SERC Lectures on $N = 1$ Supersymmetry

(With application to supersymmetric gauge theories)

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These notes discuss some aspects of non-abelian gauge theories with $N = 1$, *i.e.* minimal, supersymmetry. The following is an almost verbatim account of a set of lectures I gave in the *XV SERC School on Theoretical High Energy Physics* held at the Saha Institute of Nuclear Physics, Calcutta, during January–February, 2000. Each section was covered in one lecture, with the appendix and problems taken up in tutorial sessions.

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1 Introduction & the supersymmetry algebra

At least at a non-technical level, all of us have heard that supersymmetry (called *susy* for short), is a symmetry that mixes bosonic and fermionic degrees of freedom in a dynamical system. We will make this notion more precise during the course of these lectures; however before getting into that let us spend a few minutes to recall the motivation for the exercise we are going to undertake.

Of course the most obvious reason is that there have been some progress in understanding the dynamics of supersymmetric gauge theories. Any symmetry gives us a useful handle in analysing the behaviour of a physical system, and supersymmetry is no exception. Indeed it turns out that supersymmetry imposes powerful constraints in the way physical system can behave and consequently makes such system accessible beyond the domain of perturbation theory.

A more physical motivation — although admittedly biased by the presently held paradigm in high energy physics — is that the interactions of elementary particles well below the Planck scale is described by a local quantum field theory that is approximately supersymmetric. Also it is a non-abelian gauge theory. This fact has been established for the electroweak sector of the standard model for a long time; while a more recent evidence for the role of gauge theory in strong interactions is the announcement of evidence in favour of quark-gluon plasma. Special behaviour of supersymmetric gauge theories could hopefully therefore be directly reflected in the physics of elementary particles.

Finally, by the end of the decade, we shall know whether supersymmetry operates in nature, at least in the most expected way.

We shall not say anything more about applications of supersymmetric gauge theories in elementary particle physics. Some aspects will be discussed in the other lectures of this school.

A word about references. For the formalism of supersymmetry I have drawn heavily from the review article by Lykken[1] and the classic text by Wess and Bagger[2]. In the later parts dealing with applications and the more modern developments I have used the reviews by Peskin[3] and Seiberg[4]. All these contain much more than what I shall be able to present in nine lectures and could be used for further reading. Throughout these lectures we shall use Weyl spinors. It is also possible to formulate supersymmetry using Majorana spinors, as is done *e.g.* in Refs.[5, 6]. Lastly I have not attempted to be cite the original references to the literature in many cases. Some of the reviews in the bibliography may be consulted for this purpose.

Let us begin with supersymmetry.

The idea was first proposed by Gol'fand and Likhtman[7] in 1971. Somehow it did not gain popularity, nor was it widely known perhaps, until the 1974 paper of Wess and Zumino[8]. These authors constructed a field theory action that has a remarkable new kind of symmetry: it is invariant under infinitesimal variation of bosons (respectively fermions) that is proportional to fermions (bosons). Schematically

$$\begin{aligned} \delta_\xi \phi_B &\sim \xi \psi_F, \\ \delta_\xi \psi_F &\sim \xi \partial \phi_B, \end{aligned}$$

where, ξ is an infinitesimal fermionic parameter. That ξ is fermionic follows from matching spin and statistics of two sides of the above equations. Also we see from the first equation that ξ has mass dimension $-1/2$, which brings in the derivative of the bosonic field ϕ_B in the second. Thus already from dimensional analysis we see that such a symmetry must mix with spacetime symmetries — translation in the above. So supersymmetry is a 'spacetime symmetry' as opposed to internal symmetries that do not mix with spacetime transformations. Indeed Gol'fand and Likhtman begin by asking whether it is possible to extend the algebra of spacetime symmetries such that the Poincaré algebra (consisting of spacetime translations, rotations and boosts), is a proper subalgebra of the extended symmetry.

Before we go ahead and start writing equations, let us remind ourselves about transformation properties of spinors, *i.e.* how they are defined, since they will play a crucial role in our consideration. This will also help set up our notation and convention.

The world we live in has three space and one time directions. It is (locally) flat[1] and isotropic. Symmetries of this spacetime are

- Translations in space and time directions — infinitesimal translations generated by P_μ, $\mu = 0, 1, 2, 3$.
- Lorentz transformations (rotations and boosts) — infinitesimal Lorentz transformations generated by $M_{\mu\nu}$. ($M_{\mu\nu} = -M_{\nu\mu}$ are antisymmetric.)

They satisfy the following algebra (Poincaré algebra)

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\lambda] &= i\left(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu\right), \\ [M_{\mu\nu}, M_{\lambda\sigma}] &= i\left(\eta_{\nu\lambda}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\lambda} + \eta_{\mu\sigma}M_{\nu\lambda} - \eta_{\mu\lambda}M_{\nu\sigma}\right). \end{aligned} \tag{1.1}$$

Our convention for the metric is the standard one in particle physics: $||\eta_{\mu\nu}|| = \mathrm{diag}(+1, -1, -1, -1)$.

Quantum fields (and elementary particles described by the excitation of these fields) transform covariantly under Lorentz transformations. A trivial example is a scalar field ϕ

$$\phi(x) \to \phi'(x') = \phi(x)$$

More precisely, as

$$\begin{aligned} x^\mu &\to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \\ \phi(x) &\to \phi'(x) = \phi(\Lambda^{-1}x). \end{aligned}$$

In the above, Λ is a finite Lorentz transformation, and we have adopted the so called *active* point of view.

This is clearly the simplest possible behaviour of a field which has just one 'component'. For a multi-component field, there could be mixing between its various components. For example, for a vector V_μ

$$V^\mu(x) \to V'^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x).$$

This is the defining relation of a vector. In its infinitesimal form, $\Lambda^\mu{}_\nu = \delta^\mu_\nu - i\omega^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu$, where

$$(M_{\rho\sigma})^\mu{}_\nu = i\delta^\mu_\rho \eta_{\sigma\nu} - i\delta^\mu_\sigma \eta_{\rho\nu} \tag{1.2}$$

[1] The effect of gravity is negligible at energies far below the Planck scale — in the usual domain of elementary particle physics.

is the matrix representation of the Lorentz generators on vectors[2], and ω's are infinitesimal angles and velocities parametrising rotations and boosts. The commutation relations (1.1) are equivalent to the statements that P_μ is a vector operator, and that $M_{\mu\nu}$ defines a rank 2 anti-symmetric tensor.

In order to define a spinor, let us do the following.

Exercise: *Define the infinitesimal generators of*

$$\begin{aligned} \textit{rotations} \quad L_i &= \frac{1}{2}\epsilon_{ijk}M_{jk}, \quad \textit{and} \\ \textit{boosts} \quad K_i &= M_{0i}; \quad i,j = 1,2,3. \end{aligned} \tag{1.3}$$

Express the commutators between $M_{\mu\nu}$ in terms of the generators L_i's and K_i's. Show that the combinations

$$\mathbf{J}^{\pm} = \frac{1}{2}\left(\mathbf{L} \pm i\mathbf{K}\right) \tag{1.4}$$

commute with each other and separately satisfy the angular momentum algebra:

$$\begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk}J_k^+, \\ [J_i^-, J_j^-] &= i\epsilon_{ijk}J_k^-, \\ [J_i^+, J_j^-] &= 0. \end{aligned} \tag{1.5}$$

This exercise shows that the Lorentz algebra is (almost) a product of two independent angular momentum algebras. Transformation properties of fields which transform covariantly[3] under Lorentz algebra are determined by their behaviour under the two angular momentum algebras.

Recall that the transformation properties of fields/states under rotation in three dimensions, (that is representations of angular momentum), are labelled by *spin* j, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ can take any half-integer value. Lorentz transformation properties of fields are therefore labelled by a pair of half-integers (j_+, j_-), where $j_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ independently. For example, $(0,0)$ is the scalar representation. It corresponds to the choice $\mathbf{J}^{\pm} = \mathbf{0}$.

If we take $J_i^+ = \frac{1}{2}\sigma_i$, ($\sigma_i, i = 1,2,3$ are the three Pauli matrices); and $\mathbf{J}^- = \mathbf{0}$, the corresponding field transforms in the $(\frac{1}{2}, 0)$ representation. This two-component field is called a *left chirality spinor*. We shall label this as $\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Under a Lorentz transformation

$$\psi_\alpha \to {\psi'}_\alpha = \left(\delta_\alpha^\beta - i\omega^{\rho\sigma}(M_{\rho\sigma})_\alpha{}^\beta\right)\psi_\beta. \tag{1.6}$$

On the other hand, if we choose $\mathbf{J}^+ = \mathbf{0}$ and $J_i^- = \frac{1}{2}\sigma_i$, the field transforms as $(0, \frac{1}{2})$ representation. Again we have a two-component field called a *right chirality spinor*. We shall adopt a convention in which this field is denoted by $\bar{\chi}^{\dot\alpha} = \begin{pmatrix} \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix}$. Under a Lorentz transformation

$$\bar{\chi}^{\dot\alpha} \to \bar{\chi}'^{\dot\alpha} = \left(\delta_{\dot\beta}^{\dot\alpha} - i\omega^{\rho\sigma}(M_{\rho\sigma})^{\dot\alpha}{}_{\dot\beta}\right)\bar{\chi}^{\dot\beta}. \tag{1.7}$$

[2]On scalars $M_{\mu\nu} = 0$, *i.e.* $\Lambda = \mathbf{1}$.

[3]Fields that transform covariantly are said to be in a *representation* of the algebra.

The left and right chirality spinor representations are complex as is evident from the definition (1.4) of the generators.

One may now work out, with the help of the above exercise, that for a left chirality spinor ψ_α (respectively right chirality spinor $\bar{\chi}^{\dot{\alpha}}$), the Lorentz generators are given by the following matrices

$$(M_{\mu\nu})_\alpha{}^\beta \equiv (\sigma_{\mu\nu})_\alpha{}^\beta = \frac{i}{4}\left[(\sigma_\mu)_{\alpha\dot{\gamma}}(\bar{\sigma}_\nu)^{\dot{\gamma}\beta} - (\sigma_\nu)_{\alpha\dot{\gamma}}(\bar{\sigma}_\mu)^{\dot{\gamma}\beta}\right],$$
$$(M_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}\left[(\bar{\sigma}_\mu)^{\dot{\alpha}\gamma}(\sigma_\nu)_{\gamma\dot{\beta}} - (\bar{\sigma}_\nu)^{\dot{\alpha}\gamma}(\sigma_\mu)_{\gamma\dot{\beta}}\right]. \quad (1.8)$$

In the above we have introduced the notation

$$\sigma^\mu = \bar{\sigma}_\mu = (\mathbf{1},\ \vec{\sigma})$$
$$\bar{\sigma}^\mu = \sigma_\mu = (\mathbf{1}, -\vec{\sigma}),$$

where $\mathbf{1}$ is the 2×2 identity matrix.

The more familiar Dirac spinor is made up of one left and one right chirality spinor

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix}.$$

On a Dirac spinor the Lorentz generators take the form $M_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$, where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

The nomenclature left- and right-chirality spinors used above may now be related to the familiar notion of chirality.

Notice the index structure of the matrices σ_μ and $\bar{\sigma}_\mu$. The former has undotted-dotted indices while the latter has dotted-undotted ones.

Exercise: *Write the Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

in terms of the σ^μ and $\bar{\sigma}^\mu$ matrices.

Exercise: *Show that*

$$(\sigma_{\mu\nu})^\dagger = \bar{\sigma}_{\mu\nu}$$

and

$$\sigma_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\lambda\rho}\sigma_{\lambda\rho},$$
$$\bar{\sigma}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu}{}^{\lambda\rho}\bar{\sigma}_{\lambda\rho}, \quad (1.9)$$

where, $\epsilon_{0123} = +1$ in our convention. The eqns.(1.9) mean that the rank 2 antisymmetric tensor $\sigma_{\mu\nu}$ ($\bar{\sigma}_{\mu\nu}$ respectively) is (anti-)self-dual.

A Lorentz vector transforms in the $(\frac{1}{2},\frac{1}{2})$ representation. Therefore one can make a vector by combining left and right chirality spinors. In other words, a vector may be thought to have two spinor indices, one undotted (left type) and one dotted (right type). The transition to this description from the more familiar one is done with the help of the σ^μ matrices, (which may be thought of as Clebsch-Gordon coefficients):

$$V_\mu \rightarrow V_{\alpha\dot{\beta}} = V_\mu (\sigma^\mu)_{\alpha\dot{\beta}}.$$

Exercise: *Since a vector index can be traded with a pair of spinor indices, for a second rank tensor $T_{\mu\nu}$ we may define*

$$T_{\mu\nu} \rightarrow T_{\alpha\beta\dot{\alpha}\dot{\beta}} = T_{\mu\nu}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}.$$

Breaking the above in symmetric and antisymmetric pieces one has

$$\begin{aligned} T_{\alpha\beta\dot{\alpha}\dot{\beta}} &= T_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + T_{[\alpha\beta](\dot{\alpha}\dot{\beta})} + T_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} + T_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} \\ &\equiv \epsilon_{\dot{\alpha}\dot{\beta}} T_{(\alpha\beta)} + \epsilon_{\alpha\beta} T_{(\dot{\alpha}\dot{\beta})} + \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} T + T_{(\alpha\beta)(\dot{\alpha}\dot{\beta})}, \end{aligned}$$

where, $T_{(\dot{\alpha}\dot{\beta})} = -\,\frac{1}{2}\epsilon^{\alpha\beta}T_{\alpha\beta\dot{\alpha}\dot{\beta}}$, etc. Show that the first (respectively second) term on the RHS above correspond to antisymmetric (anti-)self-dual part of the the tensor $T_{\mu\nu}$, while last term is the traceless symmetric part and the third term is the trace.

Show that from the anti-symmetric Lorentz generators $M_{\mu\nu}$, we get two sets of tensors $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ in terms of which the commutation relations read as follows:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \frac{1}{2}\left(\epsilon_{\alpha\gamma}M_{\beta\delta} + \epsilon_{\alpha\delta}M_{\beta\gamma} + \epsilon_{\beta\gamma}M_{\alpha\delta} + \epsilon_{\beta\delta}M_{\alpha\gamma}\right),$$

and similarly for $[\bar{M}_{\dot{\alpha},\dot{\beta}}, \bar{M}_{\dot{\gamma},\dot{\delta}}]$ while $[M,\bar{M}] = 0$. Also show that

$$M_{\alpha\beta}\psi_\gamma = \frac{1}{2}\left(\epsilon_{\gamma\alpha}\psi_\beta + \epsilon_{\gamma\beta}\psi_\alpha\right),$$

and a similar relation for $\bar{M}\bar{\psi}$.

There is one last thing we need to do before we get back to supersymmetry. Recall that one can define tensors by taking products of vectors. These have multiple indices. Similarly, one can define 'spinor-tensors' that have multiple spinor indices and transform like products of spinors. Indeed, the above 'redefinition' of vector is such an example. Now consider the tensor $\epsilon_{\alpha\beta}$,

$$||\epsilon_{\alpha\beta}|| = -\,||\epsilon^{\alpha\beta}|| = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.10}$$

in our convention.

Exercise: *Show that $\epsilon_{\alpha\beta}$ is a (numerically) invariant tensor under Lorentz transformation.*

With the help of this ϵ-tensor, we can write the transpose of the spinor ψ_α

$$\left(\psi^T\right)_\alpha \equiv \psi^\alpha,$$

by 'raising the index' as:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad \Rightarrow \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta = -\psi^\beta\epsilon_{\beta\alpha}. \tag{1.11}$$

Notice that the combination $\psi^T\chi = \psi^\beta\chi_\beta = \epsilon^{\alpha\beta}\psi_\alpha\chi_\beta$ is Lorentz invariant. herefore $\epsilon^{\alpha\beta}$ behaves like the 'metric' for the left-chirality spinors.

Similarly one can define the 'metric' $\epsilon_{\dot\alpha\dot\beta}$ on the dotted spinors[4], and lower/raise indices by

$$\bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} \qquad \Rightarrow \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} = -\bar\psi_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}. \tag{1.13}$$

Notice that due to the antisymmetry of the spinors under exchange, one has to be careful in ordering them while contracting indices. In the convention we shall adopt, undotted indices are contracted from NW to SE, *i.e.* in the $\searrow$ direction,

$$\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha;$$

and dotted indices from SW to NE, *i.e.* in the $\nearrow$ direction

$$\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha}$$

respectively.

In defining supersymmetry, we augment the infinitesimal generators of the Poincaré algebra (P_μ and $M_{\mu\nu}$), by the fermionic generators

$$\begin{array}{ll} Q_\alpha, & \alpha = 1,2, \\ \bar Q^{\dot\alpha}, & \dot\alpha = 1,2; \end{array}$$

that is by a left-chirality spinor Q_α and its hermitian conjugate

$$(Q_\alpha)^\dagger = \bar Q_{\dot\alpha}.$$

These generators obey the following (anti-)commutation relations.

$$\begin{aligned} [Q_\alpha, P_\mu] = [\bar Q_{\dot\alpha}, P_\mu] &= 0, \\ [Q_\alpha, M_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \\ [\bar Q_{\dot\alpha}, M_{\mu\nu}] &= \frac{1}{2}\bar Q_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}, \\ \{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} &= 0, \\ \{Q_\alpha, \bar Q_{\dot\beta}\} &= 2\sigma^\mu{}_{\alpha\dot\beta}P_\mu. \end{aligned} \tag{1.14}$$

[4] In our convention,

$$||\epsilon_{\dot\alpha\dot\beta}|| = -\,||\epsilon^{\dot\alpha\dot\beta}|| = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.12}$$

.

In the above, the second and third line simply define left- and right-chirality spinors. So the last three equations are really the new relations. The last of these states that the effect of two successive supersymmetry transformations is the same as that of a spacetime translation. Recall that we had a glimpse of this fact earlier from our dimensional consideration.

Notice that, in conformity with the spin-statistics theorem, the supersymmetry generators anticommute.

There are a few things that a standard course on supersymmetry would have discussed in more detail. We shall just gloss over them.

- The first of this is the fact there is not much option in extending the Poincaré algebra. In 1967 Coleman and Mandula proved a theorem that may roughly be stated as follows[9]:

  > If the infinitesimal generators of the symmetries of a quantum field theory are bosonic, *i.e.* they obey commutation relations, then under some reasonable physical assumption, the corresponding symmetry algebra is a product of the Poincaré algebra and an internal symmetry algebra.

  Haag, Łopuszanskı and Sohnius[10] showed that if in addition, anticommuting spinor generators are allowed, (*extended)* supersymmetry is the only possible generalisation of the Poincaré algebra.

- In extended supersymmetry there are N sets of spinor generators Q^A_α, $\bar{Q}_{A\dot{\alpha}}$, $A = 1, 2, \cdots, N$; which satisfy the following modified relations

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} &= 2\delta^A_B \sigma^\mu_{\alpha\dot{\beta}} P_\mu \\ \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \\ \{\bar{Q}_{A\dot{\alpha}}, \bar{Q}_{B\dot{\beta}}\} &= -\,\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{AB}. \end{aligned} \tag{1.15}$$

  The generators Z^{AB} commute with all the other generators of the extended supersymmetry algebra.

- In addition, the spinors Q^A transform in the N dimensional *i.e.* defining representation of the group of N dimensional unitary matrices $\mathrm{U}(N)$ ($\mathrm{SU}(4)$ for $N = 4$). For our case $N = 1$, there is a $\mathrm{U}(1)$ symmetry. The generators Q_α and $\bar{Q}_{\dot{\alpha}}$ may be assigned charges $+1$ and -1 respectively under this symmetry. If R is the generator of this $\mathrm{U}(1)$, we have

$$\begin{aligned} [R, Q_\alpha] &= Q_\alpha \\ [R, \bar{Q}_{\dot{\alpha}}] &= -\,\bar{Q}_{\dot{\alpha}}, \end{aligned} \tag{1.16}$$

  and $[R, P_\mu] = 0$, $[R, M_{\mu\nu}] = 0$. This is a chiral symmetry, and is in general anomalous. We shall make use of this symmetry in our discussion of effective field theories.

- The supersymmetry algebra is a generalisation of a Lie algebra, and hence is constrained by (generalised) *Jacobi identities.* The structure of these identities are as follows.

$$\begin{aligned} (-1)^{\varepsilon_A \varepsilon_C} \left[[A, B\}, C\right\} &+ (-1)^{\varepsilon_B \varepsilon_A} \left[[B, C\}, A\right\} \\ &+ (-1)^{\varepsilon_C \varepsilon_B} \left[[C, A\}, B\right\} = 0, \end{aligned}$$

where, ε is 0 (respectively 1) for bosonic (fermionic) operators, and the mixed bracket notation $[\cdot,\cdot\}$ stands for an anticommutator when both the operators are fermionic and a commutator otherwise.

2 Representations of supersymmetry on states

We shall now discuss the irreducible representations of the supersymmetry algebra, *i.e.* a collection of bosonic and fermionic states/fields that transform covariantly under supersymmetry. First we shall discuss particle states as supersymmetry representations, and come back to the representation on (quantum) fields in the next lecture.

To begin with let us recall that particle representations of the Poincaré algebra are labelled by the eigenvalues of the following Casimir operators

- $P^2 = P_\mu P^\mu$ with eigenvalue m^2 (mass square),
- $W^2 = W_\mu W^\mu$ where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the Pauli-Lubanskí vector.

Exercise: *Show that*

1. *for massive particles,* i.e. $m^2 \neq 0$, $W^2 = -\, m^2 j(j+1)$, *where* $j = j_+ + j_-$*;*
2. *for massless particles,* i.e. $m^2 = 0$, $W_\mu = \lambda P_\mu$, *where* λ *is the helicity.*

Therefore mass and spin/helicity are the quantum numbers that label particle states. In order to describe which quantum numbers label representations of supersymmetry, notice that

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot\alpha}] = 0.$$

Hence P^2 commute with the new generators, and continue to be a Casimir of the enlarged symmetry. However,

$$[M_{\mu\nu}, Q_\alpha] \neq 0 \qquad\qquad [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] \neq 0,$$

and W^2 is no longer a Casimir of the supersymmetry algebra. We need to make the following modification. Define

$$\begin{aligned} B_\mu &= W_\mu - \frac{1}{4}\,\bar{Q}_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\beta}Q_\beta, \\ C_{\mu\nu} &= B_\mu P_\nu - B_\nu P_\mu. \end{aligned} \tag{2.1}$$

Exercise: *Show that* $[C_{\mu\nu}, Q_\alpha] = 0$.

Therefore the second Casimir of the supersymmetry algebra is $C^2 = C_{\mu\nu}C^{\mu\nu}$. Supersymmetry multiplets are labelled by mass and eigenvalue of the operator C^2.

We are now ready to construct the representations of the supersymmetry algebra on particle states, *i.e.* on asymptotic on-shell physical states.

First, let us consider massive states, *i.e.* $m^2 \neq 0$. In this case, one can go to the rest frame and make the choice $P_\mu = (m, \mathbf{0})$. With this choice, we find that

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu \\ &= 2m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

or, explicitly in terms of the components

$$\begin{aligned}\{Q_1, \bar{Q}_{\dot{1}}\} &= 2m \\ \{Q_2, \bar{Q}_{\dot{2}}\} &= 2m \\ \{Q_1, \bar{Q}_{\dot{2}}\} &= 0 = \{Q_2, \bar{Q}_{\dot{1}}\}.\end{aligned}$$

We have here two pairs of fermionic creation/annihilation operators.

We shall (arbitrarily) choose the dotted spinorial generators $\bar{Q}_{\dot{\alpha}}$, ($\dot{\alpha} = 1, 2$), to be the creation operators and the undotted ones annihilation operators. Further we may rescale Q's by $1/\sqrt{2m}$ to define conventionally normalised creation/annihilation operators

$$\begin{aligned}a_\alpha &= \frac{1}{\sqrt{2m}}\, Q_\alpha, \qquad \alpha = 1, 2; \\ a^\dagger_\alpha &= \frac{1}{\sqrt{2m}}\, \bar{Q}_{\dot{\alpha}}, \qquad \dot{\alpha} = 1, 2.\end{aligned} \tag{2.2}$$

Let us define a state $|\Omega\rangle$ such that

$$a_\alpha|\Omega\rangle = 0, \quad \text{for } \alpha = 1, 2.$$

$|\Omega\rangle$ is a (Clifford) vacuum state with respect to the fermionic creation/annihilation operators.

What are the quantum numbers that label this state? Thanks to our discussion on Casimirs, we know the answer to this question. One quantum number is of course the mass m. To find the other one:

> **Exercise:** *Show that on massive states, in the rest frame,*
>
> $$B_i = -\, m\left(L_i - \frac{1}{4m}\bar{Q}\sigma_i Q\right) \equiv -\, m\tilde{L}_i$$
>
> *and hence,*
>
> $$\begin{aligned}C_{0i} &= -\, mB_i = m^2\tilde{L}_i \\ C_{ij} &= 0.\end{aligned}$$
>
> *Therefore,* $C^2 = 2C_{0i}C^{0i} = 2m^4\tilde{L}_i\tilde{L}^i$.

It is easy to check that the generators $\tilde{L}_i$ obey the angular momentum algebra. So the eigenvalues of $\tilde{L}^2$ may be labelled by $\tilde{\jmath}(\tilde{\jmath}+1)$, where $\tilde{\jmath}$ can take any half-integral value. Actually, with our choice of $|\Omega\rangle$,

$$\tilde{L}_i|\Omega\rangle = L_i|\Omega\rangle.$$

Hence, acting on $|\Omega\rangle$, $\tilde{\jmath} = j$ label the spin $j = j_+ + j_-$. Being an eigenstate of spin, $|\Omega\rangle$ is labelled by

$$|\Omega\rangle = |m; j, j_3\rangle, \qquad j_3 = -j, -j+1, \cdots, j-1, j.$$

The (Clifford) vacuum $|\Omega\rangle$ is $(2j+1)$-fold degenerate.

The excitations over the vacuum $|\Omega\rangle$ are defined by the fermionic creation operators $a^\dagger_1$ and $a^\dagger_2$. We have

$$\begin{array}{ccc} & |\Omega\rangle & \\ a^\dagger_1|\Omega\rangle & & a^\dagger_2|\Omega\rangle \\ & a^\dagger_1 a^\dagger_2|\Omega\rangle & \end{array} \tag{2.3}$$

i.e. a total of $4(2j+1)$ states in the supersymmetry mutiplet.

Recall that $a^\dagger_\alpha \sim \bar{Q}_{\dot\alpha}$ transforms as a $\left(0, \frac{1}{2}\right)$ spinor. In particular, (by a choice of convention), $a^\dagger_1$ (respectively $a^\dagger_2$) has L_3 eigenvalue $+\frac{1}{2}$ $(-\frac{1}{2})$. The spins of the different states in a massive supermultiplet are

$$\begin{array}{ccccc} \text{state} & |\Omega\rangle & a^\dagger_1|\Omega\rangle & a^\dagger_2|\Omega\rangle & a^\dagger_1 a^\dagger_2|\Omega\rangle \\ \text{spin} & j_3 & j_3+\frac{1}{2} & j_3-\frac{1}{2} & j_3 \end{array} \tag{2.4}$$

If j is an integer, the first and the last states are bosonic, and the second and third ones are fermionic. The statistics is opposite when j is a half odd integer. As an example, consider $j=0$. There are then four states in the supersymmetry multiplet, two of these have spin $j_3=0$ and the other two have $j_3=\pm\frac{1}{2}$. The spin zero states may be combined into a scalar and a pseudo-scalar while the spin half states describe the degrees of freedom of a Weyl fermion.

This matching of bosonic and fermionic degrees of freedom in a multiplet is a remarkable property of supersymmetry. We can easily prove the following

Theoerem: *Every representation of supersymmetry algebra contains an equal number of bosonic and fermionic states.*

Proof: Let us define the operator $(-1)^{N_F}$ whose eigenvalues are $+1$ on bosonic and -1 on fermionic states. By definition

$$(-1)^{N_F}Q_\alpha = -\,Q_\alpha(-1)^{N_F}.$$

Now taking a trace of the representation, (which we assume to be finite dimensional for it to be well defined), we have

$$\begin{aligned} \mathrm{tr}\left[(-1)^{N_F}\{Q_\alpha,\bar{Q}_{\dot\beta}\}\right] &= \mathrm{tr}[(-1)^{N_F}Q_\alpha\bar{Q}_{\dot\beta} \\ &\quad +(-1)^{N_F}\bar{Q}_{\dot\beta}Q_\alpha\}] \\ \textit{i.e.} \qquad \mathrm{tr}\left[(-1)^{N_F}P_\mu\right] &= 0, \end{aligned}$$

where we have used the supersymmetry algebra in the LHS and the cyclic property of trace and the identity involving $(-1)^{N_F}$ and Q_α in the RHS. For a fixed value of P_μ in a given multiplet, we then have, $\mathrm{tr}[(-1)^{N_F}]=0$, which proves the assertion.

Now let us discuss the supersymmetry representation on massless states. We can choose a reference frame such that

$$P_\mu = (E,0,0,E)$$

Exercise: *Show that on massless states $W_0=\lambda E$, $W_3=\lambda E$, and hence*

$$\begin{aligned} B_0 &= W_0 - \frac{1}{4}\bar{Q}Q \\ B_3 &= W_3 + \frac{1}{4}\bar{Q}\sigma_3 Q. \end{aligned}$$

Also that the only non-vanishing component of $C_{\mu\nu}$ is

$$C_{03} = E(B_0-B_3) = -\,\frac{1}{2}E\bar{Q}_{\dot 2}Q_2,$$

whence, $C^2=0$.

In our chosen basis the supersymmetry algebra takes the following form

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

or, explicitly in components

$$\begin{aligned} \{Q_1, \bar{Q}_{\dot{1}}\} &= 4E \\ \{Q_2, \bar{Q}_{\dot{2}}\} &= 0 \\ \{Q_1, \bar{Q}_{\dot{2}}\} &= 0 = \{Q_2, \bar{Q}_{\dot{1}}\}. \end{aligned}$$

As in the massive case, let us define $|\Omega\rangle$ annhilated by the annihilation operators $a_1 \sim Q_1$ and $a_2 \sim Q_2$. In addition, since $\{Q_2, \bar{Q}_{\dot{2}}\} = 0$, we have

$$\langle\Omega|Q_2\bar{Q}_{\dot{2}}|\Omega\rangle = 0,$$

i.e, the excitation $\bar{Q}_{\dot{2}}|\Omega\rangle$ is a null state, or $\bar{Q}_{\dot{2}}$ is zero in the operator sense.

This leaves us with only one pair of creation/annihilation operators $a = \frac{1}{2\sqrt{E}}Q_1$ and $a^\dagger = \frac{1}{2\sqrt{E}}\bar{Q}_{\dot{1}}$, which satisfy $\{a, a^\dagger\} = 1$. The massless supersymmetry consists of the states

$$\begin{aligned} |\Omega\rangle &\quad \text{: a non-degenerate state of helicity } \lambda \\ a^\dagger|\Omega\rangle &\quad \text{: a non-degenerate state of helicity } \lambda + \tfrac{1}{2}. \end{aligned} \tag{2.5}$$

Notice that the massless supersymmetry multiplet is not a CPT eigenstate. One needs two pairs of irreducible massless multiplets, *i.e.* four states of helicity $\lambda, \lambda + \frac{1}{2}$ and $-\lambda, -\frac{1}{2} - \lambda$ to complete a CPT eigenstate.

We just finished discussing how certain bosonic and fermionic particle states form representations of supersymmetry. In other words, we have a set of states which transforms covariantly under supersymmetry variation. These are asymptotic on-shell states. In order to construct a quantum field theory, however, we need to know how general off-shell states form representations of supersymmetry. This can be done by considering bosonic and fermionic fields and studying their behaviour under supersymmetry variation. A far more economic and elegant approach is in terms of what will be called *superfields*. Different bosonic and fermionic fields that mix under supersymmetry transformations can be thought of as components of this single superfield. This approach is also advantageous from a practical point of view, as many properties of supersymmetry are manifest when expressed in terms of superfields. These concepts were introduced by Salam and Strathdee[11].

To do this, however, we need to make a digression to discuss the algebra and calculus of Grassmann variables. To this end, let us introduce spinor parameters θ^α, $\bar{\theta}_{\dot{\alpha}}$ $(\alpha, \dot{\alpha} = 1, 2)$ — (notice the index assignment) — which satisfy the relations

$$\begin{aligned} \{\theta^\alpha, \theta^\beta\} &= 0, \\ \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} &= 0, \\ \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} &= 0, \end{aligned} \tag{2.6}$$

as also $[x^\mu, \theta^\alpha] = 0$, $[x^\mu, \bar{\theta}_{\dot{\alpha}}] = 0$. These are anticommuting analogues of a complex variable z, and are called Grassmann numbers or variables. The pair $(\theta, \bar{\theta})$ can be taken to parametrise infinitesimal supersymmetry variation

$$\delta_{\text{susy}} = (\theta^\alpha Q^\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}) \equiv (\theta Q + \bar{\theta}\bar{Q}). \tag{2.7}$$

Due to the anticommuting nature of the parameters $(\theta, \bar{\theta})$, the combinations $\theta Q \equiv \theta^\alpha Q_\alpha$ and $\bar{\theta}\bar{Q} \equiv \theta_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$ satisfy the *commutation* relations

$$\begin{aligned} [\theta Q, \bar{\theta}\bar{Q}] &= 2(\theta\sigma^\mu\bar{\theta})P_\mu, \\ [\theta Q, \theta Q] &= 0 = [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}]. \end{aligned} \tag{2.8}$$

All the algebraic relations of supersymmetry are now expressed in terms of commutators. (We shall also consider the replacement $P_\mu \to -\, y^\mu P_\mu$, where y^μ is an infinitesimal parameter for translation.)

The infinitesimal variations may now be exponentiated to define a finite transformation

$$G(y, \theta, \bar{\theta}) = \exp\left\{ i\left(-y^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} \right) \right\}.$$

Notice the dimensions of the parameters $[y] = M^{-1}$ and $[\theta] = [\bar{\theta}] = M^{-1/2}$.

Parenthetical comments: Actually, we should have written

$$G(y, \theta, \bar{\theta}) = \exp\left\{ i\left(-y \cdot P + \theta Q + \bar{\theta}\bar{Q} \right) \right\} \exp\left\{ -\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} \right\},$$

but we left out the Lorentz transformation part. It is consistent to set that part to identity. In other words, we are parametrising a coset space defined by the quotient of the super-Poincare group by its Lorentz subgroup.

Notice also that since the anti-commutators of Q, $\bar{Q}$ are non-zero, the following forms

$$\begin{aligned} &\exp\left\{ i\left(-y \cdot P + \theta Q \right) \right\} \exp\left\{ i\bar{\theta}\bar{Q} \right\} \\ &\exp\left\{ i\left(-y \cdot P + \bar{\theta}\bar{Q} \right) \right\} \exp\left\{ i\theta Q \right\} \end{aligned}$$

and the one we gave earlier for G are not all equivalent. It is also consistent to work with either of the above forms and get the same results. However the explicit differential operator form for the generators Q and $\bar{Q}$ will be different in each case.

Notice that in the above the Grassmann parameters $(\theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ appear in the same footing as the coordinates y^μ. So the full parameter space is labelled by

$$(y^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}); \quad \mu = 0, \cdots, 3; \quad \alpha, \dot{\alpha} = 1, 2.$$

We should think of this space as 4 normal (*i.e.* bosonic) plus 4 anti-commuting (*i.e.* fermionic) extension of our familiar spacetime. (The dimension of the extended space is sometimes written as $(4|4)$.) This is called the $N = 1$ *rigid superspace*.

Just as it is advantageous to construct relativistic quantum field theory in a manifestly Lorentz covariant formalism, it is of great advantage to formulate supersymmetric theories in superspace.

3 Superspace & superfields

We can define functions in superspace — these are going to be the superfields — and differentiate and integrate them with respect to the coordinates $(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. To explain the rules on differentiation and integration, let us consider the simpler example of a $(1|1)$ dimensional superspace. This has only two coordinates (x, θ) and $\theta^2 = 0$. Due to the nilpotence of the coordinate θ, Taylor expansion of a function $f(x,\theta)$ in superspace in terms of θ terminate after the linear term:

$$f(x,\theta) = f_0(x) + \theta f_1(x), \tag{3.1}$$

where $f_0(x)$ and $f_1(x)$ are functions of of the commuting coordinate x. Using

$$\frac{d}{d\theta}\,(\theta) = 1 \qquad\qquad \frac{d}{d\theta}\,(1) = 0, \tag{3.2}$$

it follows that

$$\frac{d}{d\theta}\,(f(x,\theta)) = f_1(x). \tag{3.3}$$

(Notice that the dimension $[d/d\theta]$ is $M^{1/2}$.)

Now since we want the integral of a total derivative to vanish, we define the following rules of integration

$$\int d\theta \;=\; 0, \qquad\qquad \int d\theta\,\theta \;=\; 1, \tag{3.4}$$

which gives

$$\int d\theta\,\frac{d}{d\theta} f(x,\theta) = \int d\theta\, f_1(x) = 0.$$

The integral so defined is invariant under translation of θ by an arbitrary constant ξ:

$$\begin{aligned}
\int d(\theta+\xi)\, f(x,\theta+\xi) &= \int d\theta\,[f_0(x) + (\theta+\xi) f_1(x)] \\
&= \int d\theta\,\theta f_1(x) \\
&= \int d\theta\, f(x,\theta).
\end{aligned}$$

It is a curious fact that integration and differentiation in θ are equivalent!

- $\frac{d}{d\theta} f(x,\theta) = f_1(x)$
- $\int d\theta\; f(x,\theta) = f_1(x)$

And consistent with these rules $[d\theta] = M^{1/2}$, unlike in ordinary space. Finally, we can define a delta function by

$$\delta(\theta) = \theta$$

leading to the expected result $\int d\theta\;\delta(\theta) = 1$.

Coming back to our (4|4) dimensional superspace, we have the following rules ($\partial_\alpha \equiv \partial/\partial\theta^\alpha$ and $\bar{\partial}^{\dot{\alpha}} \equiv \partial/\partial\bar{\theta}_{\dot{\alpha}}$):

$$\begin{aligned}
\partial_\alpha \theta^\beta &= \delta_\alpha^\beta, \\
\partial_\alpha \theta_\beta &= \partial_\alpha(\epsilon_{\beta\gamma}\theta^\gamma) \;=\; -\epsilon_{\alpha\beta}, \\
\bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} &= \delta_{\dot{\beta}}^{\dot{\alpha}}, \\
\bar{\partial}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} &= -\epsilon^{\dot{\alpha}\dot{\beta}}.
\end{aligned} \tag{3.5}$$

One can also 'raise' index of ∂_α by chain rule of differentiation

$$\partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha} = \frac{\partial\theta^\beta}{\partial\theta_\beta}\,\frac{\partial}{\partial\theta^\beta} = \partial^\alpha(\epsilon^{\beta\gamma})\,\partial_\beta = -\epsilon^{\alpha\beta}\partial_\beta.$$

The following is a compilation of useful results that will come in handy in our subsequent calculations:

$$\begin{aligned} \partial_\alpha\left(\theta^\beta\theta^\gamma\right) &= \delta^\beta_\alpha\theta^\gamma - \delta^\gamma_\alpha\theta^\beta \\ \partial_\alpha(\theta\theta) &= 2\theta_\alpha \\ \bar\partial^{\dot\alpha}(\bar\theta\bar\theta) &= 2\bar\theta^{\dot\alpha} \\ \partial^2(\theta\theta) &= 4 \\ \bar\partial^2(\bar\theta\bar\theta) &= 4 \end{aligned} \tag{3.6}$$

It is straightforward to derive the above.

As for integration, we shall define the following convention

$$\begin{aligned} d^2\theta &= -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta} = \frac{1}{2}d\theta^1 d\theta^2, \\ d^2\bar\theta &= -\frac{1}{4}d\bar\theta_{\dot\alpha}d\bar\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} = -\frac{1}{2}d\bar\theta_{\dot 1}d\bar\theta_{\dot 2} \\ d^4\theta &\equiv d^2\theta\, d^2\bar\theta; \end{aligned} \tag{3.7}$$

so that,

$$\begin{aligned} \int d^2\theta\,\theta\theta &= +1, \\ \int d^2\bar\theta\,\bar\theta\bar\theta &= +1. \end{aligned} \tag{3.8}$$

After this long detour we get back to the transformation generated by

$$G(x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}) = e^{i(-x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha})}.$$

This generator is unitary since $(\theta^\alpha Q_\alpha)^\dagger = \bar Q_{\dot\alpha}\bar\theta^{\dot\alpha} = \bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}$. If we consider two such successive transformations, the result is

$$\begin{aligned} G(x,\theta,\bar\theta)\; G(y,\xi,\bar\xi) \;=\; & \exp\Big[-i(x^\mu + y^\mu - i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\xi^{\dot\beta} \\ & i\xi^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta})P_\mu \;+\; i\,(\theta^\alpha+\xi^\alpha)\,Q_\alpha + i\,(\bar\theta_{\dot\alpha}+\bar\xi_{\dot\alpha})\,\bar Q^{\dot\alpha}\Big]. \end{aligned}$$

Exercise: *Show the above. Hint: You will need the Baker-Campbell-Hausdorf formula*

$$\begin{aligned} e^A\, e^B \;=\; & \exp\Big(A + B + \frac{1}{2!}[A,B] \\ & + \frac{1}{3!}\left(\frac{1}{2}[[A,B],B] + \frac{1}{2}[A,[A,B]]\right) + \cdots\Big). \end{aligned}$$

The successive applications of superspace transformations generate the following motion is terms of the superspace coordinates

$$(x, \theta, \bar\theta) \stackrel{g(y,\xi\bar\xi)}{\longrightarrow} (x + y + i\xi\sigma\bar\theta - \theta\sigma\bar\xi, \theta + \xi, \bar\theta + \bar\xi),$$

which is given by the following differential operators

$$\begin{aligned} y^\mu P_\mu &= i\, y^\mu \frac{\partial}{\partial x^\mu}, \\ \xi^\alpha Q_\alpha &= \xi^\alpha \left(\partial_\alpha - i\sigma^\mu_{\alpha\dot\beta} \bar\theta^{\dot\beta} \partial_\mu \right), \\ \bar\xi_{\dot\alpha} \bar Q^{\dot\alpha} &= \bar\xi_{\dot\alpha} \left(-\bar\partial^{\dot\alpha} + i\bar\sigma^{\mu\,\dot\alpha\beta} \theta_\beta \partial_\mu \right). \end{aligned}$$

Alternatively, we may write[5],

$$\begin{aligned} P_\mu &= i \frac{\partial}{\partial x^\mu}, \\ Q_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot\beta} \bar\theta^{\dot\beta} \partial_\mu, \\ \bar Q_{\dot\alpha} &= \bar\partial_{\dot\alpha} - i\theta^\beta \sigma^\mu_{\beta\dot\alpha} \partial_\mu. \end{aligned} \tag{3.9}$$

The above differential operator representation leads to the anti-commutator

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = -\,2\sigma^\mu_{\alpha\dot\beta} P_\mu,$$

in apparent disagreement with (1.14), due to the extra minus sign. There is, however, no contradiction, as what we witness here is the difference in the active and passive points of view of symmetry transformations. The differential operator representation is in terms of superspace coordinates.

We shall now define a general scalar superfield $\Phi(x, \theta, \bar\theta)$ in our (4|4) dimensional $N = 1$ rigid superspace. A scalar function satisfies

$$\Phi(x', \theta', \bar\theta') = \Phi(x, \theta, \bar\theta),$$

and hence

$$\delta_\xi \Phi = \left(\xi Q + \bar\xi \bar Q\right)\, \Phi. \tag{3.10}$$

Let us now consider the Taylor expansion of Φ in powers of θ and $\bar\theta$.

$$\begin{aligned} \Phi(x, \theta, \bar\theta) &= \phi(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + (\theta\theta)m(x) \\ &\quad + (\bar\theta\bar\theta)n(x) + (\theta\sigma^\mu\bar\theta)v_\mu(x) + (\theta\theta)\bar\theta\bar\lambda(x) \\ &\quad + (\bar\theta\bar\theta)\theta\eta(x) + (\theta\theta)(\bar\theta\bar\theta)d(x). \end{aligned} \tag{3.11}$$

Each term in the above expansion is a field in physical (Minkowski) spacetime. In particular

	Fields	*Type*	Bose dof	Fermi dof
•	$\phi(x), m(x)$ $n(x), d(x)$	scalars	4×2	0
•	$\psi_\alpha(x), \eta_\alpha(x)$	L-spinors	0	2×4
•	$\bar\chi^{\dot\alpha}(x), \bar\lambda^{\dot\alpha}(x)$	R-spinors	0	2×4
•	$v_\mu(x)$	vector	4×2	0

(3.12)

[5]Since Q and $\bar Q$ are not hermitian operators, we have rescaled them by factors of i, without anything going wrong.

The fields $\{\phi(x), \psi_\alpha(x), \bar{\chi}^{\dot{\alpha}}(x), \cdots\}$ are called the components of the superfield Φ.

Let us remark parenthetically that the above is the most general possible expansion for a scalar superfield, since *e.g.* $\bar{\theta}\bar{\sigma}^\mu\theta = -\,\theta\sigma^\mu\bar{\theta}$ and $(\sigma^\mu\bar{\theta})_\alpha(\theta\sigma_\mu\bar{\theta}) = 2\theta_\alpha(\bar{\theta}\bar{\theta})$, etc. (See Appendix A for this type of manipulations.)

Now consider the supersymmetry variation of the scalar superfield (3.10). In terms of the component fields this leads to the following relations

$$\begin{aligned}
\delta_\xi\phi &= \xi\psi + \bar{\xi}\bar{\psi}, \\
\delta_\xi\psi &= 2\xi m + (\sigma^\mu\bar{\xi})(v_\mu + i\partial_\mu\phi), \\
\delta_\xi\bar{\chi} &= 2\bar{\xi}n + (\bar{\sigma}^\mu\xi)(-\,v_\mu + i\partial_\mu\phi), \\
\delta_\xi m &= \bar{\xi}\bar{\lambda} - \frac{i}{2}(\partial_\mu\psi)\sigma^\mu\bar{\xi}, \\
\delta_\xi n &= \xi\eta + \frac{i}{2}\xi\sigma^\mu(\partial_\mu\bar{\psi}), \\
\delta_\xi v_\mu &= \xi\sigma_\mu\bar{\lambda} + \eta\sigma_\mu\bar{\xi} + \frac{i}{2}(\partial_\mu\psi)\xi - \frac{i}{2}(\partial_\mu\bar{\chi})\bar{\xi}, \\
\delta_\xi\bar{\lambda} &= 2\bar{\xi}d + i(\bar{\sigma}^\mu\xi)\partial_\mu m + \frac{i}{2}\bar{\xi}(\partial^\mu v_\mu), \\
\delta_\xi\eta &= 2\xi d + i(\sigma^\mu\bar{\xi})\partial_\mu n - \frac{i}{2}\xi(\partial^\mu v_\mu), \\
\delta_\xi d &= \frac{i}{2}\xi\sigma^\mu(\partial_\mu\bar{\lambda}) - \frac{i}{2}(\partial_\mu\eta)\sigma^\mu\bar{\xi}.
\end{aligned} \tag{3.13}$$

Exercise: *(i) Derive the above relations. (You will need to use the Fierz identities given in Appendix A.)*
(ii) Work out

$$(\delta_{\xi_1}\delta_{\xi_2} - \delta_{\xi_2}\delta_{\xi_1})\,\phi = -\,2i\,(\xi_1\sigma^\mu\xi_2 - \xi_2\sigma^\mu\xi_1)\,\partial_\mu\phi.$$

4 Chiral & vector superfields

The result obtained at the end of the last lecture shows that the general scalar superfield forms a basis for an off-shell linear representation of supersymmetry:

- Supersymmetry variations of component fields are proportional to each other (also involving derivatives).
- Therefore the supersymmetry algebra closes, *i.e.* supersymmetry variations involve only those fields present in the mutiplet and no others.

However, there are a large number of component fields. It turns out that the set is not the minimal one. In other words, the scalar superfield representation is *reducible*[6].

In an effort to reduce the number of component fields in (3.11), let us try to set one of the spinor fields, say $\bar{\chi}$ to zero. To make this consistent with supersymmetry, we should also require that its supersymmetry variation vanishes, and so on. In the end, we have to

[6]It is not fully reducible though. That is, the scalar superfield cannot be written as a direct sum of irreducible superfields.

impose the following set of constraints

$$
\begin{aligned}
\bar{\chi}(x) &= 0 & v_\mu(x) &= i\partial_\mu\phi(x) \\
& & n(x) &= 0 \\
\eta(x) &= 0 \\
\bar{\lambda}(x) &= -\tfrac{i}{2}(\partial_\mu\psi)\sigma^\mu \\
& & d(x) &= -\tfrac{1}{4}\Box\phi(x)
\end{aligned}
\tag{4.1}
$$

leading to a *reduced* scalar superfield

$$
\begin{aligned}
\Phi_R &= \phi + \theta\psi + (\theta\theta)m + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\phi \\
&\quad + \frac{i}{2}(\theta\theta)\left((\partial_\mu\psi)\sigma^\mu\bar{\theta}\right) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box\phi.
\end{aligned}
$$

> **Exercise:** *Check the mutual consistency of the constraints imposed in (4.1). This demonstrates that the reduced scalar superfield Φ_R, (with fewer number of components than the general scalar superfield Φ), defines an off-shell linear representation of the supersymmetry algebra.*

Let us now define

$$y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}. \tag{4.2}$$

Using Taylor expansion, and some spinor identities, one can rewrite the restricted superfield Φ_R as

$$\Phi_R(y,\theta) = \phi(y) + \theta\psi(y) + (\theta\theta)m(y), \tag{4.3}$$

which shows that the restricted superfield is a function of y and θ, but has no explicit dependence on $\bar{\theta}$. Had it not been for the $\bar{\theta}$ in the definition of y in (4.2), we could have concluded that Φ_R is independent of $\bar{\theta}$, that is

$$\bar{\partial}_{\dot{\alpha}}\,\Phi_R(y,\theta) \overset{!}{=} 0.$$

This is of course not the case. Another problem is that

$$[\bar{\partial}_{\dot{\alpha}}, \xi Q] = i\xi^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu,$$

that is, $\bar{\partial}_{\dot{\alpha}}$ does not commute with supersymmetry variation. Hence imposing a constraint like $\bar{\partial}_{\dot{\alpha}}\Phi_R = 0$ is not consistent with supersymmetry. In other words, $\bar{\partial}_{\dot{\alpha}}\Phi_R$ is not a superfield. Thankfully from our experience with tensor analysis, we know what to do in such a situation: define an appropriate covariant derivative to impose the constraint consistently. The covariant derivative defined as

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{4.4}$$

leads to a consistent way to impose the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{4.5}$$

on the general scalar superfield (3.11) to restrict it to Φ_R.

> **Exercise:** *Show that $[\bar{D}_{\dot{\alpha}}, \xi Q] = 0$, or equivalently $\{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0$. Also show that $\{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$, and $\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0$.*

> **Exercise:** *Show that $\bar{D}_{\dot{\alpha}}y^\mu = 0$ and $\bar{D}_{\dot{\alpha}}\theta^\beta = 0$.*

The last exercise shows that a superfield constrained by (4.5) is a function of y and θ only, and has no explicit dependence on $\bar{\theta}$.

Definition: A scalar superfield Φ constrained by the (spinorial) chirality condition $\bar{D}_{\dot{\alpha}}\Phi = 0$ is called a *chiral superfield.*

We have already found an example of a chiral superfield in (4.3):

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{4.6}$$

(In the above, a scaling $\psi \to \sqrt{2}\psi$ and a change in notation $m \to F$ has been done to conform with standard notation in the literature.) A chiral superfield has the following

- complex scalar ϕ dof $= 2$ bose $[\phi] = M$,
- complex L-spinor ψ dof $= 4$ fermi $[\psi] = M^{3/2}$,
- complex scalar F dof $= 2$ bose $[F] = M^2$.

Under infinitesimal supersymmetry transformation

$$\begin{aligned} \delta_\xi\phi &= \sqrt{2}\xi\psi \\ \delta_\xi\psi &= \sqrt{2}\xi F + \sqrt{2}i\sigma^\mu\bar{\xi}\,\partial_\mu\phi \\ \delta_\xi F &= -\sqrt{2}i\partial_\mu\psi\sigma^\mu\bar{\xi}. \end{aligned} \tag{4.7}$$

An important property of a chiral superfield is that the product of two chiral superfields is again a chiral superfield. This follows from the chain rule of covariant differentiation.

The notion of an anti-chiral superfield is immediate:

Definition: A scalar superfield that satisfies the condition $D_\alpha\Phi = 0$, where

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{4.8}$$

is an anti-chiral superfield.

> **Exercise:** *Show that* $\{D_\alpha, Q_\beta\} = 0$. *Also show that* $\{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0$, *and* $\{D_\alpha, D_\beta\} = 0$, *and finally*
>
> $$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu.$$
>
> **Exercise:** *Let* $y^\dagger = x - i\theta\sigma\bar{\theta}$. *Show that* $D_\alpha y^{\mu\dagger} = 0$ *and* $D_\alpha\bar{\theta}^{\dot{\beta}} = 0$.

An anti-chiral superfield is therefore a function of $y^\dagger$ and $\bar{\theta}$ only. In particular, if $\Phi(y,\theta)$ is a chiral superfield (4.6), $\Phi^\dagger(y^\dagger,\bar{\theta})$ is an anti-chiral superfield

$$\Phi^\dagger(y^\dagger,\bar{\theta}) = \phi^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger).$$

As before, the product of two anti-chiral superfields is again an anti-chiral superfield. However the product $\Phi^\dagger\Phi$ is neither a chiral nor an anti-chiral superfield. Same applies to the sum $(\Phi + \Phi^\dagger)$.

Exercise: *Show that the imposing the conditions* $D_\alpha\Phi = 0$ *and* $\bar{D}_{\dot\alpha}\Phi = 0$ *simultaneously on a scalar superfield reduces it to a constant.*

Exercise: *Define*

$$
\begin{aligned}
\varphi &= \Phi\Big|_{\theta=\bar\theta=0}\\
\Psi_\alpha &= D_\alpha\Phi\Big|_{\theta=\bar\theta=0}\\
\mathcal{F} &= D^\alpha D_\alpha\Phi\Big|_{\theta=\bar\theta=0}.
\end{aligned}
$$

Express φ, Ψ *and* $\mathcal{F}$ *in terms of the component fields* ϕ, ψ *and* F*. Compute the transformation laws for* φ, Ψ *and* $\mathcal{F}$ *using the differential operator representation for* $Q, \bar{Q}$*.*

Exercise: *Show that in terms of the variables* $(y, \theta, \bar\theta)$*, the covariant derivatives may be written as*

$$
\begin{aligned}
D_\alpha &= \partial_\alpha + 2i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial y^\mu},\\
\bar{D}^\alpha &= -\partial^\alpha - 2i\bar\theta_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\frac{\partial}{\partial y^\mu},\\
\bar{D}_{\dot\alpha} &= -\partial_{\dot\alpha},\\
\bar{D}^{\dot\alpha} &= \partial^{\dot\alpha}.
\end{aligned}
\tag{4.9}
$$

Exercise: *Compute the expressions for* Φ^2*,* Φ^3 *and* $\Phi^\dagger\Phi$ *in the component field expansion.*

There is another way to reduce the number of fields in a general scalar superfield. For a scalar superfield $V(x, \theta, \bar\theta)$, let us impose the covariant reality condition:

$$V(x, \theta, \bar\theta) = V^\dagger(x, \theta, \bar\theta). \tag{4.10}$$

In components, this leads to the following set of constraints

$$
\begin{array}{lll}
\phi = \phi^* & & \\
 & & (\psi_\alpha)^* = \bar\chi^{\dot\alpha}\\
m = n^* & & \\
v_\mu = (v_\mu)^* & & \\
 & & \eta_\alpha = (\bar\lambda^{\dot\alpha})^*\\
d = d^* & &
\end{array}
\tag{4.11}
$$

Therefore, a scalar superfield restricted by the reality condition has four real scalars, (which may be combined into two complex scalars), one real vector and two complex Weyl spinors, (or equivalently, one real Majorana spinor). There are altogether eight bosonic and eight fermionic degrees of freedom.

We can already construct an example of a real superfield trivially from a chiral superfield. If Λ is a chiral superfield, the sum $(\Lambda + \Lambda^\dagger)$ is a real superfield. In fact, this means that a real superfield is not uniquely determined. Given a real superfield $V(x, \theta, \bar\theta)$, it is possible to construct another one

$$V(x, \theta, \bar\theta) \to V(x, \theta, \bar\theta) + \Lambda(y, \theta) + \Lambda^\dagger(y^\dagger, \bar\theta). \tag{4.12}$$

This is a 'gauge freedom' in defining a real superfield. If we expand a real superfield in components

$$\begin{aligned} V(x,\theta,\bar\theta) &= \rho(x)+\theta\chi(x)+\bar\theta\bar\chi(x)+\theta\theta M(x)+\bar\theta\bar\theta M^*(x)\\ -(\theta\sigma^\mu\bar\theta)A_\mu(x) &+ i\theta\theta\,\bar\theta\bar\lambda(x)-i\bar\theta\bar\theta\,\theta\lambda(x)+\frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,D, \end{aligned}$$

and use the freedom in

$$\begin{aligned} \left(\Lambda+\Lambda^\dagger\right)(x,\theta,\bar\theta) &= (\phi+\phi^*)+\sqrt{2}(\theta\psi+\bar\theta\bar\psi)+\theta\theta F+\bar\theta\bar\theta F^*\\ &\quad +i\theta\sigma^\mu\bar\theta(\partial_\mu\phi-\partial_\mu\phi^*)+\frac{i}{\sqrt2}\theta\theta(\bar\theta\bar\sigma^\mu\partial_\mu\psi)\\ &\quad -\frac{i}{\sqrt2}\bar\theta\bar\theta(\theta\sigma^\mu\partial_\mu\bar\psi)-\frac{1}{4}\theta\theta\,\bar\theta\bar\theta(\Box\phi+\Box\phi^*), \end{aligned}$$

we find that

$$\begin{aligned} \delta\rho &= \phi+\phi^* \;=\; 2\,\mathrm{Re}\,\phi\\ \delta\chi &= \sqrt2\psi\\ \delta M &= F\\ \delta A_\mu &= -\,i\partial_\mu(\phi-\phi^*) \;=\; \partial_\mu(2\,\mathrm{Im}\,\phi)\\ \delta\lambda &= \frac{1}{\sqrt2}\sigma^\mu\partial_\mu\bar\psi\\ \delta D &= -\frac{1}{2}\Box(\phi+\phi^*) \;=\; 2\,\mathrm{Re}\,\phi. \end{aligned} \tag{4.13}$$

Let us first point out the most interesting aspect of the 'gauge transformation' in (4.13) above:

$$\delta A_\mu(x)=\partial_\mu(2\,\mathrm{Im}\;\phi(x))=\partial_\mu\varepsilon(x),$$

where, $\varepsilon(x)$ is a real scalar field. This is therefore the usual abelian gauge transformation for the vector field $A_\mu(x)$ in the components of the real superfield V. For this reason, a real superfield is also known as a *vector superfield*.

The more general superfield transformation (4.12) means that any superfield action invariant under the above abelian gauge transformation is also independent of several component fields of V. In particular, we may choose $\mathrm{Re}\,\phi, F$ and ψ to set ρ, m and χ to zero respectively. This partially fixes the gauge freedom in $V\to V+\Lambda+\Lambda^\dagger$, and is called the *Wess-Zumino gauge*. After this gauge fixing, the vector superfield takes the form

$$V_{WZ}=-\,(\theta\sigma^\mu\bar\theta)A_\mu+i\theta\theta\,\bar\theta\bar\lambda-i\bar\theta\bar\theta\,\theta\lambda+\frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,D. \tag{4.14}$$

It should be stressed that the Wess-Zumino gauge imposes no restriction on $\mathrm{Im}\,\phi$, therefore it does *not* fix the abelian gauge freedom of the component vector field A_μ.

There are four real bosonic degrees of freedom in the component fields of V_{WZ}: $(4-1)=3$ from the vector A_μ, and one from the real scalar D. Also there are four real fermionic ones from λ.

5 More on vector superfields

Another way to reformulate Wess-Zumino gauge fixing is to observe that, without any loss of generality, a vector superfield may be decomposed as

$$V = V_{WZ} + \Lambda + \Lambda^\dagger, \tag{5.1}$$

where Λ is a chiral superfield.

Exercise: *Work out the component expansion for V_{WZ}^2 and V_{WZ}^3.*

While the Wess-Zumino condition allows one to get rid of the superfluous fields in a vector superfield and retain only the relevant ones, it is *not* invariant under supersymmetry variation. In other words, the WZ gauge condition is not covariant. It is, however, possible to give a covariant description in which only the necessary fields in a vector superfield are retained. To this end let us define the following covariant derivatives on a vector superfield.

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\,(\bar{D}\bar{D})\,D_\alpha V(x,\theta,\bar{\theta}) \\ &= -\frac{1}{8}\,(\bar{D}\bar{D})\left(e^{-2V}D_\alpha e^{+2V}\right), \\ \bar{W}_{\dot{\alpha}} &= -\frac{1}{4}\,(DD)\,\bar{D}_{\dot{\alpha}}V(x,\theta,\bar{\theta}) \\ &= -\frac{1}{8}\,(DD)\left(e^{+2V}\bar{D}_{\dot{\alpha}}e^{-2V}\right). \end{aligned} \tag{5.2}$$

Notice that

1. The construction/definition of the W_α ($\bar{W}_{\dot{\alpha}}$) ensures that[7]

$$\bar{D}_{\dot{\alpha}}W_\alpha = 0 \qquad\qquad (\,D_\alpha\bar{W}_{\dot{\alpha}} = 0\,)\,, \tag{5.3}$$

   that is, W_α (resp. $\bar{W}_{\dot{\alpha}}$) is a (anti-)chiral superfield. Since this field also carries a spinor index it is a chiral spinor superfield.

2. However, W_α is not a general chiral superfield, since it satisfies

$$DW = \bar{D}\bar{W}. \tag{5.4}$$

3. The fields W_α and $\bar{W}_{\dot{\alpha}}$ are both invariant under the gauge transformation (4.12).

Exercise: *Prove the two statements mentioned above.*

It is important to note that the two superfields W_α and $\bar{W}_{\dot{\alpha}}$ are invariant under the full superfield gauge transformation. Therefore even the abelian gauge invariance of the component vector field A_μ (that the Wess-Zumino gauge did not fix), is no longer available. Consequently, the components of these (anti-)chiral superfields can be calculated in the Wess-Zumino gauge without any loss of generality. That is the exercise we shall do now. In order to simplify the computation, observe that being a chiral superfield, W_α is a function of y and θ only, (similarly $\bar{W}_{\dot{\alpha}}$ is a function of $y^\dagger$ and $\bar{\theta}$), with no explicit dependence on $\bar{\theta}$ (resp. θ).

[7] Since the $\bar{D}$'s anticommute and has only two independent components, $\bar{D}^3 = 0$.

Exercise: *Carry out this computation in detail to show that*

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) + i\theta^\beta \sigma^{\mu\nu}_{\beta\alpha} F_{\mu\nu}(y) \\ &\quad +(\theta\theta)\sigma^\mu_{\alpha\dot\beta}\partial_\mu \bar\lambda^{\dot\beta}(y), \\ \bar W_{\dot\alpha} &= +i\bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha} D(y^\dagger) - i\bar\theta_{\dot\beta}\bar\sigma^{\mu\nu\dot\beta}_{\dot\alpha} F_{\mu\nu}(y^\dagger) \\ &\quad +(\bar\theta\bar\theta)\bar\sigma^{\mu\beta}_{\dot\alpha}\partial_\mu\lambda_\beta(y^\dagger). \end{aligned} \tag{5.5}$$

Hint: Start by writing the real superfield in Wess-Zumino gauge V_{WZ} as a function of $(y, \theta, \bar\theta)$ and use results from (4.9).

We see that the component fields in the spinor chiral superfield W are a left-handed spinor λ, a scalar D and the field strength $F_{\mu\nu}$. It is therefore also called a *field strength supermultiplet.* Moreover, since $\sigma^{\mu\nu}$ is self-dual (1.9), only the self-dual part of $F_{\mu\nu}$ contributes to the degrees of freedom in W.

One can generalise this result in which the vector field in the vector supermultiplet has abelian gauge invariance to one with non-abelian gauge invariance. In order to do that, however, we have to rewrite the abelian gauge condition in a way that is generalisable to the non-abelian case. Recall the Wess-Zumino gauge freedom $V \to V + \Lambda + \Lambda^\dagger$. With a scaling $\Lambda \to i\Lambda$, this is equivalent to

$$e^V \to e^{-i\Lambda^\dagger}\, e^V\, e^{+i\Lambda}. \tag{5.6}$$

Let us now elevate the superfields to Lie algebra valued superfields

$$\begin{aligned} V &\to V_{ij} = t^a_{ij} V_a, \\ \Lambda &\to \Lambda_{ij} = t^a_{ij}\Lambda_a, \end{aligned}$$

where t^a are hermitian generators of some Lie algebra satisfying

$$\left[t^a, t^b\right] = i f^{abc} t^c, \tag{5.7}$$

and normalised so that

$$\begin{aligned} \mathrm{tr}\left(t^a t^b\right) &= C(R)\,\delta^{ab}, \\ C(R) &= \frac{\dim R}{\dim \mathbf{g}}\, C_2(R). \end{aligned} \tag{5.8}$$

In the above, dim R and $C_2(R)$ are the dimension and the quadratic Casimir in the representation R and dim $\mathbf{g}$ is the dimension of the Lie algebra $\mathbf{g}$.

To first order in gauge parameter superfield Λ

$$\begin{aligned} \delta V &= i(\Lambda - \Lambda^\dagger) + \frac{i}{2}\left[V, (\Lambda + \Lambda^\dagger)\right] \\ &\quad + \frac{i}{12}\left[V, [V, (\Lambda - \Lambda^\dagger)]\right] + \cdots. \end{aligned} \tag{5.9}$$

The linear term $\delta V = i(\Lambda - \Lambda^\dagger)$ still allows for a choice of WZ gauge which, as in the abelian case, does not fix the non-abelian gauge freedom of the component vector fields A^a_μ. Once we fix the WZ gauge, we have $V^a_{WZ} = (\theta\sigma\bar\theta)A^a_\mu + \cdots$, and also $(\Lambda + \Lambda^\dagger) = -2i\mathrm{Re}\,(\phi)$,

$(\Lambda - \Lambda^\dagger) = 2i(\theta\sigma\bar{\theta})\partial_\mu \text{Im}\,(\phi)$. Therefore the third term and above in (5.9) vanish in the WZ gauge as they have too many θs or $\bar{\theta}$s. The relation

$$\delta V_{WZ} = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}\left[V, (\Lambda + \Lambda^\dagger)\right] \tag{5.10}$$

implies the usual non-abelian gauge transformation for the component fields A_μ (non-abelian gauge field), λ (a spinor in the adjoint representation) and D (auxiliary scalar field in the adjoint representation).

As in the abelian case, one can define the constrained (anti-)chiral spinor superfield W_α (and $\bar{W}_{\dot{\alpha}}$) by

$$\begin{aligned} W_\alpha &= -\frac{1}{8}\bar{D}^2 e^{-2V} D_\alpha e^{2V}, \\ \bar{W}_{\dot{\alpha}} &= +\frac{1}{8}D^2 e^{+2V} \bar{D}_{\dot{\alpha}} e^{-2V}. \end{aligned} \tag{5.11}$$

These superfields transform homogeneously under gauge transformations:

$$W_\alpha \to e^{-2i\Lambda} W_\alpha e^{+2i\Lambda}, \qquad \bar{W}_{\dot{\alpha}} \to e^{-2i\Lambda^\dagger} W_\alpha e^{+2i\Lambda^\dagger}. \tag{5.12}$$

Once again one can compute the component fields of W_α and $\bar{W}_{\dot{\alpha}}$ in the WZ gauge. Explicitly

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V_{WZ} + \frac{1}{2}\bar{D}^2(V_{WZ} D_\alpha V_{WZ}) - \frac{1}{4}\bar{D}^2 D_\alpha V_{WZ}^2, \tag{5.13}$$

and similarly for $\bar{W}_{\dot{\alpha}}$. After some algebra, the result is the expected non-abelian generalisation

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) + i\theta^\beta \sigma^{\mu\nu}_{\beta\alpha} F_{\mu\nu}(y) \\ &\quad +(\theta\theta)\sigma^\mu_{\alpha\dot{\beta}} \nabla_\mu \bar{\lambda}^{\dot{\beta}}(y), \\ \bar{W}_{\dot{\alpha}} &= i\bar{\lambda}_{\dot{\alpha}}(y^\dagger) + \bar{\theta}_{\dot{\alpha}} D(y^\dagger) - i\bar{\theta}_{\dot{\beta}} \bar{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}} F_{\mu\nu}(y^\dagger) \\ &\quad -(\bar{\theta}\bar{\theta})\bar{\sigma}^{\mu\dot{\beta}}_{\dot{\alpha}} \nabla_\mu \lambda_\beta(y^\dagger), \end{aligned} \tag{5.14}$$

where, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the non-abelian field strength and ∇_μ is the gauge covariant derivative, *e.g.* $\nabla_\mu \bar{\lambda} = \partial_\mu \bar{\lambda} + i[A_\mu, \bar{\lambda}]$.

> **Exercise:** *Show the transformation property (5.12) and work out the component field expressions (5.14).*

6 Wess-Zumino model, supersymmetry breaking

There are some properties that we saw before, but did not take proper notice of, are the following. First, the supersymmetry variation of the $\theta\theta\bar{\theta}\bar{\theta}$, *i.e.* the highest, component of a scalar superfield

$$\delta_\xi d(x) = \frac{i}{2}\partial_\mu \left(\xi\sigma^\mu \bar{\lambda}(x) - \eta(x)\sigma^\mu \bar{\xi}\right) \tag{6.1}$$

is a total (spacetime) derivative. Therefore

$$\int d^4x \delta_\xi d(x) = 0. \tag{6.2}$$

In case of a vector superfield, the variation is

$$\delta_\xi D(x) = i\partial_\mu\left(\xi\sigma^\mu\bar{\lambda}(x) - \lambda(x)\sigma^\mu\bar{\xi}\right). \tag{6.3}$$

Therefore, for *any* vector superfield $V(x,\theta,\bar{\theta})$

$$2\kappa\int d^4x\int d^4\theta\, V(x,\theta,\bar{\theta}), \tag{6.4}$$

where κ is a parameter, is invariant under $N=1$ supersymmetry transformation. With this observation, we immediately have a way of constructing actions which are invariant under ($N=1$ global) supersymmetry. As an example, let us consider the chiral superfield $\Phi(y,\theta)$, for which $\Phi^\dagger\Phi$ is a vector superfield. Therefore,

$$\int d^4x\int d^4\theta\,\Phi^\dagger\Phi \tag{6.5}$$

is a supersymmetric action.

Of course, (6.4) with V a vector superfield is also an example of a supersymmetric action. However, on dimensional ground we see that the parameter κ has mass dimension one, while the action (6.5) involving $\Phi^\dagger\Phi$ does not require a dimensionful parameter. Moreover (6.4) is linear in the component fields, while (6.5) is quadratic. Nevertheless, we shall have occassion to come back to (6.4) later.

Secondly, the supersymmetric variation of the $\theta\theta$, *i.e.* again the highest, component of a chiral superfield Φ

$$\delta_\xi F(x) = -\sqrt{2}i\partial_\mu\psi(x)\sigma^\mu\bar{\xi} - \partial_\mu\left(\sqrt{2}i\psi(x)\sigma^\mu\bar{\xi}\right) \tag{6.6}$$

is a also total derivative. Therefore

$$\int d^4x\left(\int d^2\theta\Phi(y,\theta) + \int d^2\bar{\theta}\Phi^\dagger(y^\dagger,\bar{\theta})\right) \tag{6.7}$$

is invariant under ($N=1$ global) supersymmetry for *any* chiral superfield Φ.

Consider, for example the chiral superfield Φ^2 leading to

$$m\int d^4x\int d^2\theta\,\Phi^2(y,\theta) + \text{h.c.}, \tag{6.8}$$

where, the complex parameter m has mass dimension one. As before we can, and shall, consider the effect of the term

$$\lambda\int d^4x\int d^2\theta\,\Phi(y,\theta) + \text{h.c.}, \tag{6.9}$$

with $[\lambda] = M^2$, later in this lecture.

The advantage of the superfield method lies in the fact that an action written in terms of superfields in manifestly invariant under supersymmetry transformations. Let us consider the Wess-Zumino model defined by the lagrangian density[8]

$$\mathcal{L}_{WZ} = \int d^4\theta\,\Phi^\dagger\Phi - \int d^2\theta\left(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right) + \text{h.c.} \tag{6.10}$$

[8] The Wess-Zumino action defines the most general unitary renormalisable supersymmetric theory for a single chiral superfield.

When expanded in terms of the component fields, the first term above contains the canonical kinetic terms for a complex scalar field ϕ and a Weyl fermion ψ, while the others describe interactions including Yukawa interactions (see exercise in lecture 4):

$$
\begin{aligned}
\mathcal{L}_{WZ} &= \eta^{\mu\nu}(\partial_\mu\phi^*)(\partial_\nu\phi) + F^*F - \left(m\phi + g\phi^2\right)F \\
&\quad - \left(m^*\phi^* + g^*(\phi^*)^2\right)F^* - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \\
&\quad + \frac{1}{2}(m\psi\psi + m^*\bar{\psi}\bar{\psi}) + g\phi\psi\psi + g^*\phi^*\bar{\psi}\bar{\psi}.
\end{aligned}
\tag{6.11}
$$

Notice that there is no term involving the derivatives of F. It is therefore an auxiliary field with algebraic equation of motion

$$
F^* - m\phi - g\phi^2 = 0. \tag{6.12}
$$

This is readily solved and we use this to eliminate F from the action. After eliminating the auxiliary field, the bosonic part of the action S_{WZ} takes the form

$$
\mathcal{L}^B_{WZ} = (\partial_\mu\phi^*)(\partial^\mu\phi) - \mathcal{V}(\phi), \tag{6.13}
$$

where the scalar potential

$$
\mathcal{V}(\phi) = F^*F = |m|^2\phi^*\phi + (m^*g\phi + mg^*\phi^*)\phi^*\phi + |g|^2(\phi^*\phi)^2. \tag{6.14}
$$

The potential is positive definite. Consequently, the hamiltonian and the total energy are also postive definite.

The last property is actually a general property of supersymmetric theories. We can prove this from the supersymmetry algebra (1.14). Explicitly, using

$$
\begin{aligned}
\{Q_1, \bar{Q}_1\} &= 2P_0 + 2P_3, \\
\{Q_2, \bar{Q}_2\} &= 2P_0 - 2P_3,
\end{aligned}
$$

the hamiltonian $\mathcal{H} \equiv P_0$ can be expressed as

$$
\mathcal{H} = \frac{1}{4}\left(Q_1\bar{Q}_1 + \bar{Q}_1Q_1 + Q_2\bar{Q}_2 + \bar{Q}_2Q_2\right). \tag{6.15}
$$

The expectation value for the energy density in any state $|\Psi\rangle$ is thus

$$
\begin{aligned}
\langle\Psi|\mathcal{H}|\Psi\rangle &= \frac{1}{4}\left(||\bar{Q}_1|\Psi\rangle||^2 + ||Q_1|\Psi\rangle||^2 + ||\bar{Q}_2|\Psi\rangle||^2 + ||Q_2|\Psi\rangle||^2\right) \\
&\geq 0,
\end{aligned}
\tag{6.16}
$$

always positive definite.

This implies that states with vanishing energy density are supersymmetric ground states of the theory. They are ground states because with zero energy we reach the minimum possible value of energy, and they are supersymmetric because

$$
\langle\Omega|\mathcal{H}|\Omega\rangle \;\Leftrightarrow\; \begin{matrix} Q_\alpha|\Omega\rangle = 0 \\ \bar{Q}_{\dot{\alpha}}|\Omega\rangle = 0 \end{matrix}\,, \quad \text{for all } \alpha, \dot{\alpha} = 1, 2. \tag{6.17}
$$

In other words, supersymmetry is preserved in ground states with zero energy. Conversely, in ground states with non-zero (positive, as always with supersymmetry) energy, supersymmetry is *spontanously broken*.

Coming back to the Wess-Zumino model, let us assume that m and g are real for simplicity. In this case, the scalar potential is

$$\mathcal{V}(\phi) = |F|^2 = m^2|\phi|^2 + mg(\phi + \phi^*)|\phi|^2 + g^2|\phi|^4, \tag{6.18}$$

while the auxiliary field is

$$F^* = m\phi + g\phi^2 = \Big(mA + g(A^2 - B^2)\Big) + i\,(mB + 2gAB)\,, \tag{6.19}$$

where, the complex scalar $\phi = A + iB$ is written in terms of two real fields. We see that F can be made to vanish for

- $\langle B \rangle = 0$ and $\langle A \rangle = 0, -(m/g)$, (for $m^2 > 0$);
- $\langle A \rangle = -(m/2g)$ and $\langle B \rangle = \pm\left(\sqrt{-m^2}/2g\right)$, (for $m^2 < 0$).

Since $F = 0$, (which guarantees that the potential $\mathcal{V}$ vanishes), supersymmetry is unbroken in the Wess-Zumino model. The supersymmetric vacuum states are parametrised by the above choices of vacuum expectation values for the scalar fields.

The superpotential of the Wess-Zumino model

$$W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \tag{6.20}$$

goes to $-W(\Phi)$ (modulo an unimportant constant shift) under the transformation

$$\Phi \to -\frac{m}{g} - \Phi$$

which interchanges the two ground states. However, the sign of $W(\Phi)$ can be rotated away by a $\mathrm{U}_R(1)$ symmetry

$$\theta \to e^{-i\alpha}\theta \qquad \Rightarrow \qquad d^2\theta \to e^{+2i\alpha}d^2\theta. \tag{6.21}$$

(Note that the above is unlike a commuting variable x for which x and dx transform in the same way. Recall that for anti-commuting variables, differentiation and intrgration are equivalent operations.) Suppose under this rotation

$$\Phi(x,\theta) \to \tilde{\Phi}(x,\theta) = e^{2in\alpha}\Phi(x, e^{-i\alpha}\theta), \tag{6.22}$$

i.e. Φ has charge n. The condition that the action remains invariant, ($\int d^2\theta\, W(\Phi) = \int d^2\theta\, W(\tilde{\Phi})$), implies that the superpotential must have charge $+2$ under $\mathrm{U}_R(1)$ rotation

$$W(\Phi) \to e^{2i\alpha}W(\Phi). \tag{6.23}$$

For $\alpha = \pi/2$, this is a discrete $\mathbf{Z}_4$ symmetry, (under which the fermions are multiplied by a factor of i), relating the two ground states. However since a 2π rotation changes the sign of fermions, a $\mathbf{Z}_2 \subset \mathbf{Z}_4$ remains unbroken.

It is not possible to break supersymmetry spontaneously in the Wess-Zumino model involving a single chiral superfield. It turns out that for this one needs at least three chiral superfields. Before exhibiting this model, let us write the general form of a Wess-Zumino type action involving many chiral superfields

$$S = \int d^4x \left[\int d^4\theta \, \delta_{ij} \Phi_i^\dagger \Phi_j - \left(\int d^2\theta \, W\left(\{\Phi_i\}\right) + \text{h.c.} \right) \right]. \tag{6.24}$$

The superpotential

$$W\left(\{\Phi_i\}\right) = w_0 + \lambda_i \Phi_i + m_{ij} \Phi_i \Phi_j + g_{ijk} \Phi_i \Phi_j \Phi_k + \cdots \tag{6.25}$$

is a general functional of the chiral superfields only, *i.e.* it is a functional only of the Φ_i's and not the $\Phi_i^\dagger$'s. In other words, $W(\Phi)$ is an *analytic* function of Φ.

The bosonic potential in this case is

$$\mathcal{V}\left(\{\phi_i, \phi_i^*\}\right) = \sum_i |F_i|^2 = \sum_i \left| \left[\frac{\delta W}{\delta \Phi_i} \right]_{\Phi_i = \phi_i} \right|^2 . \tag{6.26}$$

In fact the name *superpotential* for $W(\Phi)$ derives from its close relation to the (bosonic) potential. Let us stress once again that the superpotential is an analytic function of the chiral superfields Φ_i. Analytic functions are also called *holomorphic*. Therefore the superpotential is a *holomorphic* function. This is not only true of the bare lagrangian, but also of the effective one, (if supersymmetry is to remain unbroken). There is, however, some subtlety in the interpretation of the last statement, and we shall return to this point later. At that time, we shall see the power of requirement of analyticity (or holomorphy), and how, for many theories, it is sufficient to determine the effective superpotential *exactly* and study its consequence on the dynamics of the theory.

Let us now study a model of spontaneous supersymmetry breaking. Consider the lagrangian (first proposed by O'Raifeartaigh[12] in 1975):

$$\mathcal{L} = \int d^4\theta \sum_{i=1}^{3} \Phi_i^\dagger \Phi_i - \int d^2\theta \left(\lambda \Phi_1 + m \Phi_2 \Phi_3 + \frac{1}{2} g \Phi_1 \Phi_2^2 \right) + \text{h.c.} \tag{6.27}$$

Notice that there is a term linear in Φ. This is of the form that was mentioned in the beginning of this lecture.

The algebraic equations of motion for the auxiliary fields are

$$\begin{aligned} F_1^* &= \lambda + \frac{1}{2} g \phi_2^2, \\ F_2^* &= m\phi_3 + g\phi_1\phi_2, \\ F_3^* &= m\phi_2. \end{aligned} \tag{6.28}$$

Clearly, F_1^* and F_3^* cannot be made to vanish simultaneously. The resulting scalar potential is

$$\begin{aligned} \mathcal{V} &= |F_1|^2 + |F_2|^2 + |F_3|^2 \\ &= \lambda^2 + (m^2 + \lambda g)\,(\text{Re } \phi_2)^2 + (m^2 - \lambda g)\,(\text{Im } \phi_2)^2 \\ &\quad + m^2 |\phi_3|^2 + 2mg(\phi_1 \phi_2 \phi_3^* + \phi_1^* \phi_2^* \phi_3) \\ &\quad + g^2 |\phi_1|^2 |\phi_2|^2 + \frac{1}{4} g^2 \left(|\phi_2|^2 \right)^2 . \end{aligned} \tag{6.29}$$

This is positive as long as $m^2 \geq \lambda g$. The potential (6.29) is minimised by choosing

$$\langle \phi_2 \rangle = 0, \quad \langle \phi_3 \rangle = 0, \quad \langle \phi_1 \rangle = \text{unconstrained}.$$

The value of the potential at the minimum is

$$\mathcal{V}(\langle \phi_1 \rangle) = \lambda^2,$$

a positive definite quantity. Therefore supersymmetry is *spontaneously broken*.

> **Exercise:** *Compute the masses of the bosons and fermions in the O'Raifeartaigh model. There is at least one massless fermion when supersymmetry is spontaneously broken. This is analogous to the massless boson (Goldstone mode) that appears when a global symmetry is spontaneously broken, and is called a* goldstino*).*
>
> *Also show that the sum of the mass-square of the bosons equal to that of the fermions. This too is a generic feature of supersymmetric models, the property being manifest in vacua with unbroken supersymmetry.*

For $\lambda = 0$, there is no supersymmetry breaking, but this case illustrates another important feature of supersymmetric models. Namely, we have

$$\mathcal{V}(\langle \phi_1 \rangle, \langle \phi_2 \rangle = 0, \langle \phi_3 \rangle = 0) = 0 \tag{6.30}$$

for arbitrary values of $\langle \phi_1 \rangle$. There are infinitely many supersymmetric vacua parametrised by the vacuum expectation value of the field ϕ_1. Thus the parameter space that labels supersymmetric vacua is the complex $\langle \phi_1 \rangle$-plane (see Fig.1(a)).

Figure 1: Moduli space of supersymmetric vacua (a) for potential (6.30) and (b) for superpotential (6.31) with singularity at the origin.

Let us further restrict to $m = 0$, *i.e.* we consider the simple model of two chiral superfields Φ_1 and Φ_2 with superpotential

$$W(\Phi_1, \Phi_2) = \frac{1}{2}\Phi_1 \Phi_2^2. \tag{6.31}$$

This corresponds to the bosonic potential

$$\mathcal{V}(\phi_1, \phi_2) = g^2 |\phi_1|^2 |\phi_2|^2 + \frac{1}{4} g^2 \left(|\phi_2|^2 \right)^2 .$$

We see that $\mathcal{V}(\phi_1, \phi_2)$ vanishes for $\langle\phi_2\rangle = 0$ with no condition on $\langle\phi_1\rangle$. In the vacuum labelled by $\langle\phi_1\rangle$ the field ϕ_2 gets an effective mass

$$m_2 = \sqrt{2}g|\langle\phi_1\rangle|.$$

The origin of the parameter space is therefore a special point — the mass of the field ϕ_2 vanishes there. This is a *singular point* (or a *singularity*) in the sense that a heavy field becomes massless here.

The name for the parameter space that labels supersymmetric vacua is the *moduli space of vacua*. Fig.1(b) is the moduli space of vacua for the model described by (6.31).

> **Exercise:** *Analyse the behaviour of the moduli space of vacua for a theory of three chiral superfields Φ_1, Φ_2 and Φ_3 with superpotential*
>
> $$W(\Phi_1, \Phi_2, \Phi_3) = g\,\Phi_1\Phi_2\Phi_3.$$
>
> *The moduli space is shown in Fig.2. It consists of three branches meeting at a singular point.*

Figure 2: Moduli space of supersymmetric vacua described by the superpotential $W = g\,\Phi_1\Phi_2\Phi_3$. The origin is a singular point.

7 Lagrangians of supersymmetric gauge theories

In the last lecture, we saw how to write actions for chiral superfields leading to supersymmetric actions involving scalars and spin-half fermions. Constructing an action for a gauge theory is also along expected lines. We have the field strength supermultiplet which is a chiral spinor superfield W_α such that $\bar{D}_{\dot{\alpha}} W_\beta = 0$. Moreover, (in the non-abelian case[9]), it transforms homogeneously

$$W_\alpha \to e^{-2i\Lambda} W_\alpha e^{+2i\Lambda}.$$

Therefore,

$$\mathrm{tr}\,(W^\alpha W_\alpha) \to \mathrm{tr}\left(e^{-2i\Lambda} W^\alpha W_\alpha e^{+2i\Lambda}\right) = \mathrm{tr}\,(W^\alpha W_\alpha)$$

[9] In the abelian case, W_α is gauge invariant and no trace need to be taken.

is both gauge and Lorentz invariant. Further, since this is a chiral superfield, we can get a gauge invariant candidate term for the lagrangian by integrating over θ's: $\int d^2\theta\, \mathrm{tr}\,(W^\alpha W_\alpha)$. In terms of the component fields

$$\int d^2\theta\, \mathrm{tr}\,(W^\alpha W_\alpha) \;=\; \mathrm{tr}\Big(D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{i}{4}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \\ -2i\lambda^\alpha \sigma^\mu_{\alpha\dot\beta}\nabla_\mu \bar\lambda^{\dot\beta}\Big).$$

The problem with the above is that there is no dependence on the gauge coupling. Also the $F\tilde{F}$ term is imaginary. Both these problems are solved by defining the lagrangian

$$\mathcal{L}_{SYM} = \frac{1}{8\pi}\mathrm{Im}\left[\tau \int d^2\theta\, \mathrm{tr}\,(W^\alpha W_\alpha)\right], \tag{7.1}$$

where,

$$\tau = \frac{\theta_{YM}}{2\pi} + i\,\frac{4\pi}{g^2}. \tag{7.2}$$

Explicitly, in terms of component fields, the above lagrangian reads as follows

$$\mathcal{L}_{SYM} = \frac{1}{g^2}\,\mathrm{tr}\left(D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - i\lambda^\alpha \sigma^\mu_{\alpha\dot\beta}\nabla_\mu \bar\lambda^{\dot\beta}\right) \\ -\frac{\theta_{YM}}{32\pi^2}\; tr\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right), \tag{7.3}$$

where, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength.

Notice that in addition to the gauge fields A_μ, the supersymmetric Yang-Mills theory contains 'matter' fields: fermions λ and auxiliary fields D, both in the *adjoint* representation. The fermions λ^a are supersymmetric partners of the gauge fields, and are called *gluinos* or *gauginos*.

As we had mentioned in the last lecture, it is possible to add a term

$$2\kappa \int d^4x \int d^4\theta V(x,\theta,\bar\theta) = \kappa \int d^4x D(x).$$

However, this is gauge invariant only for an abelian vector superfield V. Therefore such a term may be added only to the lagrangian of the supersymmetric Maxwell theory. It turns out that this term leads to spontaneous breaking of supersymmetry[13]. The parameter κ is known as the Fayet-Iliopoulos parameter.

After having constructed actions for chiral superfields (matter) and vector superfields (gauge fields), we shall now discuss the coupling of matter to gauge fields. We shall begin with the coupling of matter to an abelian gauge field.

The first point to notice is that we cannot put the matter fields in the gauge supermultiplet. This is because all fields in the this multiplet must belong to the same representation as the gauge fields, *i.e.* in the abelian case they must all be charge neutral and belong to the adjoint in the non-abelian case.

Consider the chiral superfields Φ_i, $i = 1, 2, \cdots, n$. Under a global U(1) rotation

$$\Phi_i \to {\Phi_i}' = e^{-2iq_i\lambda}\Phi_i,$$

where, q_i and λ are real constants. (In particular, $\bar{D}_{\dot{\alpha}} q_i = 0$, $\bar{D}_{\dot{\alpha}} \lambda = 0$. Therefore, $\Phi_i{}'$ is also a chiral superfield.) The Wess-Zumino type lagrangian

$$\mathcal{L} = \int d^4\theta \, \Phi_i^\dagger \Phi_i - \int d^2\theta \left(m_{ij} \Phi_i \Phi_j + g_{ijk} \Phi_i \Phi_j \Phi_k + \text{h.c.} \right)$$

is invariant under this rotation if

$$\begin{aligned} m_{ij} &= 0 \quad \text{whenever } q_i + q_j \neq 0, \\ g_{ijk} &= 0 \quad \text{whenever } q_i + q_j + q_k \neq 0. \end{aligned}$$

If we want to gauge this symmetry, *i.e.* make λ a function of x, since $\bar{D}_{\dot{\alpha}} \lambda(x) \neq 0$, we must promote the function $\lambda(x)$ to a chiral superfield $\Lambda(x,\theta)$ ($\bar{D}_{\dot{\alpha}} \Lambda = 0$) such that $\Phi_i{}' = e^{-2iq_i\Lambda} \Phi_i$ is again a chiral superfield. This makes the superpotential gauge invariant. However, the kinetic term

$$\Phi_i^\dagger \Phi_i \to \Phi_i^\dagger e^{2iq_i(\Lambda^\dagger - \Lambda)} \Phi_i$$

is no longer invariant. This is familiar from gauging of non-supersymmetric theories. In order to restore invariance under local phase rotations, we need to introduce a vector superfield $V(x,\theta,\bar{\theta})$ such that

$$\begin{aligned} e^V \to e^{V'} &= e^{-2i\Lambda^\dagger} e^V e^{2i\Lambda} \\ i.e, \quad V \to V' &= V + 2i(\Lambda - \Lambda^\dagger). \end{aligned} \tag{7.4}$$

The gauge invariant lagrangian is then

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \int d^2\theta \, W^\alpha W_\alpha + \int d^4\theta \, \Phi_i^\dagger e^{2q_i V} \Phi_i \\ &\quad - \int d^2\theta \left(m_{ij} \Phi_i \Phi_j + g_{ijk} \Phi_i \Phi_j \Phi_k \right) + \text{h.c.} \end{aligned} \tag{7.5}$$

Due to the presence of the exponential in the lagrangian, it is not clear if the theory is renormalisable. We can, however, evaluate it in the Wess-Zumino gauge where

$$\Phi^\dagger e^{2qV_{WZ}} \Phi = \Phi^\dagger \left(1 + 2qV_{WZ} + 2q^2 V_{WZ}^2 \right) \Phi.$$

In components

$$\begin{aligned} \int d^4\theta \, \Phi^\dagger e^{2qV_{WZ}} \Phi &= \eta^{\mu\nu} \left(\nabla_\mu \phi \right)^* \left(\nabla_\nu \phi \right) - i\bar{\psi} \bar{\sigma}^\mu \nabla_\mu \psi \\ &\quad + i\sqrt{2} q \left(\phi^* \lambda \psi - \phi \bar{\lambda} \psi \right) \\ &\quad + |F|^2 + qD\phi^*\phi, \end{aligned} \tag{7.6}$$

where,

$$\begin{aligned} \nabla_\mu \phi &= \partial_\mu \phi + iqA_\mu \phi \\ \nabla_\mu \psi &= \partial_\mu \psi + iqA_\mu \psi. \end{aligned}$$

The supersymmetric generalisation of QED requires at least two chiral superfields Φ_+ and Φ_- so that we may write a gauge invariant mass term. Therefore, we have

$$\begin{aligned} &\text{chiral superfields} \qquad \Phi_\pm \to e^{\mp 2ie\Lambda} \Phi_\pm, \\ &\text{vector superfield} \qquad e^{2V} \to e^{-2ie\Lambda^\dagger} e^{2V} e^{2i\Lambda}. \end{aligned}$$

The lagrangian in superfields is

$$\begin{aligned}\mathcal{L}_{SQED} &= \frac{1}{2}\int d^2\theta\, W^\alpha W_\alpha + \int d^4\theta \left(\Phi_+^\dagger e^{2eV}\Phi_+ + \Phi_-^\dagger e^{-2eV}\Phi_-\right) \\ &\quad -m\left(\int d^2\theta\, \Phi_+\Phi_- + \int d^2\bar\theta\, \Phi_+^\dagger\Phi_-^\dagger\right),\end{aligned} \tag{7.7}$$

and in components

$$\begin{aligned}\mathcal{L}_{SQED} &= \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar\lambda\bar\sigma^\mu\partial_\mu\lambda \\ &\quad +\eta^{\mu\nu}(\nabla_\mu\phi_+)^*(\nabla_\nu\phi_+) + \eta^{\mu\nu}(\nabla_\mu\phi_-)^*(\nabla_\nu\phi_-) \\ &\quad -i\bar\psi_+\bar\sigma^\mu\nabla_\mu\psi_+ - i\bar\psi_-\bar\sigma^\mu\nabla_\mu\psi_- + m\left(\psi_+\psi_- + \bar\psi_+\bar\psi_-\right) \\ &\quad +i\sqrt{2}e\left(\phi_+^*\lambda\psi_+ - \phi_+\bar\lambda\bar\psi_+ - \phi_-^*\lambda\psi_- - \phi_-\bar\lambda\bar\psi_-\right) \\ &\quad -m\left(\phi_+F_- + \phi_-F_+ + \phi_+^*F_-^* + \phi_-^*F_+^*\right) \\ &\quad +|F_+|^2 + |F_-|^2 + eD\left(\phi_+^*\phi_+ - \phi_-^*\phi_-\right).\end{aligned} \tag{7.8}$$

Exercise: *Re-express the above lagrangian in terms of the Dirac spinor* $\Psi_D = \begin{pmatrix}\psi_+ \\ \bar\psi_-\end{pmatrix}$. *What are the additional fields in the lagrangian compared to QED?*

Notice that the lagrangian of supersymmetric QED (7.8) contains two complex auxiliary fields $F_\pm$ and one real auxiliary field D. Their (algebraic) equations of motions are

$$\begin{aligned}D + e\left(\phi_+^*\phi_+ - \phi_-^*\phi_-\right) &= 0, \\ F_\pm^* &= m\phi_\mp.\end{aligned}$$

(The RHS of the first equation above is $-\kappa$ in case the Fayet-Iliopoulos term (6.4) is present.) These equations can be solved to determine the classical scalar potential

$$\begin{aligned}\mathcal{V}(\phi_\pm) &= \left(|F_+|^2 + |F_-|^2\right) + \frac{1}{2}D^2 \\ &= m^2\left(|\phi_+|^2 + |\phi_-|^2\right) \\ &\quad +\frac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2.\end{aligned} \tag{7.9}$$

The two terms above are called the F- and D-term respectively.

In the massless case, we only have the D-term. Therefore, there are infinitely many degenerate vacua labelled by, upto gauge equivalence,

$$\langle\phi_+\rangle = a = \langle\phi_-\rangle,$$

for any complex number a.

In any vacuum with $a \neq 0$, gauge symmetry is broken by the (super) Higgs mechanism. The gauge superfield becomes massive by absorbing one chiral superfield degree of freedom from the matter. One of the chiral superfield degrees of freedom, however, still remains massless. A gauge invariant description of this massless degree of freedom can be given in terms of

$$X = \Phi_+\Phi_-. \tag{7.10}$$

In the vacuum, $\langle X\rangle = a^2$ is an arbitrary complex number. This means that there is no superpotential for the chiral superfield X, at least at the classical level:

$$W_{cl}(X) = 0.$$

The above is known as the *D-flatness condition*.

The vacuum expectation value $\langle X\rangle$ gives a gauge invariant parametrisation of the moduli space of classical vacua. This space has a singularity at the origin, *i.e.* at $\langle X\rangle = 0$, corresponding to the fact that at this point the U(1) gauge symmetry is unbroken, and all the original microscopic degrees of freedom are massless there (see Fig.1(b)).

The degeneracy of the classical vacua is accidental. There is no symmetry that relates different vacua parametrised by different choices for a, indeed they are inequivalent theories. This degeneracy may therefore be lifted in the quantum theory by a dynamically generated superpotential $W_{eff}(X)$.

Recall that X is gauge invariant because $\phi_\pm$ carry equal and opposite charges

$$\phi_\pm \to e^{\mp ie\lambda}\phi_\pm \qquad \Rightarrow \qquad X \to X.$$

This freedom in defining X can formally be extended to a complexification of the gauge group $U^{\mathbf{C}}(1) \approx \mathbf{C}^*$, under which

$$\phi_+ \to \rho\phi_+, \qquad \phi_- \to \rho^{-1}\phi_-; \qquad (\rho \in \mathbf{C}^*).$$

(In the above, we have extended the gauge parameter to take non-zero complex values.) The moduli space of vacua was described by setting the (D-term of the) classical superpotential to zero, and quotienting by the gauge symmetry. This turns out to be exactly equivalent to quotienting by the compexified gauge group. Thus, the space of chiral superfields modulo the complexified gauge group may be parametrised by the gauge invariant polynomials of chiral superfields[14]. (In a more general situation, there may be relations between the possible gauge invariant polynomials.)

8 Supersymmetric QCD — classical theory

Much of what we learnt in Lecture 7 generalises to the case of non-abelian gauge theories. In the present lecture, we shall discuss that generalisation.

Consider a chiral superfield $\Phi = \{\Phi_i\}$ which belong to some representation R of a Lie algebra $\mathbf{g}$:

$$\Phi_i \to {\Phi_i}' = \left(e^{-2i\Lambda}\right)_i^{\ j}\Phi_j = \left(e^{-2i\Lambda^a t_i^{aj}}\right)\Phi_j, \tag{8.1}$$

where, $||t^a_{ij}||$ with $a = 1, 2, \cdots, \dim \mathbf{g}$, $i, j = 1, 2, \cdots, \dim R$, are matrices in the representation R. The gauge invariant kinetic term for the field Φ is

$$\left(\Phi^\dagger e^{2V}\Phi\right) = \Phi^{\dagger i}\left(e^{2V^a t^a}\right)_i^{\ j}\Phi_j = \mathrm{tr}_R\left(e^{2V^a t^a}\Phi\Phi^\dagger\right).$$

This is gauge invariant because the tensor product of the representations R, its conjugate $\bar{R}$ and the adjoint contains the singlet.

More generally the lagrangian of a Wess-Zumino type model interacting with a non-abelian gauge field is

$$\begin{aligned}\mathcal{L} \ &= \ \frac{1}{8\pi}\mathrm{Im}\left[\tau\int d^2\theta\,\mathrm{tr}\left(W^\alpha W_\alpha\right)\right] + \int d^4\theta\left(\Phi_I^\dagger e^{2V}\Phi_I\right) \\ &\quad - \int d^2\theta\left(m_{IJ}\Phi_I\Phi_J + g_{IJK}\Phi_I\Phi_J\Phi_K\right) + \text{h.c.}\end{aligned} \tag{8.2}$$

The 'mass' term is allowed only when $R_I = \bar{R}_J$. For a single chiral superfield, this in only possible for SU(2) (of all SU(n)'s). Similarly the g_{IJK} term is allowed if $R_I \otimes R_J \otimes R_K$ contains the singlet. For example, a single chiral superfield in the doublet of SU(2) can have a mass term but not a cubic self coupling.

Let us now look at the D-terms in the lagrangian

$$\begin{aligned} \mathcal{L}_D &= \frac{1}{2g^2}\,\mathrm{tr}\,D^2 + \phi^* D\phi \\ &= \frac{1}{2g^2}\,D^a D^b\,\mathrm{tr}\,(t^a t^b) + D^a \phi^{*i} t_i^{aj}\phi_j \\ &= \frac{1}{2g^2}\,C(R) D^a D^a + D^a\,\mathrm{tr}_R\,(t^a(\phi\phi^*))\,, \end{aligned} \tag{8.3}$$

where we have used the normalisation (5.8). The equation of motion for D^a is therefore

$$D^a = -\,\frac{g^2}{C(R)}\,\mathrm{tr}_R\,(t^a(\phi\phi^*))\,, \tag{8.4}$$

which leads to a scalar potential

$$\mathcal{V}_D = \left[\frac{g^2}{C(R)}\,\mathrm{tr}_R\,(t^a(\phi\phi^*))\right]^2 . \tag{8.5}$$

This potential must vanish in a vacuum which preserves supersymmetry. Thus, for unbroken supersymmetry, we require

$$\mathrm{tr}_R\,(t^a(\phi\phi^*)) = 0, \tag{8.6}$$

the D-flatness condition.

The supersymmetric version of QCD that we shall consider has

- N_f chiral superfields in the fundamental representation $\mathbf{N}_c$ of the gauge group SU(N_c):

  $$Q_i^A, \quad A = 1, 2, \cdots, N_f;\ i = 1, 2, \cdots, N_c.$$

  (The colour index will not always be displayed explicitly.) There is a global U(N_f) *flavour* symmetry between the Q's which transform in the $\mathbf{N}_f$ representation. The conjugate anti-chiral superfield $Q^\dagger$ belongs to $\overline{\mathbf{N}}_f$ of U(N_f) and $\overline{\mathbf{N}}_c$ of SU(N_c). We shall think often think of $||Q_i^A||$ and $||Q_{\tilde{A}}^{\dagger\tilde{i}}||$ as $N_f \times N_c$ matrices.

- N_f chiral superfields $\tilde{Q}_{\tilde{A}}$ in the anti-fundamental $\overline{\mathbf{N}}_c$ of the gauge group SU(N_c). These transform as $\overline{\mathbf{N}}_f$ of the flavour group. The corresponding conjugate fields denoted by $\tilde{Q}^\dagger$.

Their dynamics is specified by the lagrangian

$$\begin{aligned} \mathcal{L}_{SQCD} &= \frac{1}{8\pi}\mathrm{Im}\left[\tau\int d^2\theta\,\mathrm{tr}\,(W^\alpha W_\alpha)\right] \\ &\quad + \int d^4\theta\left[Q_{\tilde{A}}^\dagger e^{2V} Q^A + \tilde{Q}_{\tilde{A}} e^{2V}\tilde{Q}^{\dagger A}\right] \\ &\quad - \int d^2\theta\, m_A\, Q^A Q_{\tilde{A}} - \int d^2\bar{\theta}\, m_A^*\,\tilde{Q}^{\dagger A} Q_{\tilde{A}}^\dagger . \end{aligned} \tag{8.7}$$

This theory has a global $\mathrm{U}(N_f)$ *flavour* symmetry. In the massless case, $m_A = m_A^* = 0$ there are actually two flavour rotations corresponding to the fact that Q and $\tilde{Q}$ can be transformed independently in flavour space, leading to a $\mathrm{U}_\ell(N_f)\times\mathrm{U}_r(N_f)$ global symmetry for the classical theory. In the following, we shall consider the massless case.

The D-flatness conditions are

$$\sum_{A=1}^{N_f}\left[\mathrm{tr}_{\mathbf{N}_c}\left(t^a\phi_{Q^A}\phi^*_{Q^A}\right)-\mathrm{tr}_{\overline{\mathbf{N}}_c}\left(t^a\phi^*_{\tilde{Q}^A}\phi_{Q^A}\right)\right]=0, \tag{8.8}$$

for all $a = 1, 2, \cdots, N_c^2 - 1$. Since the representations $\mathbf{N}_c$ and $\overline{\mathbf{N}}_c$ are isomorphic, the two traces are the same and the D-flatness conditions reduce to

$$\sum_{A=1}^{N_f}\mathrm{tr}_{\mathbf{N}_c}\left[t^a\left(\phi_{Q^A}\phi^*_{Q^A}-\phi^*_{\tilde{Q}^A}\phi_{Q^A}\right)\right]=0,\quad \text{for all } a.$$

By Schur's lemma, this is possible when

$$\phi_{Q^A}\phi^*_{Q^A}-\phi^*_{\tilde{Q}^A}\phi_{\tilde{Q}^A}=c\,\mathbf{1}_{N_c\times N_c}, \tag{8.9}$$

where, c is a constant independent of A.

To characterise classical vacua, we need to differentiate between two cases

- $N_f < N_c$:
  By a gauge and flavour rotation, we can 'diagonalise'[10] ϕ_Q. This implies that the constant on the RHS of (8.9) must vanish. Upto gauge and flavour rotations, the solution to the D-flatness conditions are therefore given by

$$\langle\phi_Q\rangle=\langle\phi^*_{\tilde{Q}}\rangle=\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & a_2 & & & & & \\ \vdots & & \ddots & & & & \\ 0 & & & a_{N_f} & 0 & \cdots & 0\end{pmatrix}. \tag{8.10}$$

  The gauge invariant composite superfields whose lowest components parametrise the moduli space of vacua are

$${M^A}_{\tilde{B}}=Q^A\tilde{Q}_{\tilde{B}},\quad A,\tilde{B}=1,2,\cdots,N_f, \tag{8.11}$$

  called the *meson* superfields. The vacuum expectation value of $\langle M^A_{\tilde{B}}\rangle = |a_A|^2\delta_{A\tilde{B}}$. When this is non-zero, *i.e.* $\langle M^A_{\tilde{B}}\rangle \neq 0$, the gauge group is broken. The most generic behaviour is $\mathrm{SU}(N_c) \to \mathrm{SU}(N_c - N_f)$. (Notice that given the vev of M, one can determine, upto gauge and flavour rotations, the vevs of Q and $\tilde{Q}$ and vice versa.)

- $N_f \geq N_c$:
  An arbitrary A-independent constant is now allowed, so upto gauge and flavour sym-

[10] Recall that ϕ_Q is an $N_f \times N_c$ matrix, so the diagonalisation refers to the $N_f \times N_f$ block.

metry, a solution to the D-flatness conditions is

$$\begin{aligned}
\langle \phi_Q \rangle &= \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ 0 & & & a_{N_c} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix}, \\
\langle \phi^*_{\tilde{Q}} \rangle &= \begin{pmatrix} \tilde{a}_1 & 0 & \cdots & 0 \\ 0 & \tilde{a}_2 & & \\ \vdots & & \ddots & \\ 0 & & & \tilde{a}_{N_c} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix},
\end{aligned} \tag{8.12}$$

together with the restriction $|a_A|^2 - |\tilde{a}_A|^2 = c$ (a constant independent of A). Once again we have the meson chiral superfields

$$\begin{aligned}
M^A{}_{\tilde{B}} &= Q^A \tilde{Q}_{\tilde{B}}, \\
||\langle M^A_{\tilde{B}} \rangle|| &= \begin{pmatrix} a_1\tilde{a}_1 & & & \\ & a_2\tilde{a}_2 & & \\ & & \ddots & \\ & & & a_{N_c}\tilde{a}_{N_c} \end{pmatrix}.
\end{aligned} \tag{8.13}$$

This time, however, it is not possible to determine a and $\tilde{a}$ from the knowledge of $\langle M \rangle$. Moreover, there are new gauge invariant composite operators. For example, in the case $N_f = N_c$, there are two such operators

$$\begin{aligned}
B &= \epsilon_{A_1 A_2 \ldots A_{N_c}} Q_1^{A_1} \cdots Q_{N_c}^{A_{N_c}} \\
&= \frac{1}{N_c!} \epsilon_{A_1 A_2 \ldots A_{N_c}} \epsilon^{B_1 B_2 \ldots B_{N_c}} Q_{B_1}^{A_1} \cdots Q_{B_{N_c}}^{A_{N_c}}, \\
\tilde{B} &= \epsilon_{\tilde{A}_1 \tilde{A}_2 \ldots \tilde{A}_{N_c}} \tilde{Q}_1^{\tilde{A}_1} \cdots \tilde{Q}_{N_c}^{\tilde{A}_{N_c}} \\
&= \frac{1}{N_c!} \epsilon_{\tilde{A}_1 \tilde{A}_2 \ldots \tilde{A}_{N_c}} \epsilon^{\tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_{N_c}} \tilde{Q}_{\tilde{B}_1}^{\tilde{A}_1} \cdots \tilde{Q}_{\tilde{B}_{N_c}}^{\tilde{A}_{N_c}},
\end{aligned} \tag{8.14}$$

called the *baryons*. The moduli space of vacua is parametrised by $\langle M \rangle$, B and $\tilde{B}$. However, they are not all independent, but related through the relation

$$\det ||\langle M \rangle|| = B\tilde{B}. \tag{8.15}$$

Likewise, *e.g.* for $N_f = N_c + 1$, we have $2N_f$ baryons

$$\begin{aligned}
B_A &= \epsilon_{A A_1 A_2 \ldots A_{N_c}} Q_1^{A_1} \cdots Q_{N_c}^{A_{N_c}} \\
\tilde{B}^{\tilde{A}} &= \epsilon^{\tilde{A} \tilde{A}_1 \tilde{A}_2 \ldots \tilde{A}_{N_c}} \tilde{Q}^1_{\tilde{A}_1} \cdots \tilde{Q}^{N_c}_{\tilde{A}_{N_c}},
\end{aligned} \tag{8.16}$$

which satisfy the following relations

$$\begin{aligned}(\det M)\left(M^{-1}\right)_A^{\tilde{B}} - B_A\tilde{B}^{\tilde{B}} &= 0,\\ M_{\tilde{B}}^A B_A \;=\; M_{\tilde{B}}^A \tilde{B}^{\tilde{B}} &= 0.\end{aligned} \qquad (8.17)$$

This pattern is easily generalisable to higher values of N_f.

Now we would like to know how the quantum theory behaves. We shall approach this question by analysing the low energy effective action involving the degrees of freedom which are light at the energy scale in which we are interested.

We shall assume that supersymmetry is unbroken, (*i.e.* we shall work above the possible supersymmetry breaking scale). This symmetry of effective action will be made manifest by working in terms of superfields. Matter fields are combined into chiral superfields Φ_i (and their conjugates $\Phi_i^\dagger$), while gauge field is described by real/vector superfield V or its descendent chiral spinor superfield W_α.

As we have seen, the moduli space of classical vacua of a theory is parametrised by the vacuum expectation values of (the lowest componenent of) the chiral superfields. The classical superpotential for these fields either vanish, or are determined from the classical lagrangian. We would like to know whether quantum corrections generate an effective (super)potential or change its form from the classical one.

Let us write the effective superpotential as

$$\int d^2\theta\, W_{\mathit{eff}}(\{\Phi_i\},\{g_I\},\Lambda),$$

where $\{g_I\}$ are the coupling constants and Λ is the dynamically generated QCD scale. Recall that Λ is determined from

$$\Lambda^{b_0} = \mu^{b_0}\exp\left(-\frac{8\pi^2}{g^2(\mu)}\right),$$

where $b_0 = 3C_2(G) - C_2(R)$ is the coefficient in the lowest order β-function

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\frac{b_0}{16\pi^2}\, g^3.$$

However, in supersymmetric gauge theories, it is more convenient to *complexify* Λ by defining

$$\Lambda^{b_0} = \mu^{b_0}\,\exp\left(-\frac{8\pi^2}{g^2(\mu)} + i\theta_{YM}\right) = \mu^{b_0}\,\exp\left(2\pi i\tau(\mu)\right). \qquad (8.18)$$

(See (7.2) for the definition of τ.)

We have already noticed that the superpotential must be a holomorphic function of the chiral superfields Φ_i, (no explicit dependence on $\Phi_i^\dagger$). Seiberg[16] proposed that the dependence of W_{eff} on $\{g_I\}$ (and Λ when applicable) is also holomorphic. This can be motivated by thinking of the couplings $\{g_I\}$ as the vacuum expectation value of some background chiral superfields $G_I(= g_I + \cdots)$. This conjecture is further motivated from string theory where the coupling constants are actually vacuum expectation values of some chiral superfields. However, within the context of field theory alone its justification is *a posteriori.*

Moreover, if the theory possesses some symmetry in the absence of the superpotential W, *i.e.* the symmetry is broken by the terms in W, then one can formally recover the symmetry by assigning suitable transformation laws on $\{g_I\}$ and Λ, such that $W_{\mathit{eff}}(\{\Phi\}, \{g_I\}, \Lambda)$ is invariant under the combined transformations of $\{\Phi\}$, $\{g_I\}$ and Λ.

With these assumptions, it turns out that it is often possible to determine the effective superpotential *exactly*.

9 Quantum corrections & effective action

In the previous lecture we sketched an argument, (due to Seiberg), to determine the effective superpotential of a quantum theory. Let us recall that strategy.

1. Ascertain all the symmetries of a theory in the absence of the superpotential. Assign suitable (formal) transformation properties to the coupling constants such that the symmetry remains valid even in the presence of the superpotential.

2. Regard the quantum corrected effective superpotential to be a holomorphic function not only of the light chiral superfields, but also of the coupling constants $\{g_I\}$ (and Λ in the case of supersymmetric QCD).

3. The effective superpotential should match the results of perturbation theory in the appropriate limit.

These requirements are often stringent enough to determine the effective superpotential. The superpotential so determined will also include non-perturbative corrections.

Parenthetically, let us add that the kinetic term of the gauge fields

$$\int d^2\theta \, \mathrm{Im}\left(\tau_{\mathit{eff}}(\{\Phi\}, \{g_I\}, \Lambda)\, W^\alpha W_\alpha\right)$$

is also holomorphic. More precisely, τ_{eff} is holomorphic in its arguments. However, in many cases it so happens that the gauge symmetry is broken and the theory is in a Higgs or a confining phase. In case gauge symmetry remains unbroken and one is in the Coulomb phase, the same arguments may be applied to determine τ_{eff}.

As an illustration, let us apply these ideas to determine the quantum effective superpotential of the Wess-Zumino model (6.10). The classical superpotential (6.20) is

$$W(\Phi) = \frac{1}{2}\, m\Phi^2 + \frac{1}{3}\, g\Phi^3.$$

There is a U(1) R-symmetry that we encountered earlier in eqns.(6.21) and (6.23). It rotates the fermionic component of the superspace

$$\theta \to \theta' = e^{-i\alpha}\theta,$$

and under this transformation the action is invariant if

$$W(\Phi) \to e^{2i\alpha}\, W(\Phi).$$

Since the superpotential has R-charge $+2$, Φ, m and g transform under $\mathrm{U}_R(1)$ as follows:

$$\begin{array}{rcl} \Phi(x,\theta) & \rightarrow & \Phi(x, e^{-i\alpha}\theta) \\ m & \rightarrow & e^{2i\alpha} m \\ g & \rightarrow & e^{2i\alpha} g. \end{array} \tag{9.1}$$

The kinetic term in (6.10) is also invariant another U(1) transformation — let us call it S-symmetry:

$$\Phi(x,\theta) \rightarrow e^{i\beta}\Phi(x,\theta). \tag{9.2}$$

The superpotential (6.20) breaks this invariance. However, we can formally recover it by assigning the following transformation rules to the couplings

$$\begin{array}{rcl} m & \rightarrow & e^{-2i\beta} m, \\ g & \rightarrow & e^{-3i\beta} g. \end{array} \tag{9.3}$$

We now demand that W_{eff} is invariant under both $\mathrm{U}_R(1)$ as well as $\mathrm{U}_S(1)$, and also that it is holomorphic in m and g (and of course Φ). (In case m and g are real, we shall formally regard them as complex and set them to their real values at the end.) The most general form of W_{eff} consistent with these requirements is

$$\begin{array}{rcl} W_{\mathit{eff}}(\Phi, m, g) & = & g^a m^b \Phi^c \;\xrightarrow{R}\; e^{i\alpha(2a+2b)} W_{\mathit{eff}}(\Phi, m, g) \\ & & \;\xrightarrow{S}\; e^{i\alpha(-3a-2b+c)} W_{\mathit{eff}}(\Phi, m, g). \end{array}$$

Invariance under the two U(1)'s determines (say) a and b in terms of c. The most general form W_{eff} is then

$$\begin{array}{rcl} W_{\mathit{eff}}(\Phi, m, g) & = & \sum_{n=0}^{\infty} A_n g^{n-2} m^{3-n} \Phi^n \\ & = & m\Phi^2 \sum_{n=0}^{\infty} A_n \left(\frac{g\Phi}{m}\right)^n \qquad (9.4) \\ & = & m\Phi^2 f\left(\frac{g\Phi}{m}\right), \qquad (9.5) \end{array}$$

where, A_n are arbitrary constant coefficients and $f(\cdot)$ is an arbitrary function defined by the Taylor series above.

It remains to determine the function f, or equivalently the coefficients A_n. To that end, notice that for small values of g, perturbation theory is applicable. We compare this with the Taylor series

$$W_{\mathit{eff}} \overset{g\rightarrow 0}{=} m\Phi^2 \left(A_0 + A_1 \frac{g\Phi}{m} + \sum_{n=2}^{\infty} A_n \left(\frac{g\Phi}{m}\right)^n \right)$$

to find that $A_0 = 1/2$ and $A_1 = 1/3$. It follows from the properties of superspace Feynman rules[11] that the higher terms in the sum come from the tree level process shown in Fig.3. (Loops are not possible as it would increase powers of g but not of Φ.)

[11] A complete discussion of superspace perturbation theory is beyond the scope these lectures. However, all we need to know for this is the fact that the propagator for chiral superfield at zero momentum $\langle\Phi\Phi\rangle(k=0) \sim m^{-1}$.

Figure 3: Superspace Feynman diagram contributing to the sum in eqn.(9.4).

For $n \geq 2$, the Feynman diagram in Fig.3 is not one particle irreducible, and hence it cannot contribute to the effective superpotential. Therefore the effective superpotential is exactly the same as the classical superpotential (6.20)

$$W_{\mathit{eff}}(\Phi, m, g) = W(\Phi, m, g) = \frac{1}{2}\, m\Phi^2 + \frac{1}{3}\, g\Phi^3.$$

In other words, the superpotential of the Wess-Zumino model is not renormalised by quantum (and non-perturbative) corrections. This important result, first derived in perturbation theory, is called the *non-renormalisation theorem.*

There is, however, a caveat to the above argument. It was already known, (through the analysis of perturbation theory of superfields), that in case there are massless chiral superfields, the non-renormalisation theorem is invalidated. Indeed it was found[17] that the effective potential in the Wess-Zumino model with $m = 0$ gets a contribution $g^3(g^*)^2\Phi^3$ which comes from the two-loop diagram shown in Fig.4. This is clearly non-holomorphic.

Figure 4: A two loop diagram which gives a non-holomorphic contribution to the 1PI effective superpotential. (A solid line denotes a $\Phi\Phi$ propagator while a dot-dashed line stands for a $\Phi^\dagger\Phi$ or a $\Phi\Phi^\dagger$ propagator.)

Similar violation of holomorphy was found in SQCD at two loops. In this case the problem is even more serious due to supersymmetric relation between the stress-tensor $T_{\mu\nu}$ and the axial current j^5_μ. The latter does not receive any correction beyond one loop. A resolution to this apparent paradox was suggested by Shifman and Vainshtein[18], who argued that one needs to distinguish between two commonly used notions of effective action (or effective potential).

- From the microscopic lagrangian $\mathcal{L}(\{\phi\})$, one defines a (Euclidean) generating functional for the correlation functions

$$\begin{aligned} Z[\{J\}] &= e^{-F[\{J\}]} \\ &= \int [\mathcal{D}\phi] \exp\left(-\int d^4x \left(\mathcal{L}[\{\phi\}] + \sum J\phi\right)\right), \end{aligned}$$

which is a functional of the external sources $\{J\}$. A Legendre transform of this generating functional is a functional of the effective fields (called classical fields) $\{\phi_{cl}\}$

$$\Gamma[\{\phi_{cl}\}] = -F[\{J\}] - \sum \int d^4x J(x)\phi_{cl}(x),$$

where

$$\begin{aligned} \frac{\delta}{\delta J(x)} F[\{J\}] &= -\phi_{cl}(x), \\ \frac{\delta}{\delta \phi_{cl}} \Gamma[\{\phi_{cl}\}] &= -J(x). \end{aligned}$$

$\Gamma[\{\phi_{cl}\}]$ generates all the one particle irreducible graphs of the theory, and is called the (1PI) effective action. Notice that in defining this effective action one integrates out *all* the degrees of freedom.

- The alternative notion is that of a Wilsonian effective action. To define this, let us first Fourier transform to momentum space $\{\phi(x)\} \to \{\phi(k)\}$. Now divide the degrees of freedom into low energy and high energy modes[12]

$$\phi(k) = \phi_<(k) + \phi_>(k),$$

where

$$\begin{aligned} \phi_<(k) &= \begin{cases} \phi(k) & \text{for } \mu \le |k| \le M_{UV}, \\ 0 & \text{for } 0 \le |k| < \mu, \end{cases} \\ \phi_>(k) &= \begin{cases} 0 & \text{for } \mu \le |k| \le M_{UV}, \\ \phi(k) & \text{for } 0 \le |k| < \mu, \end{cases} \end{aligned}$$

where M_{UV} is some UV cut-off. The Wilsonian effective action is defined via

$$\begin{aligned} \int [\mathcal{D}\phi]\ e^{-S[\{\phi\}]} &= \int \prod_k [\mathcal{D}\phi_<(k)] \int \prod_k [\mathcal{D}\phi_>(k)]\ e^{-S[\{\phi\}]} \\ &= \int \prod_{0\le|k|<\mu} [\mathcal{D}\phi_<(k)] \exp\left(-S_W[\{\phi_<\}]\right) \end{aligned}$$

Thus in the Wilsonian effective action, one only integrates over the high energy degrees of freedom. In particular the IR region is excluded from the integrals.

Since the violation of holomorphy comes from the IR region, the Wilsonian effetcive action is free of this problem, while the 1PI effective action may, (and as we saw in general it so does), suffer from a *holomorphic anomaly.* Seiberg's arguments are therefore applicable

[12] This is really meaningful in Euclidean continuation.

to the Wilsonian effective action. (However, when there is no massless particle, the two notions are almost identical.) There are other limitations of the 1PI effective action. For example, the source term added to the superpotential may lead to supersymmetry breaking. (See Ref.[15] for such subtleties.)

Finally we are ready to discuss quantum corrections to *massless* supersymmetric QCD. To this end we need to determine the global symmetries of the theory. Recall from the last lecture that when $m = 0$, there are two independent flavour symmetry

$$\begin{aligned} \mathrm{U}_\ell(N_f) \times \mathrm{U}_r(N_f) &\approx \mathrm{SU}_\ell(N_f) \times \mathrm{SU}_r(N_f) \times \mathrm{U}_\ell(1) \times \mathrm{U}_r(1) \\ &\approx \mathrm{SU}_\ell(N_f) \times \mathrm{SU}_r(N_f) \times \mathrm{U}_B(1) \times \mathrm{U}_A(1), \end{aligned}$$

where $\mathrm{U}_B(1)$ and $\mathrm{U}_A(1)$ are the diagonal and anti-diagonal parts of the two U(1) subgroups respectively. The conserved charge arising from $\mathrm{U}_B(1)$ is called the *baryon number*, and this is a good symmetry of the quantum theory. The axial $\mathrm{U}_A(1)$ on the other hand, acts oppositely on the left- and right-moving quarks, and is *anomalous*. We shall normalise so as to assign baryon number +1 (respectively −1) to the superfields Q ($\tilde{Q}$). The various U(1) quantum numbers of the quark superfields are as follows:

	Q	$\tilde{Q}$	$Q^\dagger$	$\tilde{Q}^\dagger$
L	+1	0	−1	0
R	0	−1	0	+1
B	+1	−1	−1	+1
A	+1	+1	−1	−1

The anomaly of the axial $\mathrm{U}_A(1)$ leads to a shift of the θ-parameter: $\theta_{YM} \to \theta_{YM} - n\alpha$, where n is the coefficient of the anomaly term in the conservation law, or in other words, the number of fermion zero modes contributing to the anomaly. In this case, $n = 2N_f$, which leads to

$$\exp(2\pi i\tau) \xrightarrow{A} e^{2iN_f\alpha} \exp(2\pi i\tau).$$

In addition, we have the R-symmetry of supersymmetry discussed in lectures VI and the present one. There is no superpotential, (we are consideing the massless case[13]), the superfields Q and $\tilde{Q}$ have zero R-charge. If we write

$$Q = \phi_Q + \theta q_L + \cdots, \qquad \tilde{Q} = \phi_{\tilde{Q}} + \theta \tilde{q}_L + \cdots,$$

the $\mathrm{U}_R(1)$ transformation properties are as follows (see eqn.(6.22)):

$$\begin{aligned} \phi_Q &\to \phi_Q, & \phi_{\tilde{Q}} &\to \phi_{\tilde{Q}}, \\ q_L &\to e^{-i\alpha} q_L, & \tilde{q}_L &\to e^{-i\alpha}\tilde{q}_L, \\ && \lambda \to e^{i\alpha}\lambda. \end{aligned}$$

This symmetry is also anomalous, leading to

$$\exp(2\pi i\tau) \xrightarrow{R} e^{2i(N_c - N_f)\alpha} \exp(2\pi i\tau).$$

[13] The $m \neq 0$ case can be thought of as arising from a Higgs mechanism. This happens naturally if the $N = 1$ supersymmetric theory is embedded in the $N = 2$ theory, which has a superpotential term $\tilde{Q}\Phi Q$, where Φ is the Higgs superfield valued in the adjoint.

The two anomalous U(1) symmetries may be combined to an anomaly free $U_{\mathcal{R}}(1)$, the generator $\mathcal{R}$ of which is

$$\mathcal{R} = R + \frac{N_f - N_c}{N_f}\, A. \tag{9.6}$$

The following table shows the relevant symmetry properties of the various fields.

Field	Flavour	$U_B(1)$	$U_A(1)$	$U_R(1)$	$U_{\mathcal{R}}(1)$
L-quark q_L	$(\mathbf{N}_f, \mathbf{1})$	$+1$	$+1$	-1	$-\frac{N_c}{N_f}$
squark ϕ_Q	$(\mathbf{N}_f, \mathbf{1})$	$+1$	$+1$	0	$\frac{N_f-N_c}{N_f}$
R-quark q_R	$(\mathbf{1}, \overline{\mathbf{N}}_f)$	-1	$+1$	-1	$-\frac{N_c}{N_f}$
squark $\phi_{\tilde{Q}}$	$(\mathbf{1}, \overline{\mathbf{N}}_f)$	-1	$+1$	0	$\frac{N_f-N_c}{N_f}$
gluino λ	$(\mathbf{1}, \mathbf{1})$	0	0	$+1$	$+1$
meson $M^A_{\tilde{B}}$	$(\mathbf{N}_f, \overline{\mathbf{N}}_f)$	0	$+2$	0	$2\,\frac{N_f-N_c}{N_f}$
scale Λ	$(\mathbf{1}, \mathbf{1})$	0	$-\frac{2N_f}{3N_c-N_f}$	$\frac{2N_c-2N_f}{3N_c-N_f}$	0

In the above we have used the relation (8.18) between the QCD scale Λ and the combination τ of the coupling constant g and the θ-angle.

At present we are discussing the case $N_f < N_c$ — the only gauge invariant composite fields are the mesons.

The meson superfield matrix $M^A_{\tilde{B}}$ transforms as $(\mathbf{N}_f, \overline{\mathbf{N}}_f)$ under global flavour symmetry. However, the effective superpotential ought to be a flavour singlet and hence can only be a (holomorphic) function of $\det M$. It can also depend (holomorphically) on τ or equivalently, the complexified QCD scale Λ. We therefore start with

$$W_{\mathit{eff}}(M, \Lambda) = (\det M)^a\, \Lambda^b$$

and require that W_{eff} has the right U(1) charges. This fixes a and b:

$$W_{\mathit{eff}}(M, \Lambda) = C_{N_f N_c} \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)}, \tag{9.7}$$

where $C_{N_f N_c}$ is a constant which cannot be determined by the arguments used so far. It turns out[14] that

$$C_{N_f N_c} = N_c - N_f. \tag{9.8}$$

Notice that the effective superpotential (9.7) has the correct (mass) dimension. We should also add that the superpotential is non-perturbative and hence does not violate any perturbative non-renormalisation theorem.

Let us analyse the behaviour of this superpotential. In order to find the supersymmetric vacua, we set the F-term to zero. The F-term

$$\begin{aligned} F^*_{AB} &= \frac{\partial}{\partial M_{AB}} W_{\mathit{eff}} \\ &= (N_c - N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)} \left(M^{-1} \right)_{AB} \end{aligned} \tag{9.9}$$

Figure 5: Runaway potential of the 'meson' fields.

implies that the bosonic potential tends to zero as $\langle M_{AB} \rangle \sim a$ tends to infinity (see Fig.5).

Thus, we come to the strange conclusion that there is no stable supersymmetric vacuum (at finite vaccum expectation values of the meson superfields). This is inspite of the fact that the classical theory we started with had an infinitely degenerate vacua.

Although this seems strange, it is the only possibility consistent with supersymmetry. The alternative based on expectation from ordinary QCD is chiral symmetry breaking and confinement. In this case the quarks would condense in pairs

$$\langle q_{LA} \tilde{q}_{LB} \rangle \neq 0.$$

However, this is the F-term of M_{AB}, a non-zero value for which would lead to supersymmetry breaking. Therefore a QCD like vacuum with quark pair condensate is unstable compared to any vacuum that allows for unbroken supersymmetry.

The gaugino bilinear $\lambda\lambda$ can, however, acquire a vacuum expectation value consistent with supersymmetry. For example, in pure super-Yang-Mills theory with $N_f = 0$, the effective superpotential is

$$W_{eff}(\Lambda) = c_{N_c} \Lambda^3 = c_{N_c} \mu^3 \exp\left(\frac{2\pi i \tau_{eff}}{N_c} \right). \tag{9.10}$$

Now, $\lambda\lambda$ is the scalar component of the composite chiral superfield $W^\alpha W_\alpha$ which couples to τ. The vacuum expectation value $\langle \lambda\lambda \rangle$ can therefore be found by differentiating the effective lagrangian $\mathcal{L}_{eff}$ with respect to F_τ, the F-term of τ.

$$\begin{aligned} \langle \lambda\lambda \rangle &= 16\pi i \frac{\partial}{\partial F_\tau} \int d^2\theta \, W_{eff}(\tau) \\ &= 16\pi i \frac{\partial}{\partial \tau} W_{eff}(\tau) \\ &= -\frac{32\pi^2}{N_c} c_{N_c} \mu^3 \exp\left(\frac{2\pi i \tau}{N_c} \right), \end{aligned} \tag{9.11}$$

[14] One uses the idea of *holomorphic decoupling*, (which is illustrated through the last problem), to get a recursion relation for the constants in theories with different number of flavours.

or, $\langle\lambda\lambda\rangle \sim e^{-8\pi^2/N_c g^2}$ (for $\theta_{YM} = 0$), is the non-perturbative gaugino condensate.

Recall that the R-symmetry shifts $\lambda \to e^{i\alpha}\lambda$, hence $\theta_{YM} \to \theta_{YM} + 2N_c\alpha$. When α is a multilple of π/N_c, theta angle shifts by 2π, and a $\mathbf{Z}_{2N_c}$ subgroup of $\mathrm{U}_R(1)$ remains unbroken. However the vacuum expectation value $\langle\lambda\lambda\rangle$ is invariant under $\lambda \to -\lambda$ ($\alpha = \pi$), breaking $\mathbf{Z}_{2N_c}$ spontaneously to a $\mathbf{Z}_2$ subgroup. This leads to N_c inequivalent vacuum states in pure super-Yang-Mills theory parametrised by the vev of gaugino bilinear

$$\langle\lambda\lambda\rangle = \exp\left(\frac{2\pi i m}{N_c}\right), \quad m = 0, 1, \cdots, N_c - 1. \tag{9.12}$$

Exercise: *(Holomorphic decoupling) Consider a theory of two chiral superfields given by the superpotential*

$$W(\Phi_1, \Phi_2) = \frac{1}{2} m\Phi_2^2 + g\Phi_1^2\Phi_2.$$

Analyse the classical moduli space of this theory and show that at a generic point the field Φ_2 is heavy. *Find the effective superpotential for the* light *field Φ_1 by 'integrating out' Φ_2. (In the classical theory, this means that one ignores the kinetic term for the* heavy *field, and solves its resulting algebraic equation of motion.*
Now apply Seiberg's analysis to determine the low energy *effective superpotential assuming that the only relevant degrees of freedom are those of the* light *field Φ_1. Compare the two results.*
Identify the diagram that 'renormalises' the superpotential, (it is actually a tree diagram), and see that it does not violate the non-renormalisation theorem.

A Appendix: Fierz identities

There are some identities which involve spinors. Much of it follows from simple angular momentum addition rule.

If we take the product of a spin $s_1 = \frac{1}{2}$ representation[15] with another one $s_2 = \frac{1}{2}$, the result is a direct sum of spin-0 and spin-1 representations:

$$\begin{aligned} (s_1 = 1/2) \otimes (s_2 = 1/2) &= (s^{(1)} = 0) \oplus (s^{(2)} = 1), \\ \textit{i.e,} \qquad \mathbf{2} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{3}. \end{aligned}$$

A simpler fact is

$$\mathbf{2} \otimes \mathbf{1} = \mathbf{2}.$$

Since the Lorentz algebra is (almost) a direct sum of two angular momentum algebras (see (1.5)), the above rules for tensoring representations apply to the Lorentz algebra as well. The defining representations here are the $\left(\frac{1}{2}, 0\right)$ or $(\mathbf{2}, \mathbf{1})$ representation called the left-handed spinor ψ_α and the $\left(0, \frac{1}{2}\right)$ or $(\mathbf{1}, \mathbf{2})$ representation called the right-handed spinor $\bar{\chi}^{\dot{\alpha}}$.

There are three basic possibilities in combining these representations.

[15] The spin-$\frac{1}{2}$ representation is called the defining or fundamental representation of SU(2), as all other representations can be constructed from it.

- We can tensor two left-handed spinors

$$(\mathbf{2},\mathbf{1})\otimes(\mathbf{2},\mathbf{1})=(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1}).$$

  More explicitly in terms of the spinors

$$\psi_\alpha\chi_\beta=\frac{1}{2}\,\epsilon_{\alpha\beta}(\psi\chi)-\frac{1}{2}\,(\psi\sigma_{\mu\nu}\chi)\,\sigma^{\mu\nu}{}_{\alpha\beta}, \tag{A.1}$$

  we get a scalar and a rank-2 self-dual antisymmetric tensor.

  Notice that the first term in the RHS above is antisymmetric (in indices α and β) while the second one is symmetric, as is expected from the representations of the angular momentum algebra. The coefficients $\frac{1}{2}\,\epsilon_{\alpha\beta}$ and $-\frac{1}{2}\,\sigma^{\mu\nu}{}_{\alpha\beta}$ are the Clebsch-Gordon coefficients relating different representations.

  In order to prove (A.1), based on covariance properties one can argue that

$$\begin{aligned}\psi_\alpha\chi_\beta &= \frac{1}{2}(\psi_\alpha\chi_\beta-\chi_\beta\psi_\alpha)+\frac{1}{2}(\psi_\alpha\chi_\beta+\chi_\beta\psi_\alpha),\\ &= \frac{1}{2}\,A\epsilon_{\alpha\beta}+\frac{1}{2}\,B_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha\beta}.\end{aligned}$$

  The unknown coefficients A and $B_{\mu\nu}$ may be determined by contracting both sides of the above equation by $\epsilon^{\beta\alpha}$ and $\sigma^{\rho\lambda\beta\alpha}$ respectively. For the latter, we need to use the fact that $\sigma_{\mu\nu}$ is self-dual (1.9), and that

$$\mathrm{tr}\,(\sigma^{\rho\lambda}\sigma^{\mu\nu})=\frac{1}{2}(\eta^{\rho\mu}\eta^{\lambda\nu}-\eta^{\rho\nu}\eta^{\lambda\mu})\;+\;\frac{i}{2}\,\epsilon^{\rho\lambda\mu\nu}.$$

- Similarly, for two right-handed spinors

$$(\mathbf{1},\mathbf{2})\otimes(\mathbf{1},\mathbf{2})=(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3}).$$

  More explicitly in terms of the spinors

$$\bar\psi^{\dot\alpha}\bar\chi^{\dot\beta}=\frac{1}{2}\,\epsilon^{\dot\alpha\dot\beta}(\bar\psi\bar\chi)-\frac{1}{2}\,(\bar\psi\bar\sigma_{\mu\nu}\bar\chi)\,\bar\sigma^{\mu\nu\dot\alpha\dot\beta}, \tag{A.2}$$

  where on the RHS we find a scalar and a rank-2 anti-self-dual antisymmetric tensor.

- Finally, combining a left- and a right-handed spinor

$$\begin{aligned}(\mathbf{2},\mathbf{1})\otimes(\mathbf{1},\mathbf{2}) &= (\mathbf{2},\mathbf{2}),\\ \psi_\alpha\bar\chi_{\dot\beta} &= \frac{1}{2}\,(\psi\sigma_\mu\bar\chi)\,\sigma^\mu_{\alpha\dot\beta},\end{aligned} \tag{A.3}$$

  we get a Lorentz vector.

Using these relations, following from basic group theoretic facts, it is easy to derive the following relations.

$$\begin{aligned}\psi_\alpha\psi_\beta &= \frac{1}{2}\,\epsilon_{\alpha\beta}(\psi\psi)\\ \bar\chi_{\dot\alpha}\bar\chi_{\dot\beta} &= -\frac{1}{2}\,\epsilon_{\dot\alpha\dot\beta}(\chi\chi)\end{aligned}$$

$$
\begin{aligned}
(\theta\psi)(\theta\chi) &= -\frac{1}{2}(\theta\theta)(\psi\chi) \\
(\bar{\theta}\psi)(\bar{\theta}\chi) &= -\frac{1}{2}(\bar{\theta}\bar{\theta})(\bar{\psi}\bar{\chi}) \\
\psi\sigma^{\mu}\bar{\chi} &= -\bar{\chi}\sigma^{\mu}\psi \\
\psi\sigma^{\mu}\bar{\sigma}^{\nu}\chi &= \chi\sigma^{\nu}\bar{\sigma}^{\mu}\psi \\
(\sigma^{\mu}\bar{\theta})_{\alpha}(\theta\sigma^{\nu}\bar{\theta}) &= \frac{1}{2}\eta^{\mu\nu}\theta_{\alpha}(\bar{\theta}\bar{\theta}) - i(\sigma^{\mu\nu}\theta)_{\alpha}(\bar{\theta}\bar{\theta}) \\
(\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) &= \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta}) \\
(\theta\psi)(\bar{\theta}\bar{\chi}) &= \frac{1}{2}(\theta\sigma^{\mu}\bar{\theta})(\psi\sigma_{\mu}\bar{\chi})
\end{aligned} \tag{A.4}
$$

These are known as *Fierz identities*, and are very useful in simplifying expressions involving many spinor fields.

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