilde{\theta}^{\alpha} W_{\alpha}(\tilde{y},\theta) + \tilde{\theta}^{\alpha}\tilde{\theta}_{\alpha}\left(\Phi^{\dagger}(y',\theta,\bar{\theta})\, e^{2eV(y',\theta,\bar{\theta})}\right)_{\bar{\theta}\bar{\theta}}$$

where $\tilde{y}^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} + i\tilde{\theta}\sigma^{\mu}\bar{\tilde{\theta}}$ and $y'^{\mu} = x^{\mu} + i\tilde{\theta}\sigma^{\mu}\bar{\tilde{\theta}}$.

- In terms of Ψ, $N=2$ gauge theory Lagrangian is

$$\mathcal{L}_{N=2} = \frac{1}{4\pi}\, \mathrm{Im}\, \mathrm{Tr} \int d^2\theta\, d^2\tilde{\theta}\, \frac{1}{2}\tau \Psi^2$$

- In fact, the most general $N=2$ Lagrangian for the gauge fields involves an arbitrary holomorphic function $\mathcal{F}(\Psi)$.

$$\mathcal{L} = \frac{1}{4\pi}\, \mathrm{Im}\, \mathrm{Tr} \int d^2\theta\, d^2\tilde{\theta}\, \mathcal{F}(\Psi)$$

$$= \frac{1}{8\pi}\, \mathrm{Im}\left(\int d^2\theta\, \mathcal{F}_{ab}(\Phi)\, W^{a\alpha} W^{b}_{\alpha} + 2\int d^2\theta\, d^2\bar{\theta}\, (\Phi^{\dagger} e^{2eV})^{a} \mathcal{F}_{a}(\Phi)\right)$$

where, $\mathcal{F}_a(\Phi) = \partial \mathcal{F}/\partial \Phi^a$ and $\mathcal{F}_{ab}(\Phi) = \partial^2 \mathcal{F}/\partial\Phi^a \partial\Phi^b$

- The function $\mathcal{F}$ is referred to as $N=2$ prepotential.
- Renormalizable action $\Rightarrow$ $\mathcal{F} = \frac{1}{2}\tau\Psi^2$
- Low-energy effective action $\Rightarrow$ complicated form for $\mathcal{F}$ but consistent with symmetries.
- Exact determination of $\mathcal{F}$ is the work of Seiberg and Witten

- We can also write down Lagrangian for coupling of hypermultiplet to vector multiplet.

$$\mathcal{L}_H = \int d^4\theta \left(Q_i^\dagger e^{-2V} Q_i + \tilde{Q}_i e^{2V} \tilde{Q}_i^\dagger \right) + \int d^2\theta \left(\sqrt{2}\, \tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i \right)$$
$$+ \int d^2\bar\theta \left(\sqrt{2}\, Q_i^\dagger \Phi^\dagger \tilde{Q}_i^\dagger + m_i^* Q_i^\dagger \tilde{Q}_i^\dagger \right)$$

where, Φ, Q_i and $\tilde{Q}_i$ are $N=1$ chiral superfields and V is $N=1$ vector superfield.

- Let us go back to $N=2$ SUSY gauge theory without matter fields.

⑤ $N=2$ SUSY Gauge Theory

- In what follows we will analyse $N=2$ $SU(2)$ Super Yang-Mills theory in detail.

- Before we do that let us quickly analyse ~~the~~ structure of vacuum for a general gauge group G.

- Potential for the scalar field in the classical action is

$$V = \frac{1}{2e^2} \operatorname{Tr}\left([\phi^\dagger \phi]^2 \right)$$

- The vacuum is, therefore, defined by $[\phi^\dagger, \phi] = 0$.

  $\Rightarrow$ ϕ takes values in the Cartan subalgebra of the gauge group G. $\boxed{\Phi = \phi_i H^i}$ and the gauge group G is broken to H.

- Different values of ϕ within the Cartan subalgebra correspond to different physical theories.

- Coset elements G/H act as gauge transformations and hence do not take ϕ out of the Cartan subalgebra
- Some of the elements of the coset G/H generate Weyl reflections. It is therefore best to parametrize vacua, i.e., moduli space in terms of Weyl invariant functions rather than ϕ.

- Since Weyl reflections act on ϕ by conjugation we can obtain Weyl invariants from the characteristic equation

$$\det(\lambda - \phi) = 0$$

- Let $a_1, \cdots a_M$ be the roots of the characteristic equation then for, say, $SU(M)$ gauge group, we have $Tr(\phi) = \sum a_i = 0$. For generic values of a_i, $SU(M)$ is broken to $U(1)^{M-1}$.

- In terms of a_i, the characteristic eqn is

$$\lambda^M + \lambda^{M-2} \sum_{i<j} a_i a_j + \cdots\cdots + (-1)^M \prod_{i=1}^{M} a_i = 0$$

- For $SU(2)$, $\phi = \frac{1}{2} a \sigma_3$ and Weyl invariant is

$$u = Tr(\phi^2) = \frac{1}{2} a^2$$

- For $SU(3)$, the Weyl invariants are

$$u = \frac{1}{2} Tr(\phi^2) = -(a_1 a_2 + a_1 a_3 + a_2 a_3)$$

$$v = -\frac{1}{3} Tr(\phi^3) = a_1 a_2 a_3$$

$\boxed{\text{Use relation } Tr(\phi) = 0.}$

- At low energies $SU(M)$ is broken to $U(1)^{M-1}$ and effective Lagrangian for $(M-1)$ massless vector multiplets takes the form

- $\mathcal{L} = \frac{1}{4\pi} \mathrm{Im}\left[\int d^4\theta \, \frac{\partial \mathcal{F}}{\partial A^i} \bar{A}^i + \int d^2\theta \, \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial A_i \partial A_j} W^{\alpha i} W_\alpha^j \right]$

The Kähler potential is given by $K = \mathrm{Im}\left(\bar{A}_i \frac{\partial \mathcal{F}}{\partial A_i}\right)$, where A_i are neutral chiral superfields whose scalar component is a_i.

- The metric on the moduli space is given by

$$ds^2 = g_{i\bar{j}} \, da^i d\bar{a}^{\bar{j}} = \mathrm{Im} \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da^i d\bar{a}^{\bar{j}}$$

- Let us now restrict ourselves to $SU(2)$ gauge group.

$V(\phi) = 0 \Rightarrow \phi = \frac{1}{2} a \sigma_3 \Rightarrow SU(2)$ is broken to $U(1)$.

- This theory has $U(2)$ R-symmetry which rotates two supercharges into each other. Action of this R-symmetry on the components of vector superfield is as follows.

- Under $SU(2)_R$ symmetry, the gauge field A_μ^a and the scalar field ϕ^a is invariant and two Weyl fermions transform as a doublet of $SU(2)_R$.

- Under $U(1)_R$: $A_\mu^a \to A_\mu^a$, $\lambda_i^a \to e^{i\alpha} \lambda_i^a$, $\phi^a \to e^{2i\alpha} \phi^a$

- All these symmetries are not independent. $U(1)_R$ transformation with $\alpha = \pi$ is same as the Z_2, center of $SU(2)$, generated by $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$.

- Classically the Lagrangian ~~has~~ is conformally invariant and has $U(2)$ R-symmetry. This can be nicely incorporated into superconformal symmetry $SU(2,2|2)$.

- Scale invariance is broken in the quantum theory which is manifested by non-zero β-function.

- This is related by supersymmetry to $U(1)_R$, which also becomes anomalous in the quantum theory.

- At a generic point in the moduli space, the massless fields are A_μ^3, λ_i^3, ϕ^3. They can be written in terms of $N=1$ multiplet $W_\alpha \equiv (A_\mu^3, \lambda_1^3)$ and $A \equiv (\underset{a}{\phi^3}, \lambda_2^3)$.

- $N=2$ SUSY Lagrangian for light fields is

$$\mathcal{L} = \frac{1}{4\pi}\left[\int d^2\theta\, d^2\bar\theta\; K(A,\bar A) + \int d^2\theta\; \mathrm{Im}\left(\tfrac{1}{2}\tau(A) W^\alpha W_\alpha\right)\right]$$

where $K(A,\bar A) = \mathrm{Im}\left(\bar A \frac{\partial \mathcal{F}}{\partial A}\right)$ and $\tau(A) = \frac{\partial^2 \mathcal{F}}{\partial A \partial A} = \frac{\theta_{eff}}{2\pi} + i\frac{4\pi}{e_{eff}^2}$

- Full effective prepotential $\mathcal{F}$ can be written as,

$$\mathcal{F} = \underbrace{\mathcal{F}_0}_{\text{Tree level}} + \underbrace{\mathcal{F}_1}_{\text{one-loop}} + \underbrace{\mathcal{F}_2}_{\text{two-loop}} + \cdots + \text{instanton effects.}$$

$$\mathcal{F}_0 = \mathcal{F}_{cl} = \frac{1}{2}\tau_{cl}\, a^2 \qquad \tau_{cl} = \frac{\theta_{cl}}{2\pi} + \frac{4\pi i}{e_{cl}^2}.$$

- 1-loop effective action: To derive 1-loop effective action let us first understand meaning of θ_{eff}.

- $U(1)_R$ acts as a chiral symmetry on the Dirac fermion

$$\Psi_D = \begin{pmatrix}\lambda_1 \\ \bar\lambda_2\end{pmatrix}. \qquad U(1)_R : \Psi_D \to e^{i\alpha\gamma_5}\Psi_D$$

This chiral symmetry becomes anomalous in the quantum theory.

- For $SU(2)$ gauge theory, the anomalous conservation law is

$$\partial_\mu J_R^\mu = -\frac{1}{4\pi^2} F_{\mu\nu}\tilde F^{\mu\nu}$$

- Therefore, under $U(1)_R$ transformation, the effective Lagrangian changes by

$$\delta \mathcal{L}_{eff} = -\frac{\alpha}{4\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$\boxed{\begin{array}{l} \frac{1}{32\pi^2}\int F\tilde{F} = \text{integer} \\ \Rightarrow \alpha = \frac{\pi}{4} n \quad n \in \mathbb{Z} \\ U(1)_R \text{ breaks to } \mathbb{Z}_8 \end{array}}$

whereas it transforms the gauge kinetic term as

$$\frac{1}{16\pi} \mathrm{Im}\left[\frac{\partial^2 \mathcal{F}}{\partial A^2}(e^{2i\alpha}A)(-F^2 + iF\tilde{F})\right]$$

~~Recall Z_4 is common both SU(2) and U(1)~~ ~~So unbroken sym. is only Z_4~~ ~~$\alpha = (\pi/4)\, n, n \in \mathbb{Z}$~~

- i.e.,

$$\frac{1}{16\pi} \mathrm{Im}\left[\frac{\partial^2 \mathcal{F}}{\partial A^2}(e^{2i\alpha}A)(-F^2 + iF\tilde{F})\right] = \frac{1}{16\pi}\mathrm{Im}\left[\frac{\partial^2 \mathcal{F}}{\partial A^2}(-F^2 + iF\tilde{F})\right] - \frac{\alpha}{4\pi^2} F\tilde{F}$$

Identifying coefficients of $F\tilde{F}$ we get

$$\frac{\partial^2 \mathcal{F}}{\partial A^2}(e^{2i\alpha}A) = \frac{\partial^2 \mathcal{F}}{\partial A^2} - \frac{4\alpha}{\cancel{4}\pi^{\cancel{2}}} F\tilde{F}$$

- For infinitesimal α, we get

$$\frac{\partial^3 \mathcal{F}}{\partial A^3} = +\frac{2i}{\pi}\frac{1}{A}$$

Integrating this gives us one-loop prepotential

$$\mathcal{F}_{1\text{-loop}}(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}$$

$\Lambda \to$ dynamically generated scale.

- Alternatively, notice that the transformation

$$a \to e^{2i\alpha} a\,, \quad \lambda_i \to e^{i\alpha}\lambda_i\,, \quad A_\mu \to A_\mu \quad \text{and} \quad \theta \to \theta - 8\alpha$$

leaves the action invariant

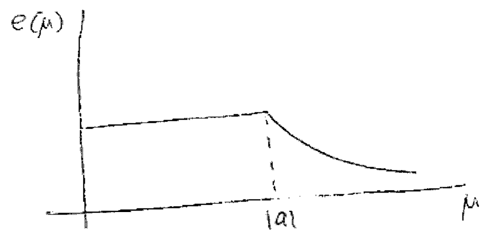
$$\Rightarrow \theta_{eff} = -4 \arg(a) + \theta_0$$

θ_0 can be absorbed in the definition of a.

- Since conformal current is anomalous, the coupling constant e is a function of the scale μ.

- Being an asymptotically free gauge theory β-function for $SU(2)$ gauge theory is negative.

However, $SU(2)$ breaks to $U(1)$ at a scale $|a|$ therefore below the scale $|a|$ we have non-interacting ~~gau~~ $U(1)$ gauge theory whose coupling doesnot run.



As $|a|$ reduces e_{eff} increases.
One-loop β-$f^{\underline{n}}$ for $\mathcal{N}=2$ $SU(2)$ gauge theory is given by

$$\mu \frac{de}{d\mu} = \beta(e) = -\frac{1}{4\pi^2} e^3$$

Integrating this relation gives $\quad \frac{1}{e_{eff}^2} = \frac{1}{2\pi^2} \ln \frac{|a|}{\Lambda}$

- Since $\tau(A)$ is a holomorphic function of A, results for θ_{eff} and e_{eff} should conspire to ensure holomorphicity of $\tau(A)$.

$$\tau(A) = \frac{\theta_{eff}}{2\pi} + i\,\frac{4\pi}{e_{eff}^2} = -\frac{4}{2\pi}\arg(a) + i\,\frac{4\pi}{2\pi^2}\ln\frac{|a|}{\Lambda}$$

$$= \frac{2i}{\pi}\ln\frac{a}{\Lambda} \quad \text{where} \quad a = |a|\, e^{i\arg(a)}$$

- Since $\tau(A) = \frac{\partial^2 \mathcal{F}}{\partial A^2} \quad \Rightarrow \quad \mathcal{F}_{\text{1-loop}}(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}$

- There are no higher loop corrections to $\mathcal{F}$.
  Higher loop terms are down by $(e_{eff}^2)^n$

$$(e_{eff}^2)^n \sim \frac{1}{\left(\ln \frac{|a|}{\Lambda}\right)^n}$$

Using holomorphicity we can write

$$(e_{eff}^2)^n \sim \frac{1}{\left(\ln \frac{a}{\Lambda}\right)^n}$$

But this gives new contributions to $U(1)$ anomaly which contradicts Adler-Bardeen theorem, which says that one loop relation bet$^{\underline{n}}$ θ_{eff} & $\arg(a)$ is exact.

- Instanton terms are given by the factor

$$e^{-\frac{8\pi^2 k}{e_{eff}^2}} \quad \text{where } k \text{ is instanton number.}$$

$$e^{-\frac{8\pi^2}{e_{eff}^2}k} = e^{-4\ln|a|} \quad \text{Using holomorphicity we write}$$

$$e^{-4\ln a} = \frac{1}{a^4}$$

- Remember θ is defined modulo 2π. Since $\theta_{eff} = -4\arg(a)$

$\Rightarrow$ any physical quantity must be invariant under $a \to a e^{i\pi/2}$

- Higher loop corrections are not invariant under this transformation but instanton terms are.

- The prepotential with non-perturbative corrections is given by

$$\mathcal{F} = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \frac{\Lambda^{4k}}{A^{4k-2}}$$

$\mathcal{F}_k$ are field independent coefficients.

- Recall Kähler potential is given by $K(A,\bar{A}) = \mathrm{Im}\left(\frac{\partial \mathcal{F}}{\partial A}\bar{A}\right)$

and the Kähler metric is $g_{a\bar{a}} = \partial_a \partial_{\bar{a}} K$

- Under coordinate change $K(a,\bar{a}) \to K(a,\bar{a}) + F(a) + \bar{F}(\bar{a})$, the metric is invariant.

- Consider a holomorphic function

$$a \to a' = f(a)$$

Question: Is it possible to write $\mathcal{F}'(a')$ such that

$$K'(a',\bar{a}') = \frac{1}{2i}\left(\frac{\partial \mathcal{F}'}{\partial a'}\bar{a}' - a'\frac{\partial \bar{\mathcal{F}}'}{\partial \bar{a}'}\right) = K(a,\bar{a}) + F(a) + \bar{F}(\bar{a})?$$

Answer: Not possible in general. It is possible only in case of special Kähler manifolds (special coordinates)

- Define $a_D = \frac{\partial \mathcal{F}}{\partial a}$

Transformations which preserve special coordinates are

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D' \\ a' \end{pmatrix} = \underbrace{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}_{SL(2,\mathbb{R})} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad p,q,r,s \in R \ \ \& \ ps - qr = 1$$

- Usually we choose $a = \langle \phi^3 \rangle$ but in general $\langle \phi^3 \rangle$ need not be a special coordinate. We will therefore reserve a to denote special coordinate and use $u = \frac{1}{2}\langle \phi^3 \rangle^2$ as a gauge invariant variable.

- For large u, $a = \sqrt{2u}$ a good approximation in the perturbative limit.

- In the strong coupling limit a as well as a_D are complicated functions of u.

$$a \equiv a(u) \quad \text{and} \quad a_D \equiv \frac{\partial \mathcal{F}}{\partial a} \equiv a_D(u)$$

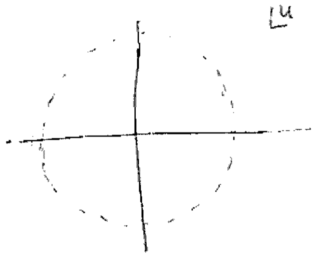
- Our task is to find $a(u)$ and $a_D(u)$ such that for large $|u|$, they reduce to $a \sim \sqrt{2u}$ and

$$a_D \simeq \frac{\partial \mathcal{F}}{\partial a} = \frac{i}{\pi}\sqrt{2u}\,\ln\frac{2u}{\Lambda^2} + \frac{i}{\pi}\sqrt{2u}$$

$$\boxed{\begin{array}{l}\text{BPS bound} \\ m^2 \geqslant (Q_{e_1}^2 + Q_m^2)\frac{|a|^2}{e_{eff}^2} \\ = |n_e a + n_m a_D|^2\end{array}}$$

⑥ Monodromies and invariant particle spectrum

Let us draw a large circle in the u-plane.



Traversing along this circle takes $u \to e^{2\pi i} u$. This takes

$$a(u) \to -a(u) \quad \text{and}$$

$$a_D(u) \to -a_D + 2a$$

Thus

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \underbrace{\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}_{M_\infty} \begin{pmatrix} a_D \\ a \end{pmatrix} \to \text{Monodromy at } \infty.$$

- A puzzle:

$$g_{a\bar{a}} = \partial_a \partial_{\bar{a}} K = \mathrm{Im}\left(\frac{\partial^2 \mathcal{F}}{\partial a^2}\right) = \mathrm{Im}(\tau(a))$$

+ve definiteness of metric implies +ve definiteness of $\mathrm{Im}(\tau(a))$.

$\tau(a)$ is a holomorphic function $\Rightarrow$ $\partial_a \partial_{\bar{a}} \tau(a) = 0$ Laplace's eqn.

Which means $\partial_a \partial_{\bar{a}} \mathrm{Im}(\tau(a)) = 0$.

- $\mathrm{Im}(\tau(a))$ cannot have local maximum or minimum in u-plane.

- For large $|a|$,

$$\mathrm{Im}(\tau(a)) = \frac{2}{\pi} \ln \frac{|a|}{\Lambda} = \frac{2}{\pi} \ln \frac{|\sqrt{2u}|}{\Lambda}$$

As $|u|$ reduces $\mathrm{Im}(\tau(a))$ reduces. +ve definiteness of $\mathrm{Im}(\tau(a))$ implies it reaches minimum at some value of $|u|$.
This contradicts the earlier result.

- Only way out of this is to assume that $\mathrm{Im}(\tau(a))$ is not a single valued function.

- Now on we will consider $a(u)$ and $a_D(u)$ as multivalued functions of u. $\Rightarrow$ branch points in u-plane.

- If we have a meromorphic function with +ve imaginary part we can always associate a torus to it.

- Let us find out how many singularities we have in the u-plane.

  We have anomalous $U(1)_R$ symmetry $\phi^a \to e^{2i\alpha}\phi^a$, $\theta \to \theta - 8\alpha$

- Take $\alpha = \pi/4$, $\phi^a \to e^{i\pi/2}\phi^a = i\phi^a$ and $\theta \to \theta - 2\pi$

  Since this transformation belongs to anomaly free subgroup of $U(1)_R$, it is a symmetry. Thus $\phi^a \to i\phi^a$ must be a symmetry $\Rightarrow$ $u \sim \langle \phi^a \phi^a \rangle$ $u \to -u$ is a symmetry.

- The singularity structure should also respect this symmetry.

  $\Rightarrow$ Singularity at u_0 must be accompanied by one at $-u_0$.

- Let us assume that there is only one singularity at $u=0$

Then monodromy at $u=0$ is same as that at $u=\infty$

$$M_0 = M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}$$

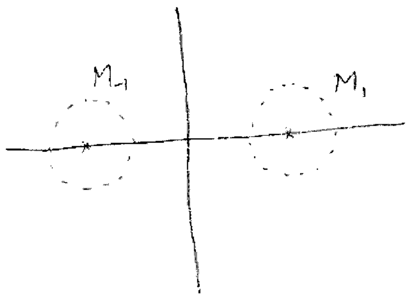
$$\tau(a) = \frac{\partial a_D}{\partial a} \Rightarrow \tau(a) \to \tau(a)+2 \quad \Rightarrow \mathrm{Im}(\tau(a)) = \mathrm{Im}(\tau(a))$$

- In this case, $\mathrm{Im}(\tau(a))$ is single valued everywhere which contradicts our earlier result.

- Therefore, one singularity hypothesis is ruled out.

- Let us assume there are two singularities at $u=\pm u_0$.

  By appropriately choosing Λ and rotation of coordinates we set $u_0=1$.



If M_1 is the monodromy matrix at $u=1$ and M_{-1} is at $u=-1$ then they are related to monodromy at infinity M_∞ by

$$M_1 M_{-1} = M_\infty$$

- What is the physics of a singularity?

In the Wilsonian approach, we have been integrating out all heavy modes to derive low energy effective action.

As we move in the moduli space our assumption about heavy modes may break down. That is, at some points in the moduli space some state(s) are massless, but we have integrated out these states at a generic point in the moduli space.

- Singularity is a signal ~~of~~ a massless state. ~~It~~ It can be removed by reincorporating that light state in the neighbourhood of the singularity.

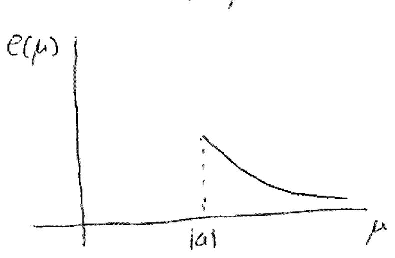
- What kind of particles become massless?

Let us suppose that at $u=1$ an electrically charged particle $(n_e \neq 0, n_m = 0)$ becomes massless. i.e. as $u \to 1$, $a \to 0$.

$$m^2 \geq |n_e a + n_m a_D|^2$$

From the mass formula it follows that as $a \to 0$ this state becomes massless.

In the neighbourhood of $u=1$ we can approximate $a \sim (u-1)^{\alpha}$, $\alpha > 0$. Let us look at the RG equation



$$\mu \frac{de(\mu)}{d\mu} = \beta_0 e^3 + \cdots$$

$$\frac{1}{e_{eff}^2} = -2\beta_0 \ln \frac{|a|}{\Lambda}$$

We see that $Im(\tau(a)) = \frac{1}{e_{eff}^2}$ is always a function of $|a|$ and hence is single valued.

- Thus massless electrically charged particles do not satisfy our requirement of multivalued $Im(\tau(a))$.

Reason: We know that $Im(\tau(a))$ is single valued at infinity. If it is single valued at $u=1$, due to massless electrically charged state then from the relation between monodromy matrices it is clear that $u=-1$ monodromy will also keep $Im(\tau(a))$ single valued.

So we conclude that both at $u=1$ and $u=-1$ we have a particle becoming massless with $n_m \neq 0$.

- Suppose that a state with charge $(n_e, n_m = 1)$ exists for large $|u|$ and suppose that this state becomes massless as $u \to 1$.

- What happens if we take this state around along a loop at infinity?
- The BPS mass formula $M^2 = |n_e a + n_m a_D|^2$ is invariant under $SL(2,\mathbb{Z})$ transformations

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} p & -q \\ -r & s \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

where $ps - qr = 1 \quad p,q,r,s \in \mathbb{Z}$

- Monodromy matrix at infinity is $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$
- It's action on $(n_e, n_m = 1)$ gives us $(n_e - 2, n_m = 1)$ state
- Before taking around the loop the state $(n_e, n_m = 1)$ was massless at $u = 1$. After taking it around the loop we find $(n_e - 2, n_m = 1)$ state becomes massless.
- There is no invariant meaning to the electric charges of these states. Only distinguishing factor is n_e = even vs. n_e = odd.

<u>Dual Gauge Theory at $u=1$</u>

- Let us choose a convention a state with n_e = even becomes massless at $u = 1$ and n_e = odd becomes massless at $u = -1$.
- At $u = 1$, monopole with quantum numbers $(n_e = 0, n_m = 1)$ becomes massless. Since monopole becomes massless, it implies low energy theory is in the strong coupling regime. It is easier to change variables so that the monopole becomes an elementary particle. In the process we transform photon to dual (magnetic) photon, which has local coupling to the monopole.

- For us this transformation amounts to exchanging a and a_D. This can be done by the transformation which also inverts coupling constant.

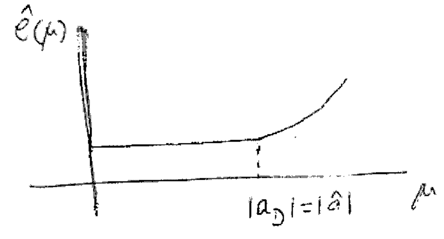
$$\begin{pmatrix} \hat a_D \\ \hat a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -a \\ a_D \end{pmatrix}$$

The charges transform as

$$\begin{pmatrix} \hat n_e \\ \hat n_m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} n_m \\ -n_e \end{pmatrix}$$

$$\begin{pmatrix} n_e = 0 \\ n_m = 1 \end{pmatrix} \to \begin{pmatrix} \hat n_e = 1 \\ \hat n_m = 0 \end{pmatrix} \quad \text{and } \hat\tau(\hat a) = \frac{\partial \hat a_D}{\partial \hat a} = -\frac{1}{\frac{\partial a_D}{\partial a}} = -\frac{1}{\tau(a)}$$

- $\hat\tau$ is the coupling of the magnetic theory.
- Near $u=1$, effective low energy theory is weakly coupled in dual variables where we have a light electrically charge hypermultiplet coupled to photon.
- This theory is infrared free, so in the region near $u=1$ the β-fn behaves as follows.



The coupling constant reduces as μ reduces. Since mass of the monopole is proportional to $|a_D|$, it acts as a cutoff. There are no charged states below this scale, and hence coupling does not run below $|a_D|$.

- The β-fn eqn is

$$\mu \frac{d\hat e}{d\mu} = \beta(\hat e) = \beta_0 \hat e^3 + \cdots$$

$$\Rightarrow \frac{1}{\hat e_{eff}^2} = -2\beta_0 \ln\mu \quad \Rightarrow \operatorname{Im}(\hat\tau(\hat a)) = \frac{4\pi}{e_{eff}^2(\hat a)} = -8\pi\beta_0 \ln\left|\frac{\hat a}{\Lambda}\right|$$

- In the limit $|\hat{a}| \to 0$, $Im(\hat{\tau}(\hat{a}))$ is very large, i.e., we are in the weak ~~s~~ coupling limit. (Recall, the original theory in this limit is strongly coupled.)

- Using holomorphicity we write
$$\hat{\tau}(\hat{a}) = -i 8\pi \beta_0 \ln \frac{\hat{a}}{\Lambda} = -\frac{i}{\pi} \ln \frac{\hat{a}}{\Lambda} \quad \text{since } \beta_0 = \frac{1}{8\pi^2}$$
for $U(1)$-theory.

- Due to infrared free nature of this theory these results are more and more trustworthy as $|a_D| = |\hat{a}| \to 0$

- Now let us look at the coupling of the original theory
$$\tau(a) = -\frac{1}{\hat{\tau}(\hat{a})} = -i\pi \frac{1}{\ln \frac{a_D}{\Lambda}}$$

Since $\tau(a) = \frac{da_D}{da} \Rightarrow a = \frac{i}{\pi}\left(a_D \ln \frac{a_D}{\Lambda} - a_D\right) + \text{const.}$

- As $a_D \to 0$, $u \to 1$. Let us assume a_D has a simple zero at $u = 1$, i.e., $a_D = k_1(u-1)$

Then as $(u-1) \to e^{2\pi i}(u-1)$ $\quad a_D \to a_D$
$$a \to a - 2a_D$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}}_{M_1} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad \text{where} \quad a_D = k_1(u-1)$$
$$a = \frac{i}{\pi}\left(k_1(u-1) \ln \frac{k_1(u-1)}{\Lambda} - k_1(u-1)\right)$$

$$\tau(a) \to \frac{\tau(a)}{-2\tau(a)+1}$$

- Let us look at the theory near $u = -1$

This can be done by finding $SL(2,\mathbb{Z})$ transformation which takes $(n_e = 0, n_m = 1)$ state to $(n_e = +1, n_m = 1)$ state.

- The matrix which does that is $\begin{pmatrix} +1 & +1 \\ 0 & +1 \end{pmatrix}$

- It transforms $\begin{pmatrix} a_D \\ a \end{pmatrix}$ to

$$a = \frac{i}{\pi}\left((a_D - a) \ln \frac{a_D - a}{\Lambda} - (a_D - a) \right)$$

$$a_D - a = K_{-1}\,(u+1)$$

- Monodromy in this case is $(u+1) \to e^{2\pi i}(u+1)$

$$a_D - a \longrightarrow a_D - a$$

$$a \longrightarrow a - 2(a_D - a) = 3a - 2a_D$$

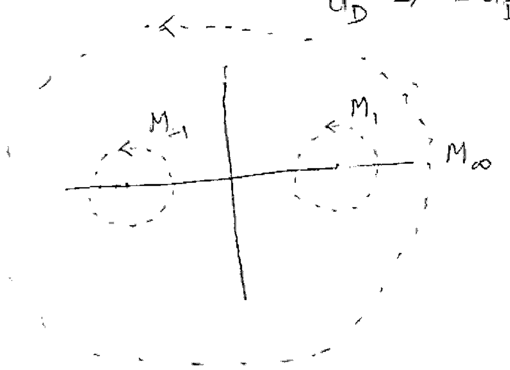
$$\Rightarrow \begin{pmatrix} a_D \\ a \end{pmatrix} \to \underbrace{\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}}_{M_{-1}} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

- To understand this matrix treat $a_D - a = \hat{a}_D$ invariant under monodromy and define inverse relation as

$$a_D = \hat{a}_D + \hat{a}$$

Using monodromy of $\hat{a}_D$ and $\hat{a}$ it is clear that

$$a_D \to -a_D + 2a.$$



Recall

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

It is easy to check $M_1 M_{-1} = M_\infty$

- What if we take product of monodromies in the opposite order?

$$M_{-1} M_1 \neq M_\infty$$

- Is it possible to fine $SL(2,\mathbb{Z})$ transformation which relates M_∞ to M_1 and M_{-1} as

$$M_{-1}M_1 = U M_\infty U^{-1} \quad ?$$

- In fact, there is such a transformation

$$M_{-1}M_1 = M_{-1}M_1 M_{-1}(M_{-1})^{-1} = M_{-1}M_\infty (M_{-1})^{-1}$$

$$\Rightarrow \quad U = M_{-1}$$

- Let us summarise

i) For large $|u|$, $a = \sqrt{2u}$, $a_D = \frac{i}{\pi}\sqrt{2u}\ln\frac{2u}{\Lambda^2} + \frac{i}{\pi}\sqrt{2u}$

ii) For $u = 1$, $a_D = k_1(u-1)$, $a = \frac{i}{\pi}\left(a_D \ln\frac{a_D}{\Lambda_0} - a_D\right) + c_1$

iii) For $u = -1$, $a_D - a = k_{-1}(u+1)$, $a = \frac{i}{\pi}\left[(a_D - a)\ln\frac{a_D - a}{\Lambda_0} - (a_D - a)\right] + c_1$

iv) $\mathrm{Im}(\tau(a)) = \mathrm{Im}\left(\frac{\partial a_D/\partial u}{\partial a/\partial u}\right)$ must be positive.

- These results are obtained by writing

$$a = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}}$$

$$a_D = \frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}}$$

- To understand this let us start with the condition $\mathrm{Im}(\tau) \geq 0$. This condition can be implemented by associating a torus for each value of u.

- $[\mathrm{Im}(\tau(a))]^{-1}$ becomes singular at $u = \pm 1$.

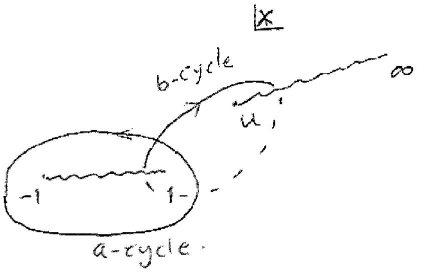
- The modular parameter of the associated torus ~~becomes~~ should become real at $u = \pm 1$

- At $u = \infty$, $\mathrm{Im}(\tau(a))$ becomes singular $\Rightarrow$ the modular parameter of the associated torus becomes effectively purely imaginary, and divergent.

- We denote this torus by an algebraic equation

$$y^2 = (x-1)(x+1)(x-u)$$

- In complex x-plane we have two branch cuts.



This is a representation of a torus as a branched cover of sphere.

The algebraic curve is invariant under $x \to -x$, $u \to -u$ & $y \to iy$.

Monodromy around any single point, $x = u$, $x = 1$, $x = -1$ or $x = \infty$ takes $y \to -y$.

- Let ω be a holomorphic differential form.
Using behaviour of $\mathrm{Im}(\tau)$ we define a-cycle as a loop encircling the branch cut between $x = -1$ and $x = 1$ and b-cycle as a loop encircling $x = 1$ and $x = u$, and write

$$\tau = \frac{\oint_b \omega}{\oint_a \omega}$$

- Basis for closed 1-forms is given by

$$\omega = \frac{dx}{y} \quad \text{and} \quad \tilde{\omega} = \frac{x\,dx}{y}$$

- 'Integrals' of these two 1-forms along a-cycle define two period integrals of the torus.

- In our case, $y^2 = (x+1)(x-1)(x-4)$

$$\omega = \frac{dx}{\sqrt{(x+1)(x-1)(x-4)}} \quad \text{and} \quad \tilde{\omega} = \frac{x\,dx}{\sqrt{(x+1)(x-1)(x-4)}}$$

- Recall $\tau(a) = \dfrac{da_D/du}{da/du}$

- Let us define an appropriate linear combination of ω and $\tilde{\omega}$ for expressing a and a_D

$$a(u) = \oint_a \Omega \quad \text{and} \quad a_D(u) = \oint_b \Omega$$

where $\Omega = p(u)\,\omega_1 + q(u)\,\cancel{\omega}_2$

- Comparing with asymptotic behaviour of a ~~at~~ and a_D we can determine $p(u)$ and $q(u)$

$$\Omega = \frac{\sqrt{2}}{\pi}\left(\cancel{dx}\,\tilde{\omega} - u\omega\right) = \frac{\sqrt{2}}{\pi}\,\frac{\sqrt{x-u}\,dx}{\sqrt{(x+1\cancel{x})(x-1)}}$$

- Since on the algebraic curve a loop encircling $x=1$ and $x=-1$ is an a-cycle and the one encircling $x=1$ and $x=u$ is b-cycle we find

$$a(u) = \oint_a \Omega = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \frac{\sqrt{x-u}\,dx}{\sqrt{(x+1)(x-1)}}$$

$$a_D(u) = \oint_b \Omega = \frac{\sqrt{2}}{\pi}\int_{1}^{u} \frac{\sqrt{x-u}\,dx}{\sqrt{(x+1)(x-1)}}$$

⑦ U(1) gauge theory coupled to monopole multiplet

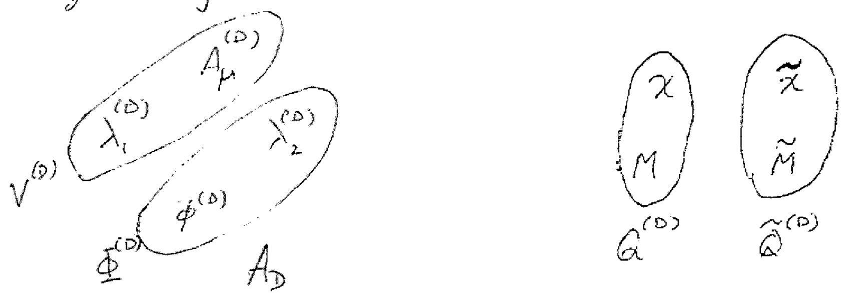
We will now consider Wilsonian effective action near $u=1$ where a monopole hypermultiplet becomes massless.

- The low energy theory contains a vector multiplet A and a hypermultiplet $(M, \tilde{M})$.

- Since $(M, \tilde{M})$ have monopole charge, A_D has a local coupling to them and hence it is better to work in terms of dual variables A_D.

$$A_D = \frac{\partial \mathcal{F}}{\partial A}$$

- In $N=1$ SUSY notation, the multiplets A_D and M are given by



- Recall, Lagrangian for the hypermultiplet contains superpotential. In case of massless hypermultiplet this superpotential takes the form

$$W = \Phi^{(D)} Q^{(D)} \tilde{Q}^{(D)}$$

- The scalar potential is

$$V = \left| \frac{\partial W}{\partial \phi^{(D)}} \right|^2 + \left| \frac{\partial W}{\partial M} \right|^2 + \left| \frac{\partial W}{\partial \tilde{M}} \right|^2$$

- Minimisation of V implies

$$\frac{\partial W}{\partial \phi^{(D)}} = \frac{\partial W}{\partial M} = \frac{\partial W}{\partial \tilde{M}} = 0$$

$$\Rightarrow \quad M\tilde{M} = 0\,, \quad \phi^{(D)}\tilde{M} = 0 \quad \text{and} \quad \phi^{(D)}M = 0$$

- Apart from these F-terms we also have a D-term constraint,

$$(M^\dagger M - \tilde{M}\tilde{M}^\dagger) = 0$$

$$\Rightarrow \quad |M| = |\tilde{M}|$$

- Solution to ~~F-terms~~ F-terms is

$$M = \tilde{M} = 0 \quad \text{and} \quad \phi^{(D)} \text{ arbitrary}$$

- Putting this back in the potential gives

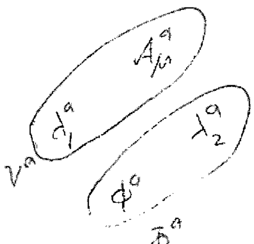
$$\left|\frac{\partial W}{\partial M}\right|^2 = \phi^{(D)^2} M^2$$

Thus mass of the monopole is $m_{mon} \sim \phi^{(D)} \equiv a_D$

i) Explicit breaking of $N=2$ to $N=1$

Let us consider explicit breaking of $N=2$ SUSY to $N=1$ SUSY by adding a small mass term to Φ in the original theory.

- $SU(2)$ vector multiplet is,



The new action is

$$S_{N=2} + \underbrace{\tfrac{1}{2} m \int d^2\theta \; \Phi^a \Phi^a + h.c.}_{\text{This term breaks } N=2 \text{ to } N=1}$$

- Since we are using $u = \frac{1}{2}\langle \phi^a \phi^a \rangle$ we will write $\Phi^a\Phi^a = U$ where, $U = u + \theta\sigma_\epsilon + \theta\theta F$ which was a part of $N=2$ vector multiplet.

In terms of this field the action is

$$S_{N=2} + m \int d^2\theta \; U + h.c.$$

- Total superpotential in the effective theory is

$$W = \Phi^{(D)} Q^{(D)} \tilde{Q}^{(D)} + mU$$

Notice, the scalar component of U is u whereas that of $\Phi^{(D)}$ is a_D. We know that a_D is a function of u. Therefore, we can writee

$$W = \Phi^{(D)}(U)\, Q^{(D)} \tilde{Q}^{(D)} + mU$$

ii) Monopole condensation and confinement

Recall near $u = 1$ we have $a_D \sim k_1 (u-1)$

$$\Rightarrow \quad \Phi^{(D)}(U) = k_1 (U-1)$$

With this the superpotential becomes

$$W = k_1 (U-1)\, Q^{(D)} \tilde{Q}^{(D)} + mU$$

- Minimising Bosonic potential gives

$$\frac{\partial W}{\partial u} = 0 \quad \Rightarrow \quad k_1 \tilde{M} M + m = 0$$

$$\frac{\partial W}{\partial M} = 0 \quad \Rightarrow \quad k_1 (u-1) \tilde{M} = 0$$

$$\frac{\partial W}{\partial \tilde{M}} = 0 \quad \Rightarrow \quad k_1 (u-1) M = 0$$

and the D-term constraint is $|M| = |\tilde{M}|$

- Solution to these set of equations is

$$\cancel{u=1}, \quad \cancel{M = \sqrt{k_1 m}\, e^{i\alpha}}, \quad \cancel{\tilde{M} = \sqrt{k_1 m}\, e^{-i\alpha}}$$

$$u = 1, \quad M = \sqrt{\frac{m}{k_1}}\, e^{i\alpha}, \quad \tilde{M} = -\sqrt{\frac{m}{k_1}}\, e^{-i\alpha}$$

- This means u is pinned at 1, entire moduli space has disappeared, and monopoles have vev $|M| = \sqrt{\frac{m}{k_1}}$

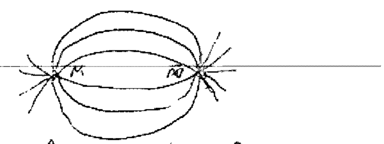
- Vev for monopole field implies dual photon is massive and $U(1)$ gauge symmetry is completely broken.

• Since monopoles have condensed, in dual picture, i.e. the electric picture, we have confinement. That is we have linearly rising potential between charges.

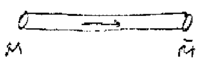
• To understand this let us consider the example of superconductivity.

In superconductivity we have U(1) symmetry ~~spont~~ broken by the Higgs mechanism.

In the unbroken symmetry phase, lines of force from, say, a monopole to an anti monopole are spread around.



In the broken symmetry phase, electric charges condense to form the Cooper pairs. In this phase, we have Vortex solution carrying magnetic flux.



All the magnetic field flux is confined to the vortex, which has constant energy per unit length. Thus

String (vortex) tension = energy/unit length

produces linear confining potential for monopoles

• In our case, monopoles are condensing therefore we find electric charges in the confining phase. String tension in this case is proportional to m.