(1) To

FAX +91-22-2280 4610

or +91-22-2280 4611

Derivation of the Schwarzschild soln

~~ds~~ = -

Radially symmetry.

$$ds^2 = -A(r,t)\,dt^2 + B(r,t)\,dr^2 + C(r,t)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Area is $4\pi r^2$

(No cross term: can be diagonalized as fn of r & t)

To simplify 1st do $C(r,t) = r^2$

$$ds^2 = -A(r,t)\,dt^2 + B(r,t)\,dr^2 + r^2(d\theta^2 + r^2\sin^2\theta\, d\varphi^2)$$

$e_0 = \sqrt{A}\,dt \quad ; e_r = \sqrt{B}\,dr \quad ; e_\theta = r\,d\theta \quad ; e_\varphi = r\sin\theta\, d\varphi$

Extract ω from

$$de_a + \omega_{ab} \wedge e^b = 0$$

3 eq.

$t \to \quad \frac{1}{2\sqrt{A}} \partial_r A\, dr \wedge dt + \omega_{0r} \wedge e^r + \omega_{0\theta} \wedge e^\theta + \omega_{0\varphi} \wedge e^\varphi = 0$

$r \to \quad \frac{1}{2\sqrt{B}} \partial_t B\, dt \wedge dr + \omega_{r0} \wedge e^0 + \omega_{r\theta} \wedge e^\theta + \omega_{r\varphi} \wedge e^\varphi = 0$

$\theta \to \quad dr \wedge d\theta + \omega_{\theta 0} \wedge e^0 + \omega_{\theta r} \wedge e^r + \omega_{\theta\varphi} \wedge e^\varphi = 0$

$\varphi \to \quad \sin\theta\, dr \wedge d\varphi + r\cos\theta\, d\theta \wedge d\varphi + \omega_{\varphi 0} \wedge e^0 - \omega_{0r} \wedge e^r + \omega_{\varphi\theta} \wedge e^\theta$

Soln. by inspection $\omega_{\theta r} = \frac{1}{\sqrt{B}}\, d\theta = \frac{e_\theta}{\sqrt{B}\, r}$.

$\omega_{\varphi r} = \frac{1}{\sqrt{B}} \sin\theta\, d\varphi = \frac{1}{\sqrt{B}\, r}\, e_\varphi$

$$\omega_{\varphi\theta} = \cos\theta\, d\varphi \qquad \frac{1}{r\sin\theta} e_\varphi$$ (2) II.2

$$\omega_{0r} = \frac{1}{2}\frac{\partial_t B}{\sqrt{AB}}\, dr + \frac{1}{2\sqrt{AB}}\partial_r A\, dt$$

Work in local (orthonormal) coordinates

$$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}.$$

$$R_{0r} = \left\{ \frac{1}{2}\partial_t \left[\frac{\partial_t B}{\sqrt{AB}}\right] + \frac{1}{2}\partial_r\left[\frac{\partial_r A}{\sqrt{AB}}\right]\right\} \frac{1}{\sqrt{AB}} e_0 \wedge e_r$$

$$\boxed{R_{0r0r} = -\left\{\frac{1}{2}\partial_t\left(\frac{\partial_t B}{\sqrt{AB}}\right) + \frac{1}{2}\partial_r\left(\frac{\partial_r A}{\sqrt{AB}}\right)\right\}\frac{1}{\sqrt{AB}}}$$

$$R_{0\theta} = d\omega_{0\theta} + \omega_0{}^r \wedge \omega_{r\theta}$$

$$= \frac{-1}{2}\frac{\partial_t B}{\sqrt{A}\, B^{3/2}}\cdot\frac{1}{r}\, e_r \wedge e_\theta - \frac{1}{2}\frac{\partial_r A}{AB}\cdot\frac{1}{r}\, e_0 \wedge e_\theta$$

$$\boxed{\begin{aligned} R_{0\theta 0\theta} &= \frac{1}{2}\frac{\partial_r A}{AB}\,\frac{1}{r}. \\ R_{0\theta r\theta} &= \frac{-1}{2}\cdot\frac{\partial_t B}{\sqrt{A}\, B^{3/2}}\cdot\frac{1}{r}. \end{aligned}}$$

Similarly
$$\boxed{\begin{aligned} R_{0\varphi 0\varphi} &= \frac{1}{2}\frac{\partial_r A}{AB}\cdot\frac{1}{r} \\ R_{0\varphi r\varphi} &= \frac{-1}{2}\frac{\partial_t B}{\sqrt{A}\, B^{3/2}}\frac{1}{r} \end{aligned}}$$

from $R_{0\varphi}$ = equation

③ ~~(3)~~

$R_{\cancel{r\theta\theta\theta}}$ ⊘

$$\boxed{\begin{aligned} R_{r\theta r\theta} &= \frac{\partial_r B}{2B^2 r} \\ \text{also} \quad R_{r\theta 0\theta} &= -\frac{\partial_t B}{2B^{3/2}\sqrt{A}\, r} \end{aligned}} \quad \Big\}$$

~~fear~~ from $R_{r\theta}$ eq.

consistent with previous calc

$$\boxed{\begin{aligned} R_{r\varphi r\varphi} &= \frac{\partial_r B}{2B^2 r} \\ R_{r\varphi 0\varphi} &= -\frac{\partial_t B}{2B^{3/2}\sqrt{A}\, r} \end{aligned}} \quad \Big\}$$

from $R_{r\varphi}$ eq.

$$\boxed{R_{\theta\varphi\theta\varphi} = \frac{1}{r^2} - \frac{1}{Br^2}}$$

from $R_{\theta\varphi}$ eq

Work in local Lorentz coordinates

$$R_{00} = R_0{}^r{}_{r0} + R_0{}^\theta{}_{\theta 0} + R_0{}^\varphi{}_{\varphi 0}.$$

$$= -\left\{\frac{1}{2}\frac{1}{\sqrt{AB}}\right\}\left\{\partial_t\left(\frac{\partial_t B}{\sqrt{AB}}\right) + \partial_r\left(\frac{\partial_r \cancel{A}}{\sqrt{AB}}\right)\right\}.$$

$$+ \frac{1}{2}\times 2 \cdot \frac{\partial_r A}{AB}\cdot\frac{1}{r} = 0 \qquad ①$$

$$R_{0r} = R_0{}^\theta{}_{\theta r} + R_0{}^\varphi{}_{\varphi r}$$

$$= -\frac{1}{2}\times 2 \cdot \frac{\partial_t B}{\sqrt{A}\; B^{3/2}}\cdot\frac{1}{r} = 0 \qquad — ②$$

$$\Rightarrow \underline{\partial_t B = 0}$$

$$R_{0\theta} = R_0{}^r{}_{r\theta} + R_0{}^\varphi{}_{\varphi\theta} = 0$$

$$R_{0\varphi} = 0.$$

$$R_{rr} = R_r{}^{0}{}_{0r} + R^{\theta}{}_{r\theta r} + R^{\varphi}{}_{r\varphi r} \quad (4) \quad \text{I}$$

$$= \frac{1}{2}\frac{1}{\sqrt{AB}}\partial_r\left(\frac{\partial_r A}{\sqrt{AB}}\right) + 2\cdot\frac{\partial_r B}{2B^2 r} = 0 \quad (3).$$

using $\partial_t B = 0$

$$R_{r\theta} = 0 = R_{r\varphi}$$

$$R_{\theta\theta} = R^{0}{}_{\theta 0\theta} + R^{r}{}_{\theta r\theta} + R^{\varphi}{}_{\theta\varphi\theta}$$

$$= -\frac{1}{2}\frac{\partial_r A}{AB}\cdot\frac{1}{r} + \frac{\partial_r B}{2B^2 r} + \frac{1}{r^2} - \frac{1}{Br^2} \quad (4)$$

$$R_{\theta\varphi} = 0$$

$$R_{\varphi\varphi} = R_{\theta\theta}$$

Adding (1) + (3)

$$\rightarrow \quad \frac{\partial_r A}{AB}\cdot\frac{1}{r} + \frac{\partial_r B}{B^2 r} = 0$$

$$\partial_r \ln A = -\partial_r \ln B \quad \rightarrow \quad A = \frac{K}{B}.$$

K fn of time only.

B fn of r only.

Rescale t s.t. $K = 1$.

$\rightarrow$ A fn of r only.

eq 4.

$$-\partial_r A + \frac{1}{r} - \frac{A}{r} = 0$$

$$\frac{\partial}{\partial r}(rA) = 1$$

$$A = 1 + \frac{C}{r}$$

$\rightarrow$ we will fix c $\rightarrow$ ~~...~~ Now.

C is fixed by looking at a test particle force on moving in ~~r5~~ (5)
this geometry.

$$ds^2 = -\left(1+\frac{C}{r}\right)dt^2 + \frac{dr^2}{\left(1+\frac{C}{r}\right)} + r^2 d\Omega^2$$

$r \to \infty$ almost flat space.

→ Test particle moves on a geodesic
obtained form

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{dt}.$$

EOM

$$\to \quad \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) - \frac{1}{2}\frac{g_{\rho\sigma}}{dx^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$\to g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{dx^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$\to \frac{d^2x^\nu}{d\lambda^2} + g^{\nu\mu}\left[\frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{2\,\partial x^\mu}\right]\cdot\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$\Gamma^\nu_{\rho\sigma} = \frac{1}{2} g^{\nu\mu}\left[\frac{\partial g_{\mu\rho}}{\partial x^\sigma} + \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\mu}\right]$$

Now. Asymtotically for the Schwarz geometry.
choose $\lambda = t$

$i = r, \theta, \varphi$

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} + 2\Gamma^i_{0i}\frac{dx^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

~~leading order~~ as $r \to$
say a radially moving particle.

→ Fixing the Integration Constant

→ Consider a Test particle in schwarzschild geometry

→ moves on geodesic

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$$

→ can't can't move faster than light

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1 \quad (\text{time like})$$

(Radially moving & $dr \to 0$: slowly moving) $\left(\frac{dr}{dt}\right) << 1$

Then $\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 = -1$

$$\Rightarrow \text{ for } \quad d\lambda = dt \sqrt{1 - \left(\frac{dr}{dt}\right)^2}$$

for slowly moving $\lambda = t$ $\quad \left(\frac{dr}{dt}\right)^2 \approx 0$

→ $\mu = r$ eq

$$\frac{d^2 r}{d\lambda^2} + \Gamma^r_{00} + 2\Gamma^r_{0r} \frac{dr}{dt} + \Gamma^r_{rr} \left(\frac{dr}{dt}\right)^2 = 0$$

$$\Gamma^r_{00} = \frac{g^{rr}}{2} \left[\partial_0 g_{r0} + \partial_0 g_{r0} - \partial_r g_{00} \right]$$

$$= \frac{1}{2} \left(1 - \frac{2M}{r}\right) \partial_r \left[1 - \frac{2M}{r} \right]$$

$r \to \infty$

$$= \frac{M}{r^2}$$

$$\frac{d^2 r}{dt^2} = \frac{-M}{r^2} \quad / \text{ Newton's Law}$$

with 4-paper RN sch ⑥

~~To~~

fax: 0091-22-22804610

Singularity at $r=0$

$$R^{abcd}R_{abcd} = \mathcal{R}$$

$$R_{\theta r\theta r} = \frac{-2M}{r^3}, \quad R_{\theta\theta\theta\theta} = \frac{M}{r^2}; \quad R_{\theta\varphi\theta\varphi} = \frac{M}{r^2}; \quad R_{r\theta r\theta} = \frac{M}{r^3}$$

$$R_{r\varphi r\varphi} = -\frac{M}{r^3}$$

$$R = \frac{32M^2}{r^6}$$ / blows up. at $r=0$.

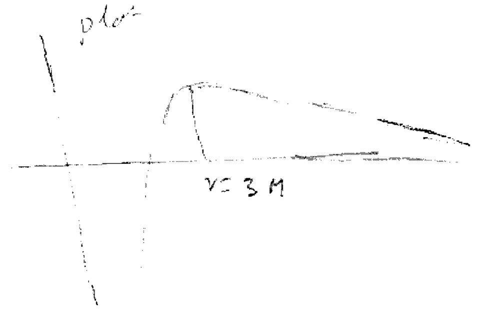
L4 O : Null geodesics in Swarzschild geometry (8)

Eq

$$\frac{\dot{r}^2}{2} + \underbrace{\frac{L^2}{2r^2}\left(1-\frac{2m}{r}\right)}_{\text{effective potential}} = \frac{E^2}{2}$$

$\frac{L^2}{2}\left[\frac{1}{2r^2} - \frac{2m}{r^3}\right.$

- Shape of potential independent of L



V_{max} at $\quad -\frac{2}{r^3} + \frac{6M}{r^4} = 0$

$r = 3M$

Minimum Energy to cross barrier

$$E^2 = \frac{L^2}{9M^2}\left(1 - \frac{2}{3}\right) = \frac{L^2}{27M^2}$$

$E = |P|$ for photon

$L = |P| \cdot b \qquad \therefore\ b$ impact parameter (at $r \to \infty$)

$$b = \frac{L}{E} = \sqrt{27}\,M$$

so for $b < \sqrt{27}M$: light ray will be captured !

cross-sect $\sigma = \pi b_c^2 = 27\pi M^2$.

0 Time like geodesics in Swarzschild geometry (9)

$$\frac{\dot{r}^2}{2} + \underbrace{\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+1\right)}_{V_{eff}} = \frac{E^2}{2}$$

V_{eff} as $r\to\infty$ $V \to \frac{1}{2}$; $r \to 0$ $-\infty$

V_{min} at $\frac{2M}{r^2}\left(\frac{L^2}{r^2}+1\right) + \left(1-\frac{2M}{r}\right)\left(-\frac{2L^2}{r^3}\right) = 0$

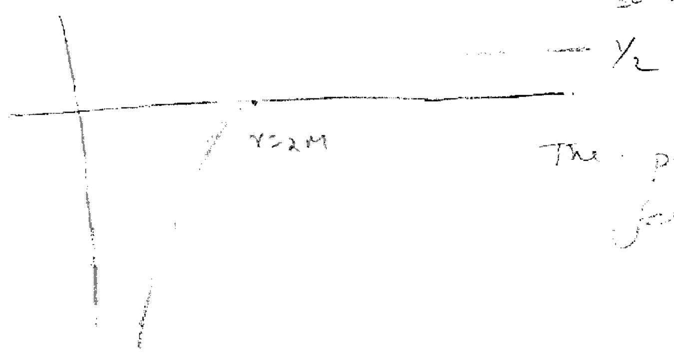
$2ML^2 + 2Mr^2 - 2L^2(r) + 4ML^2$

$\longrightarrow$ $Mr^2 - L^2 r + 3ML^2 = 0$

$D = L^4 - 12M^2L^2 = L^2(L^2-12M^2)$

for $L^2 < 12M^2$ No real Root

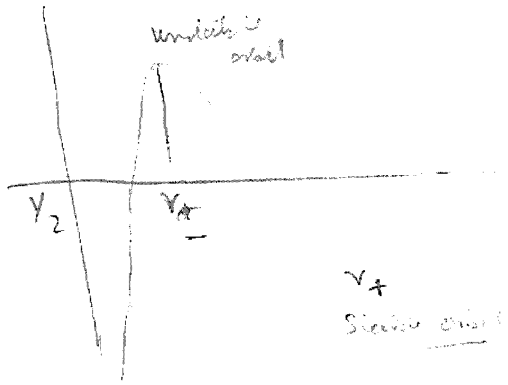
so No minima of potential



The particle will always fall into the BH if $\dot{r}<0$

if $L^2 > 12M^2$ 2 minima

$$r_\pm = \frac{L^2 \pm L^2\sqrt{1-\frac{12M^2}{L^2}}}{2M} = \frac{L^2}{2M}\left[1 \pm \sqrt{1-\frac{12M^2}{L^2}}\right]$$



Note $r_+ = \frac{L^2}{M}$ for $L \gg M$

force $\frac{1}{r^2}$ forcing

Mass describing circular orbit in $\frac{1}{r}$ potential

r of stable orbits as $L^2 > 12M^2$.

$$r_+ > 6M$$

~~governa stable~~

Range of unstable orbits

$$r_- = \frac{L^2}{2M}\left[1 - \sqrt{1 - \frac{12M^2}{L^2}}\right] = \frac{L^2}{2M}\left[\frac{12M^2}{2L^2} + \text{ve terms}\right]$$

$$3M < r_- < 6M.$$

Behavior of particles in unstable orbits when perturbed slightly ($\dot{r} = 0$ evaluate E)

$$E = \frac{R-2M}{\sqrt{R(R-3M)}}$$ at $R = 4M$ asymptotic $E = 1$

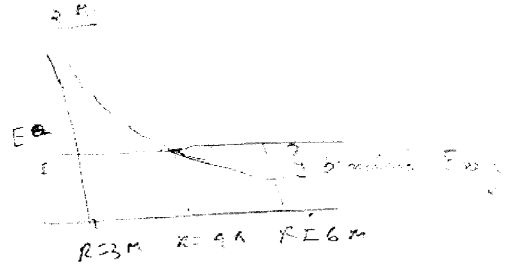
$R = 3M$ $E \to \infty$

$R = 4M$: $E^2 = 1$

$R = 6M$

$\frac{4M}{6M \cdot 3M}$

$$E = \frac{2\sqrt{2}}{3} = \left(\frac{8}{9}\right)^{1/2} < 1$$



$\therefore$ for unstable orbits in $3M < r_- < 4M$
→ when perturbed → particle goes out to ∞

but $4M < r_- < 6M$; as Energy is less than Asymptotic Energy particle falls inside

Now if a particle stays at stable orbit $r_+ > 6M$
→ loses Energy by gravitational waves
→ Reach 6M → falls inside

(11)

For later purposes we need the trajectory of a radial time like geodesic.

Therefore we solve the Equation

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right) = E^2 \qquad / \text{ Can be Integrated}$$

soln is given by (parametric form).

$$r = \frac{R}{2}(1+\cos\eta) \qquad \tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta + \sin\eta)$$

→ lets verify that it's a soln

$$\frac{dr}{d\tau} = -\frac{R}{2}\sin\eta \qquad \frac{d\tau}{d\eta} = \sqrt{\frac{R^3}{8M}}\,(1+\cos\eta) = \sqrt{\frac{R}{2M}}\,r$$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{R^2}{4}(1+\cos\eta)(1-\cos\eta) \qquad \left(\frac{d\tau}{d\eta}\right)^2 = \frac{R}{2M}\,r^2$$

→ dividing

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\frac{R^2}{4}\left(2 - \frac{2r}{R}\right)\frac{2r}{R}}{\frac{R}{2M}\,r^2}$$

$$= \frac{2M}{r} - \frac{2M}{R}$$

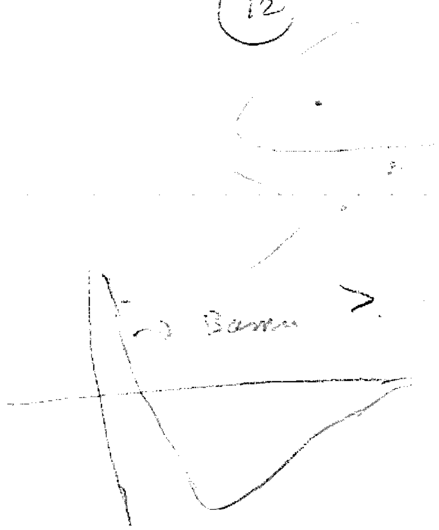
identification

$$E^2 = 1 - \frac{2M}{R}$$

→ Sort out the orbit business

Newtonian case. $T + V$

$$\frac{1}{2} m \dot{r}^2 + \left(\frac{L^2}{2m^2 r^2} - \frac{1}{r} \right)$$



→

Lets see how this is reproduced

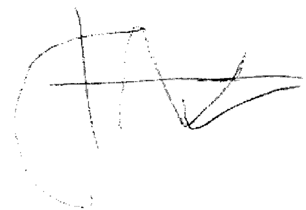
$$r = \frac{L^2}{2m}\left[1 - \sqrt{1 - \frac{12 m^2}{L^2}} \right]$$

$L \to$ low

$$\frac{L^2}{2m} \quad \frac{1}{2} \cdot \frac{12 m^2}{L^2}$$

$= 3m$

but what is the Energy



~~E = R - 2M~~

$$E = \frac{r - 2M}{\sqrt{r \cdot r - 3M}}$$

$\frac{-r\,dr}{2m-r}$ 13

Radial Null geodesics for $r < 2M$.

1st solve.

$$dt = \frac{dr}{\frac{2m}{r} - 1} \qquad dt = \frac{-dr}{\frac{2m}{r} - 1}$$

$$dt = -dr - 2m \ln\left[1 - \frac{r}{2m}\right] \qquad t = dr + 2m \ln\left[1 - \frac{r}{2m}\right]$$

t decreases r decreases + C (~~ingoing~~ out) − A

t increases / r decreases + C (~~outgoing~~ in)

Note that $dt = \frac{r\,dr}{2m - r}$

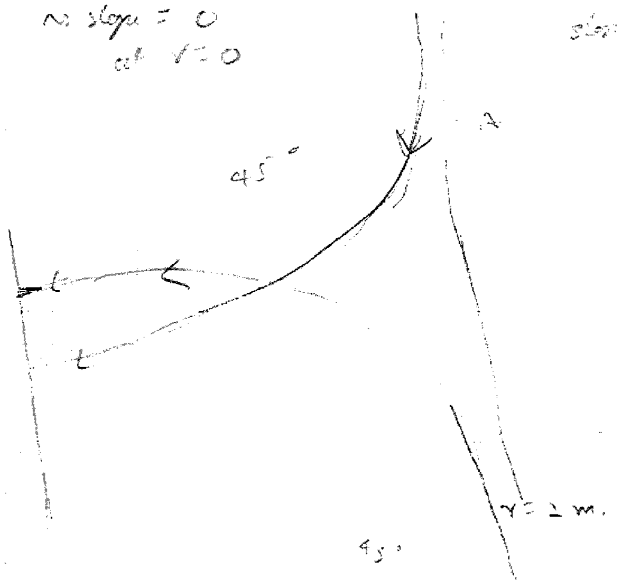
$\therefore$ slope = 0 at r = 0

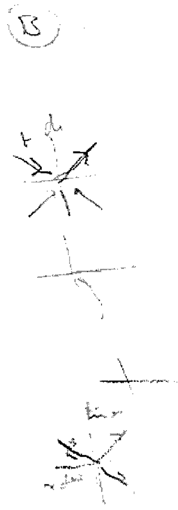
$dt = \frac{-r\,dr}{2m - r}$ by r decreases

slope = 0 at r = 0

Both depends r decreases

plotting this





Geodesics in E F coordinates for $r < 2m$ (14)

$v = t + r_*$ as we have seen

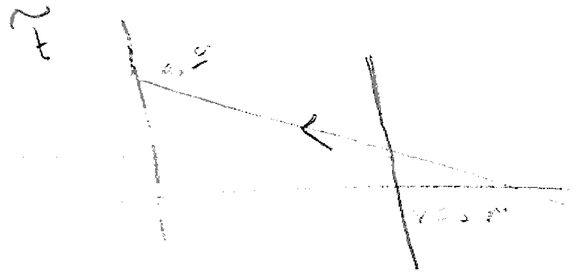
$v = \tilde{t} + r$ r^* for $r < 2m$ $r^* = r + 2m \ln\left[1 - \frac{r}{2m}\right]$

$\tilde{t} = t + 2m \ln\left[1 - \frac{r}{2m}\right]$ from integrating

$v = \tilde{t} + r$

$\tilde{t} = t + r$ $\left(dr^* = -\frac{dr}{\frac{2m}{r} - 1} \right)$

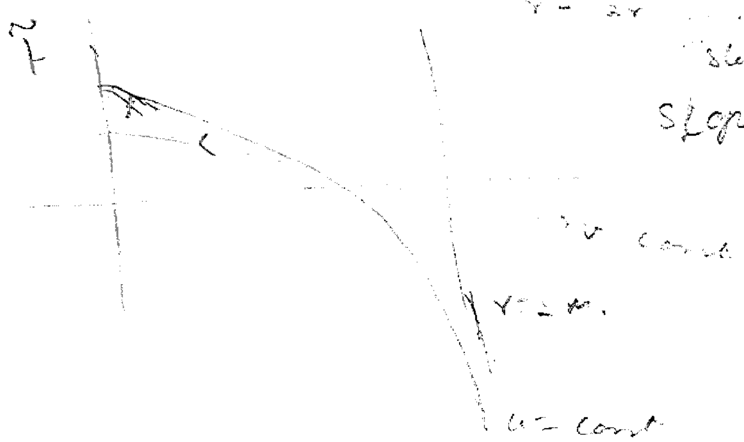
so ingoing geodesic looks like



but how does u look

$$u = t - r^* = \tilde{t} - r - 4m \ln\left(1 - \frac{r}{2m}\right)$$

so u = constant $\tilde{t} = r + 4m \ln\left(1 - \frac{r}{2m}\right)$



Embedding the 3-surface in R^4

metric on R^4 $\quad dr^2 + r^2 d\Omega^2 + dz^2$

$$z = z(r).$$

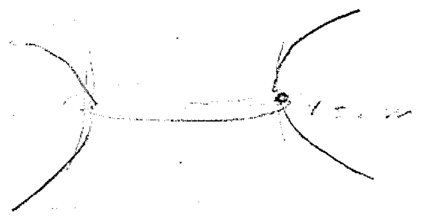
$$\left[\left(\frac{dz}{dr}\right)^2 + 1\right] dr^2 + r^2 d\Omega^2$$

$$\left(\frac{dz}{dr}\right)^2 + 1 = \frac{1}{1-\frac{2m}{r}}$$

$$\left(\frac{dz}{dr}\right)^2 = \frac{2m/r}{1-\frac{2m}{r}} = \frac{2m}{r-2m}$$

$$\frac{dz}{dr} = \pm\sqrt{\frac{2m}{r-2m}} \qquad z = \pm\frac{\sqrt{\cancel{2}m}}{2}\sqrt{r-2m}$$

(2 copies)



The derivation of the RN metric. (16)

1st The soln of Maxwell's eq in spherical geometry.

→ Assume $ds^2 = -f\,dt^2 + g\,dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$

→ Static Assum $F_{\mu\nu} = A\sqrt{fg}\; dt \wedge dr$ → $dF = 0$.

$= A(r)\, e_0 \wedge e_r$.

→

Maxwell's Eq: $D_\mu F^{\mu\nu} = J^\nu$

→ $\frac{1}{\sqrt{g}}\partial_\mu [\sqrt{g} F^{\mu\nu}] = J^\nu$ vacuum.

$= 0$ → No source.

$F^{0r} = -\frac{A}{\sqrt{fg}}$.

$\sqrt{g}\, F^{0r} = r^2 \sin\theta \cdot A$ → A (fn of r only)

→ $A = \frac{-e}{r^2}$ → soln / $r \neq 0$

To find the Net charge

$$\frac{1}{4\pi}\int_{2\text{-sphere at } \infty} {}^*F$$

$(^*F)_{\mu_1\mu_2} = \frac{1}{L2}\, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_3\mu_4}$

$\varepsilon_{0r\theta\varphi}$ → Normalization

$(^*F) = \sqrt{g} \cdot \frac{e}{r^2\sqrt{fg}}$

$\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$

$= \sqrt{fg}\; r^2 \sin\theta\; \frac{e}{r^2\sqrt{fg}}$

→ choose Normalization so that $\varepsilon_{0r\theta\varphi} = \sqrt{g}$.

→ $\frac{1}{4\pi}\int_{2\text{-sphere}} {}^*F = e$

whats the stress Energy tensor / We will work with local Lorentz Coordinates (17)

$$F_{0r} = \frac{-e}{r^2}$$

The stress energy tensor is given by

$$T_{ab} = \frac{1}{4\pi}\left[F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{de}F^{de}\right]$$

$$T_{00} = \frac{1}{8\pi}\frac{e^2}{r^4}$$

$$T_{rr} = -\frac{1}{8\pi}\frac{e^2}{r^4} \qquad ; T_{\theta\theta} = \frac{1}{8\pi}\frac{e^2}{r^4} = T_{\varphi\varphi}$$

→ Now $T = g^{ab}T_{ab} = 0.$

→ ∴ Einstein Eq.

$$R_{\mu\nu} = 8\pi\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]$$

$$= 8\pi T_{\mu\nu}$$

→ we have already computed R curvature components. for $ds^2 = -f\,dt^2 + g\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$.

~~for~~ Lets recall the curvature components.

~~Roo~~ equation
from $R_{00} + R_{rr}$ we get the same condition as
$= T_{00} + T_{rr}$ before $g = \frac{1}{f}$.
$= \underline{\underline{0}}$.

→ Now:

for R_{00} & $R_{\varphi\varphi}$: The Equation reduces to.

$$-\partial_r f + \frac{1}{r} - \frac{f}{r} = \frac{e^2}{r^3}$$

$$\rightarrow -\frac{\partial_r}{r}\left[rf\right] + \frac{1}{r} = \frac{e^2}{r^3}$$

$$\rightarrow \partial_r (r f) = 1 - \frac{e^2}{r^2}$$

(15) ~~25~~ (20)

$$r f = r + \frac{e^2}{r^2} - 2M$$

$$f = 1 + \frac{e^2}{r^2} - \frac{2M}{r}$$

we do

$$ds^2 = -\left[1 + \frac{e^2}{r^2} - \frac{2M}{r}\right] dt^2 + \frac{1}{\left(1 + \frac{e^2}{r^2} - \frac{2M}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi)$$

also.

$$F = -\frac{e}{r^2} \quad dt \wedge dr$$

(19)

The curvature Calculate in RN

$A = 1 + \frac{e^2}{r^2} - \frac{2m}{r}$

$\partial_r A = -\frac{2e^2}{r^3} + \frac{2m}{r^2}$

$\partial_r^2 A = +\frac{6e^2}{r^4} - \frac{4m}{r^3}$

Rem

$R_{0r0r} = -\frac{1}{2} \partial_r^2 A$

$= \frac{1}{2}\left(-\frac{6e^2}{r^4} + \frac{4m}{r^3}\right) = -\frac{3e^2}{r^4} + \frac{2m}{r^3}$

$R_{0\theta0\theta} = \frac{1}{2r}\left[-\frac{e^2}{r^3} + \frac{2m}{r^2}\right] = -\frac{e^2}{2r^4} + \frac{m}{r^3}$

$R_{0\varphi0\varphi} = R_{0\theta0\theta}$

$R_{r\theta r\theta} = -\{ \quad \}$

~ blowing along radial ~ $\frac{1}{r^8}$

Lets get the. Interior dust, soln
pressureless dust

$P = 0$

The stress Energy tensor is $T_{00} = \rho$
$T_{ij} = 0$

$\rightarrow ds^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + \underbrace{\sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)}_{\text{metric of unit sphere.}}]$

$e_0 = d\tau \; : \; e_\chi = a\,d\chi \; : \; e_\theta = a\sin\chi\, d\theta; \; e_\phi = a\sin\chi\sin\theta\, d\phi$

$\rightarrow$ spin connection

$de_a + \omega_a{}^b \wedge e_b = 0.$

$\cancel{de_0} + \omega_0{}^b \wedge e_b = 0$; '0' eq

$d\, \dot{a}\, d\tau \wedge d\chi + \omega_\chi{}^b \wedge e_b = 0.$; χ eq

$\dot{a}\sin\chi\, d\tau\wedge d\theta + a\cos\chi\, d\chi\wedge d\theta + \omega_\theta{}^b \wedge e_b = 0$; θ eq

$\dot{a}\sin\chi\sin\theta\, d\tau\wedge d\phi + a\cos\chi\sin\theta\, d\chi\wedge d\phi + a\sin\chi\cos\theta\, d\theta\wedge d\phi + \omega_\phi{}^b \wedge e_b = 0$

$\omega_\chi{}^0 = \dot{a}\, d\chi$

$\omega_\theta{}^0 = \dot{a}\sin\chi \cdot d\theta.$

$\omega_\phi{}^0 = \dot{a}\sin\chi\sin\theta\, d\phi.$

$\omega_\theta{}^\chi = \cos\chi\, d\theta$

$\omega_{\chi\theta} = -\cos\chi\, d\theta$

$\omega_\phi{}^\chi = \cos\chi\sin\theta\, d\phi.$

$\omega_\phi{}^\theta = \cos\theta\, d\phi$

$\omega_{\chi 0} = -\dot{a}\, d\chi$

$\omega_{\theta 0} = -\dot{a}\sin\chi\, d\theta$

$\omega_{\phi 0} = -\dot{a}\sin\chi\sin\theta\, d\phi$

(21)

$R_{\chi\theta} = d\omega_{\chi\theta} + \omega_{\chi}{}^{a} \wedge \omega_{a\theta}$

$= \sin\chi \, d\chi \wedge d\theta + \dot{a} \, d\chi \wedge \dot{a} \sin\chi \, d\theta$

$+ (-\cos\chi \sin\theta \, d\varphi) \wedge (\cos\theta \, d\varphi)$

$= \frac{1}{a^2} e_\chi \wedge e_\theta + \frac{\dot{a}^2}{a^2} e_\chi \wedge e_\theta$

$R_{\chi\theta} = \left(\frac{1+\dot{a}^2}{a^2}\right) e_\chi \wedge e_\theta$

$R_{\chi\varphi} = d\omega_{\chi\varphi} + \omega_{\chi}{}^{a} \wedge \omega_{a\varphi}$

$= \sin\chi \sin\theta \, d\chi \wedge d\varphi - \cos\chi \cos\theta \, \cancel{d\theta} \wedge d\varphi$

$+ \dot{a} \, d\chi \wedge \dot{a} \sin\chi \sin\theta \, d\varphi$

$+ (-\cos\chi \, \cancel{d\theta}) \wedge (-\cos\theta) \, d\varphi$

$\boxed{R_{\chi\varphi} = \left(\frac{1+\dot{a}^2}{a^2}\right) \cdot e_\chi \wedge e_\varphi}$

$R_{\theta\varphi} = d\omega_{\theta\varphi} + \omega_{\theta}{}^{a} \wedge \omega_{a\varphi}$

$= + \sin\theta \, d\theta \wedge d\varphi + \dot{a} \sin\chi \, d\theta \wedge \dot{a} \sin\chi \sin\theta \, d\varphi$

$+ \cos\chi \, d\theta \wedge (-\cos\chi \sin\theta \, d\varphi)$

$= \sin^2\chi \sin\theta \, d\theta \wedge d\varphi + \dot{a}^2 \sin^2\chi \sin\theta \, d\theta \wedge d\varphi$

$= \frac{(1+\dot{a}^2)}{a^2} e_\theta \wedge e_\varphi$

$$R_{0\chi} = d\omega_{0\chi} + \omega_0{}^a \wedge \omega_{a\chi}$$

$$= \ddot{a}\, dt \wedge d\chi + \omega_0{}^\theta \cancel{\wedge \omega_{\theta\chi}} + \omega_0{}^\varphi \cancel{\wedge \omega_{\varphi\chi}}$$

$$= \ddot{a}\, dt \wedge d\chi = \frac{\ddot{a}}{a} e_0 \wedge e_\chi$$

$$R_{0\theta} = d\omega_{0\theta} + \omega_0{}^a \wedge \omega_{a\theta}$$

$$= \ddot{a} \sin\chi\, dt \wedge d\theta + \dot{a} \sin\chi\, d\chi \wedge d\theta + \omega_0{}^\chi \wedge \omega_{\chi\theta} + \omega_0{}^\varphi \wedge \omega_{\varphi\theta}$$

$$= \ddot{a} \sin\chi\, dt \wedge d\theta + \dot{a} \cancel{\cos\chi\, d\chi \wedge d\theta} + \dot{a}\, d\chi \wedge (-\cos\chi) d\theta + \dot{a} \sin\chi \sin\theta\, d\varphi \wedge \cos\theta\, d\varphi$$

$$= \ddot{a} \sin\chi\, dt \wedge d\theta$$

$$= \frac{\ddot{a}}{a} e_0 \wedge e_\theta$$

$$R_{0\varphi} = d\omega_{0\varphi} + \omega_0{}^a \wedge \omega_{a\varphi}$$

$$= \ddot{a} \sin\chi \sin\theta\, dt \wedge d\varphi + \dot{a} \cos\chi \sin\theta\, d\chi \wedge d\varphi + \dot{a} \sin\chi \cancel{\cos\theta\, d\theta \wedge d\varphi}$$

$$+ \dot{a}\, d\chi \wedge (-\cos\chi \sin\theta\, d\varphi) + \dot{a} \sin\chi\, d\theta \wedge \cos\theta\, d\varphi$$

$$= \frac{\ddot{a}}{a} e_0 \wedge e_\varphi$$

So we have

$$R_{0\chi0\chi} = -\frac{\ddot{a}}{a} \quad ; \quad R_{0\theta0\theta} = -\frac{\ddot{a}}{a} \quad ; \quad R_{0\varphi0\varphi} = -\frac{\ddot{a}}{a}$$

$$R_{\chi\theta\chi\theta} = \left(\frac{1+\dot{a}^2}{a^2}\right) \quad ; \quad R_{\chi\varphi\chi\varphi} = \left(\frac{1+\dot{a}^2}{a^2}\right)$$

$$R_{\theta\varphi\theta\varphi} = \left(\frac{1+\dot{a}^2}{a^2}\right).$$

$$R_{00} = R^{\chi}{}_{0\chi0} + R^{\theta}{}_{0\theta0} + R^{\varphi}{}_{0\varphi0}$$

$$= -\frac{3\ddot{a}}{a}$$

$$R_{0\chi} = R^{\varphi}{}_{0\varphi\chi} + R^{\theta}{}_{0\theta\chi} \quad 0 \quad \text{same of } R_{0i}$$

$$R_{\chi\chi} = R^{0}{}_{\chi0\chi} + R^{\theta}{}_{\chi\theta\chi} + R^{\varphi}{}_{\chi\varphi\chi}$$

$$= \frac{\ddot{a}}{a} + 2\left(\frac{1+\dot{a}^2}{a^2}\right).$$

$$R_{\chi\theta} = \underbrace{\cancel{R^{0}{}_{\chi0\theta}} + R^{\varphi}{}_{\chi\varphi\theta}} = 0$$

$$R_{\theta\theta} = \frac{\ddot{a}}{a} + 2\left(\frac{1+\dot{a}^2}{a^2}\right)$$

$$R_{\varphi\varphi} = \frac{\ddot{a}}{a} + 2\left(\frac{1+\dot{a}^2}{a^2}\right)$$

Einstein's Eq

$$R_{\mu\nu} = 8\pi\left[T_{\mu\nu} - \frac{g_{\mu\nu}T}{2}\right].$$

$$-3\frac{\ddot{a}}{a} = (8\pi)\left(+\frac{\rho}{2}\right)$$

$$T_{00} = \rho$$

$$T = -\rho$$

$$T_{00} + \frac{T}{2} = \rho - \frac{\rho}{2}$$

$$= \frac{\rho}{2}$$

$$\boxed{\frac{3\ddot{a}}{a} = -4\pi\rho}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{1+\dot{a}^2}{a^2}\right) = 8\pi\left[-\frac{\cdot}{2}\,[\rho]\right]$$

$\chi\chi$ ..]

$$-\frac{4\pi\rho}{3} + 2\frac{(1+\dot{a}^2)}{a^2} = 4\pi\rho.$$

$4\pi\rho$

$$-\frac{2\pi\rho}{3} + \left(\frac{1+\dot{a}^2}{a^2}\right) = 2\pi\rho$$

$-\frac{4\pi\rho}{3}$

$$\rightarrow \boxed{\left(\frac{1+\dot{a}^2}{a^2}\right) = \frac{8\pi\rho}{3}.}$$

$\frac{1+3}{3}$

or

$$\frac{1+\dot{a}^2}{a^2} = -\frac{2\ddot{a}}{a}$$

$$\rightarrow$$

$$\cancel{a^2 + \dot{a}^2 = -2a\ddot{a}}$$

Can be solved by

$$\boxed{\begin{aligned} a &= \frac{a_m}{2}(1+\cos\eta) \\ \tau &= \frac{a_m}{2}(\eta + \sin\eta) \end{aligned}}$$

check

from this $\rho = \frac{3}{\pi a_m^2}\left(\frac{1}{1+\cos\eta}\right)$

Match this with Schwarschild Exterior

The surface of star say. R → have out a Radial time like geodesic.

→ To match

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \sin^2\chi_0\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)$$

$$K_{\eta\eta} = K_{\eta\theta} = K_{\eta\varphi} = K_{\theta\varphi} = 0$$

$$K_{\theta\theta} = \frac{K_{\varphi\varphi}}{\sin^2\theta} = -a(\eta)\sin\chi_0\cos\chi_0$$

matching

$$C = 2\pi R = \frac{2\pi R_i}{2}(1+\cos\eta)$$

$$\tau = \left(\frac{R_1^3}{8M}\right)^{1/2}(\eta + \sin\eta)$$

geodesics

max of $R_1 = a_m \sin\chi_0$

$$M = \tfrac{1}{2} a_m \sin^3\chi_0$$

$\chi = \chi_0$ matching

$\Rightarrow$

This requires

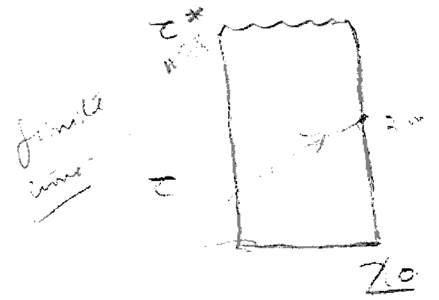
$$\boxed{R = a_m \sin\chi_0}$$

Radius

$$M = \tfrac{4}{3}\pi\rho(\ldots)\, a_m^3 \sin^3\chi_0$$

$$= \tfrac{4}{3}\pi \frac{3}{\pi a_m^2}\frac{1}{8}\, a_m^3 \sin^3\chi_0$$

1^{st} Understand the penrose diagram of the collapse.



Note that $a = \frac{a_m}{2}(1+\cos\eta)$

$$\tau = \frac{a_m}{2}(\eta + \sin\eta).$$

$\boxed{\eta = +\pi}$ $\underline{a = 0}$

$$\tau^* = \frac{a_m}{2}(+\pi)$$

for $a = 0$ what's the curvature.

Blows up / Ricci indeed! $\rho = \frac{3}{8\pi a_m}\left(\frac{1}{1+\cos\eta}\right)^3$

Check $R_{00} \sim T_{00} \sim \rho$

here show by $-10^2\left[\frac{dt^2}{a^2} + d\chi^2\right]$

$$= a^2\left[-(d\tilde{\tau})^2 + d\chi^2\right]$$

Equation $d\tilde{\tau} = \int \frac{d\tau}{a} = \int \frac{d\tau}{\eta}\frac{1}{a}\,d\eta$

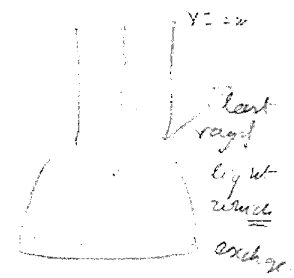
$= \int d\eta$ and $a = \frac{a_m}{2}(1+\cos\eta)$

$$\tau = \frac{a_m}{2}(\eta + \sin\eta)$$

~~Conformal $\chi = \chi_0$ boundary~~ of dust.

→ Now we attach to this the schwarzschild soln.

What is this last ray of light



a sur η_{2m}. The sum a

$$\frac{a_m}{2}(1+\cos\eta_{2m}) = 2m$$

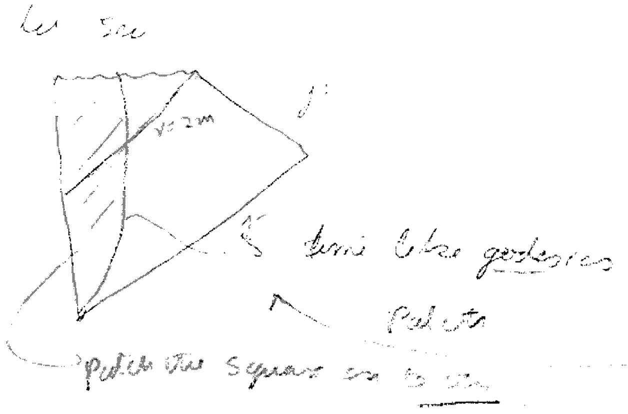
even horizon

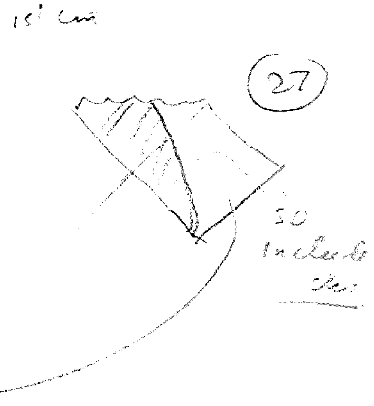
$$\frac{a_m}{2}(1+\cos\eta_{2m}) = \frac{1}{2}a_m\sin^3\chi_0$$

Can solve for $\boxed{\eta_{2m}}$

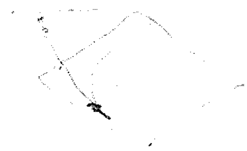
Son and the light ray back (last light

Now





(27)



To understand the Kruskal Completion of the RN soln

28

$$ds^2 = -\left(1+\frac{e^2}{r^2}-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1+\frac{e^2}{r^2}-\frac{2m}{r}\right)} + r^2 d\Omega^2$$

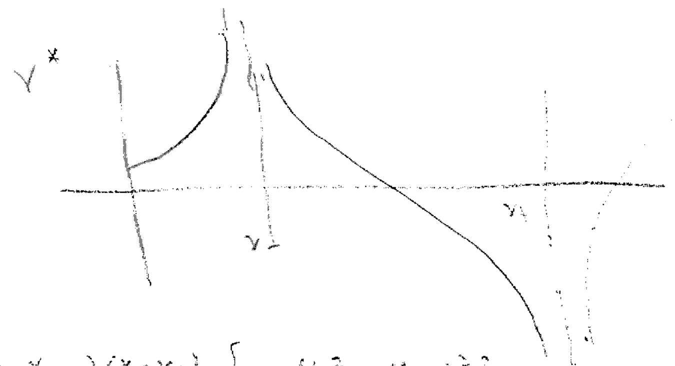
$$dr^* = \frac{dr}{1+\frac{e^2}{r^2}-\frac{2m}{r}}$$

$$r_+ = m + \sqrt{m^2-e^2}$$

$$r_- = m - \sqrt{m^2-e^2}$$

$$r^* = r + \frac{r_+^2}{r_+-r_-}\ln\left(\frac{r}{r_+}-1\right) - \frac{r_-^2}{r_+-r_-}\ln\left(\frac{r}{r_-}-1\right)$$

plot



$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}\left[-dt^2 + (dr^*)^2\right]$$

→ To figure out the kruskal coordinates

~~for~~ for a Radial Null geodesic we have

$$\frac{(r-r_+)(r-r_-)}{r^2}\frac{dt}{d\lambda} = E$$

$$d\lambda = \frac{1}{E}\frac{(r-r_+)(r-r_-)}{r^2}\,dt$$

again define $u = t - r_*$ & $v = t + r_*$ & write it in terms of u, v

$$\ln\left(\frac{r-r_+}{r_+}\right) = \left(\frac{r_+-r_-}{r_+^2}\right)\cdot\left[\frac{v-u}{2} - r\right] + \frac{r_-^2}{r_+^2}\ln\left(\frac{r-r_-}{r_+}\right)$$

from here ... motivated to define $\alpha = \frac{1}{2}\frac{r_+-r_-}{r_+^2}$

$$\beta = \frac{r_-^2}{r_+^2}$$

Near ~~horiz~~ horizon.

$$\frac{r-r_+}{r_+} = e^{\alpha(v-u)} \quad \text{so } dr = e^{\alpha(v-u)}\,dt$$

: upto a constant

~~λ =~~ ~~Constants~~

$$\boxed{\begin{aligned} \lambda_{\text{Constant } v} &= -e^{-\alpha u} \\ \lambda_{\text{Constant } u} &= e^{\alpha v} \end{aligned}}$$

Now ~~In these coordinates~~ to go over to penrose

$$\tan u = -e^{-\alpha u} = -e^{-\alpha(t-r_*)}$$

$$= -e^{-\alpha t}\left\{ e^{\alpha r} \left|\frac{r-r_+}{r_+}\right|^{1/2} \left|\frac{r-r_-}{r_-}\right|^{-\beta/2} \right\}$$

$$\tan v = +e^{\alpha v} = +e^{\alpha(t+r_*)}$$

$$= e^{\alpha t}\left\{ e^{\alpha r} \left|\frac{r-r_+}{r_+}\right|^{1/2} \left|\frac{r-r_-}{r_-}\right|^{-\beta/2} \right\}$$

Solve for r from

$$\tan u \tan v = -e^{2\alpha r} \left|\frac{r-r_+}{r_+}\right| \left|\frac{r-r_-}{r_-}\right|$$

Now lets go over to region $r_- < r < r_+$

we choose. $u = r_* + t$ & $v = r_* - t$

[remember $r_* = \frac{u+v}{2}$ hence ~~to...~~ → so λ must change ~~sign~~]

$$\tan u = +e^{+\alpha(t+r_*)} = e^{\alpha t}\left\{ e^{\alpha r} \left|\frac{r-r_+}{r_+}\right|^{1/2} \left|\frac{r-r_-}{r_-}\right|^{-\beta/2} \right\}$$

$$\tan v = e^{\alpha(t+r_*)} = e^{-\alpha t}\left\{ e^{\alpha r} \left|\frac{r-r_+}{r_+}\right|^{1/2} \left|\frac{r-r_-}{r_-}\right|^{-\beta/2} \right\}$$

→ here

~~tan~~

$$\text{So } \tan u \tan v = +e^{2\alpha r} \left|\frac{r-r_+}{r_+}\right| \left|\frac{r-r_-}{r_-}\right|^{-\beta} \quad r_- < r < r_+$$

In region $\mathcal{E}$ ~~it is~~ so $r < r_-$ Same as 1st

Now lets try to write the metric

(30) ②

→ The metric
$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}\left(-dt^2 + dr_*^2\right)$$

→ so $\sec^2 u \, du = \alpha \, e^{-\alpha(t - r_*)}(dt - dr_*)$

$$\sec^2 v \, dv = \alpha \, e^{\alpha(t + r_*)}(dt + dr_*)$$

$$\sec^2 u \sec^2 v \, du \, dv = \alpha^2 \, e^{2\alpha r_*}\left(dt^2 - dr_*^2\right)$$

$$\tan u \tan v = -e^{2\alpha r_*}$$

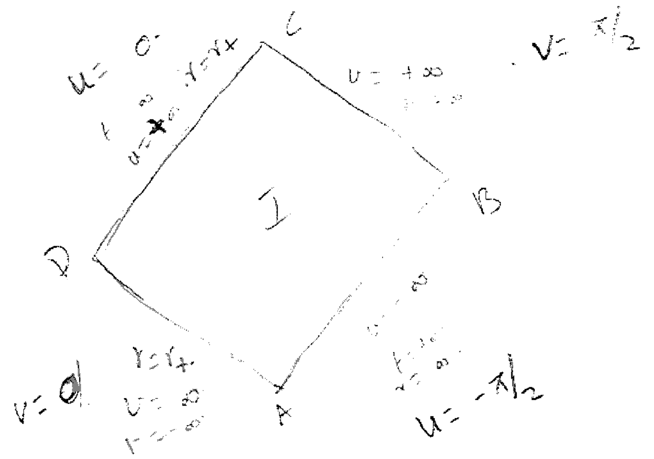
So divided

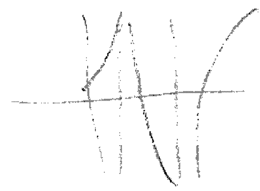
$$\frac{1}{\alpha^2 \sin 2u}\frac{4}{\sin 4u} du\, dv \frac{(r-r_+)(r-r_-)}{r^2} = \left(-dt^2 + dr_*^2\right)\frac{(r-r_+)(r-r_-)}{r^2}$$

The penrose diagram (31) (4)

In region I $r > r_+$

given $u = t - r_*$ $\quad v = t + r_*$



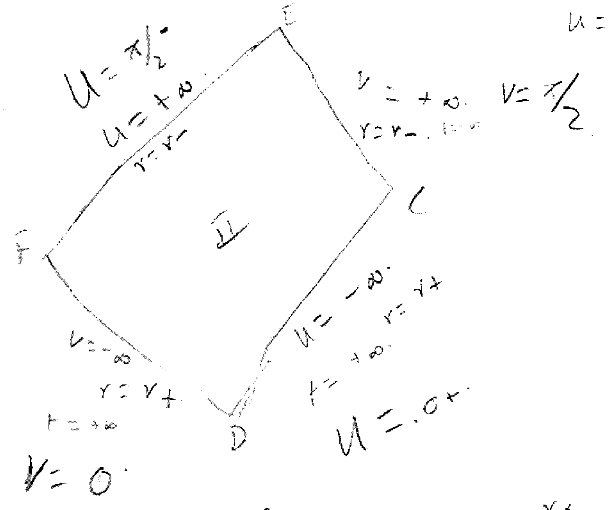


$\tan u = -e^{-\alpha u}$

$\tan v = e^{\alpha v}$

In region II $r_- < r < r_+$

$u = t + r_*$; $v = r_* - t$



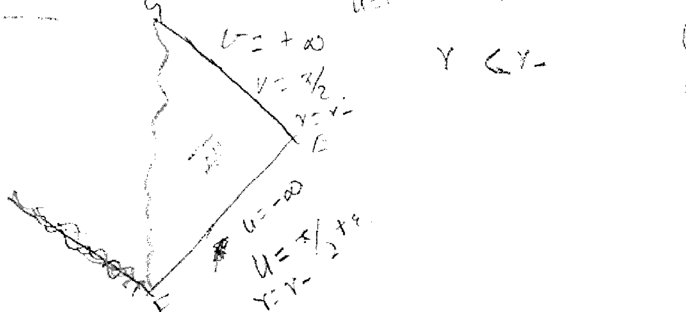
$\tan u = e^{\alpha u}$

$\tan v = e^{\alpha v}$

In region III

$u = t - r_*$; $v = t + r_*$

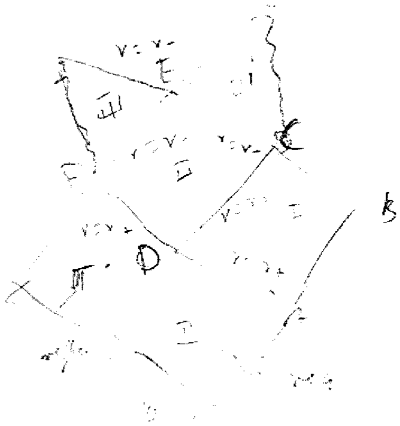
$r < r_-$



$\tan u = -e^{-\alpha u}$

$\tan v = e^{\alpha v}$

The glued diagram



Now when $r_+ = r_- = m$. horizon when $e^2 = m^2 \to r^2 + m^2 - 2mr$ perfect square

$$r^* = r + 2m \ln\left(\frac{r}{m} - 1\right) - \frac{m}{\left(\frac{r}{m} - 1\right)}$$

plot of r^*

$$r^* = r + m \log\left[\frac{r^2}{m^2} - 2\frac{r}{m} + \frac{e^2}{m^2}\right] + \frac{2m^2}{e^2/m^2 - 1} \tan^{-1}\left(\frac{r/m}{e^2/m^2 - 1}\right)$$

To make the singularity explicit $e=0$ (33)

$$(ds^2) = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dt^2 - \frac{2a\sin^2\theta\,(2mr)}{\rho^2}\,dt\,d\varphi$$

$$+ \frac{\rho^2}{\Delta}dr^2 + \left[(r^2+a^2)\sin^2\theta + \frac{2mr}{\rho^2}a^2\sin^4\theta\right]d\varphi^2 + \rho^2\,d\theta^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 - 2mr \rightarrow$$

$4m^2 - 4a^2$

$a^2 > m^2$

No real roots

The ring singularity is explicit

simplest case $a^2 > m^2$: metric singular only at $r=0$, $\theta = \pi/2$: to see it

we go over to another coordinate system $(x, y, z, \tilde{t})$ given by

$$x + iy = (r + ia)\sin\theta \exp i\left[\varphi + i\int\frac{a\,dr}{r^2+a^2-2mr}\right]$$

$$z = r\cos\theta \; : \; \tilde{t} = \int\left[dt + \frac{r^2+a^2}{(r^2+a^2-2mr)}dr\right] - r.$$

→ The metric reduces to

$$ds^2 = dx^2 + dy^2 + dz^2 - d\tilde{t}^2$$

$$+ \frac{2mr^3}{r^4+a^2z^2}\left[\frac{r(x\,dx+y\,dy) - a(x\,dy - y\,dx)}{r^2+a^2} + \frac{z\,dz}{r} + d\tilde{t}\right]^2$$

→ we determine r from

$$x^2+y^2 = (r^2+a^2)\sin^2\theta, \quad z = r\cos\theta$$

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$$

$$\rightarrow r^4 - (x^2+y^2+z^2-a^2)r^2 - a^2z^2 = 0.$$

$x = \sqrt{r^2+a^2}\sin\theta\cos\varphi$

$y = \sqrt{r^2+a^2}\sin\theta\sin\varphi$

$r=0 \rightarrow z=0$

$\theta = \pi/2$

from the metric it is shown The singularity is at $r=0$, $z=0$.

$$r^2 + z^2 = a^2 - \frac{a^2z^2}{r^2} + r^2$$

$$x^2 + \left(1+\frac{a^2}{r^2}\right)z^2 = a^2 + r^2$$

Surfaces of constant r are. are ellipsoids in the (34) (tl)
(x, y, z) plane but at $r=0$, $z=0$ → degenerate to disc

as: one needs from the eq
for $r \underline{<0}$ to be satisfied

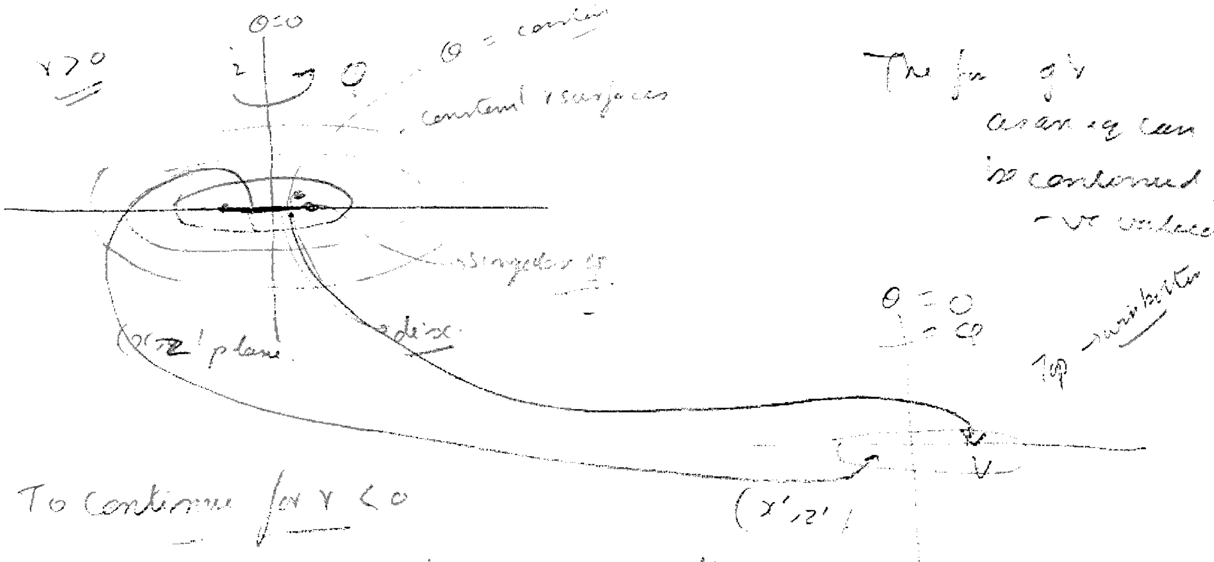
but $x^2+y^2 = a^2 \sin^2\theta$

So $\boxed{x^2+y^2 \leq a^2}$ when.

$\theta = \pi/2$ also

$x^2+y^2 = a^2$ it is a ring

→ One can also show explicitly the RiemannR^{abcd} diverges at the ring. but Not on the disc



The fn of r as an eq can be continued for $-$ve values

To continue for $r<0$
x, y on to → identify as
$r<0$ on bottom

→ so metric in Extended space is same except
$\underline{r \to -r}$ The term (mr) in Δ for small r flips sign.

near $r \to 0$ $(d\varphi^2)$ is time like.
the ring & $\theta = \pi/2$. ↪ so closed time like curves

Only time like curves from the inside of the ring do cross the boundary & go out would be outside.

How to continue it across r_+ & r_-

$$r_+ = m + \sqrt{m^2 - a^2} \quad ; \quad r_- = m - \sqrt{m^2 - a^2}$$

→ go to a Kerr coordinate

$$du_+ = dt + (r^2 + a^2)\Delta^{-1} dr \quad ; \quad d\varphi_+ = d\varphi + a\Delta^{-1} dr$$

The metric is then

$$ds^2 = \rho^2 d\theta^2 - 2a \sin^2\theta \, dr \, d\varphi_+ + 2 dr \, du_+$$

$$+ \rho^2 \frac{1}{\rho^2}\left[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta\right]\sin^2\theta \, d\varphi_+^2$$

$$- \frac{4a}{\rho^2} m r \sin^2\theta \, d\varphi_+ \, du_+ - \left(1 - \frac{2mr}{\rho^2}\right) du^2$$

↳ a genuine singularity at $r = 0$

⟶ seeming form → etc.

The Kerr metric along the symmetry axis is given by

$$ds^2 = \frac{\Delta}{(r^2+a^2)}\left[-dt^2 + \frac{(r^2+a^2)}{\Delta^2}{}^2 dr^2\right]$$

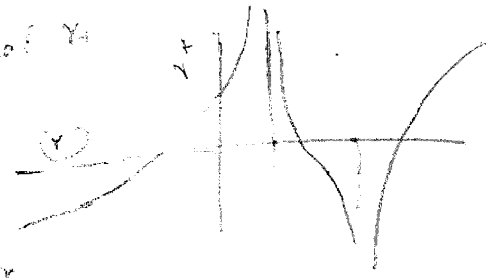
$$\Delta = r^2+a^2-2mr = (r-r_-)(r-r_+) \; ; \; \frac{2m \pm \sqrt{4m^2-4a^2}}{2}$$

$u = t - r_*$; $v = t + r_*$

$$r_* = r + \frac{r_+^2+a^2}{r_+-r_-}\log\left(\frac{r}{r_+}-1\right) - \frac{r_-^2+a^2}{r_+-r_-}\log\left(\frac{r}{r_-}-1\right)$$

look at affine parameter close to horizon

plot r_*



$$\frac{(r_*-r_-)(r-r_+)}{r_+^2+a^2}\frac{dt}{d\lambda} = E$$

~ so from this we see

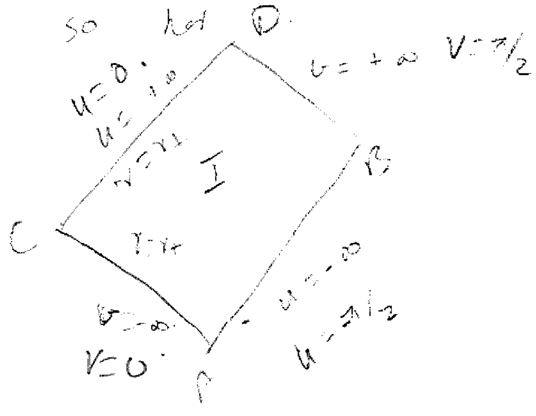
$$d\lambda = e^{\frac{1}{2}\left(\frac{v-u}{2}\right)\frac{r_+-r_-}{(r_+^2+a^2)}}$$

⟶ motivates us to define $\alpha = \dfrac{r_+-r_-}{2(r_+^2+a^2)}$

⟶ $\text{Lin } u = -e^{-\alpha u}$ $\qquad$ $\text{Lin } v = e^{\alpha v}$

$u = t - r_*$; $v = t + r_*$ $\qquad$ in $r_+ < r < \infty$

so had



Consider the 2D Cosmological background. (Massive scalar field)

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^2\right)$$

$$\frac{1}{\sqrt{g}}\partial_\eta\left[\sqrt{g}\, g^{\eta\eta}\partial_\eta \varphi\right] + \frac{1}{\sqrt{g}}\partial_x\left[\sqrt{g}\, g^{xx}\partial_x \varphi\right] - m^2\varphi = 0$$

$\rightarrow$ $\sqrt{g} = a^2$

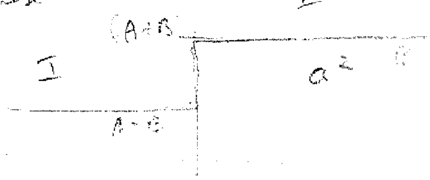
$$-\partial_\eta^2\varphi + \partial_x^2\varphi - m^2\varphi\sqrt{g}$$

$$\varphi = \frac{1}{\sqrt{2\pi}} e^{ikx}\chi_k(\eta)$$

$$-\partial_\eta^2\varphi - k^2\varphi - m^2a^2\varphi = 0$$

$$\rightarrow \quad \frac{d^2\varphi}{d\eta^2} + \left(k^2 + m^2a^2\right)\varphi = 0$$

Lets solve it for a simple case



Soln in I $\varphi = \frac{1}{\sqrt{2\pi}\sqrt{2\omega_I}} e^{-i\left(\sqrt{k^2+m^2a^2}\right)\eta} e^{ikx}$ +ve frequency $\rightarrow i\frac{\partial}{\partial\eta}$

Soln in II
$$\varphi = \frac{\alpha_k}{\sqrt{2\pi}\sqrt{2\omega_{II}}} e^{-i\omega_{II}\eta} e^{ikx} + \frac{\beta_k}{\sqrt{2\pi}\sqrt{2\omega_{II}}} e^{+i\omega_{II}\eta} e^{ikx}$$

$$\omega_{I(k)} = \sqrt{k^2 + (A-B)m^2}$$

$\rightarrow$ demand soln should be smooth across η. we get the following

$$\frac{1}{\sqrt{2\omega_I}} = \frac{\alpha_k + \beta_k}{\sqrt{2\omega_{II}}}$$

(38) (2)

~~so~~: & we have

$$\frac{1}{\sqrt{2\omega_I}}\omega_I = \frac{\omega_{II}}{\sqrt{2\omega_{II}}}\alpha_k - \frac{\omega_{II}}{\sqrt{2\omega_{II}}}\beta_k$$

$$\sqrt{\frac{\omega_{II}}{\omega_I}} = \alpha_k + \beta_k$$

$$\sqrt{\frac{\omega_I}{\omega_{II}}} = \alpha_k - \beta_k$$

$$\therefore \alpha_k = \frac{1}{2}\left[\sqrt{\frac{\omega_{II}}{\omega_I}} + \sqrt{\frac{\omega_I}{\omega_{II}}}\right]$$

$$\beta_k = \frac{1}{2}\left[\sqrt{\frac{\omega_{II}}{\omega_I}} - \sqrt{\frac{\omega_I}{\omega_{II}}}\right]$$

Note $|\alpha_k|^2 - |\beta_k|^2 = \frac{1}{4}\cdot 4 = 1$

→ from here how to get [illegible]

, coefficient → coefficient

$$a_k \Big|_{in} = \alpha_k a_{k\,out} + \beta_k a_{k\,out}^\dagger$$

$$a_k^\dagger \Big|_{in} = \alpha_k^* a_{k\,out}^\dagger + \beta_k^* a_{k\,out}$$ Explain

so

$$a_{k\,out}^\dagger = \alpha_k^* a_k - \cancel{\beta_k a_k^*}$$

$$\beta_k^* a_k - \alpha_k a_k^\dagger\Big|_{in} = - a_{k\,out}^\dagger$$

$$a_k^\dagger\Big|_{out} = \alpha_k a_k^\dagger\Big|_{in} - \beta_k^* a_k\Big|_{in}$$

$$a_k\Big|_{out} = \alpha_k^* a_k\Big|_{in} - \beta_k a_k^\dagger\Big|_{in}$$

$$\therefore \langle in | a_k^\dagger a_k\Big|_{out} | in\rangle$$

$$= \beta_k^* \beta_k$$

$$= \frac{1}{4}\frac{(\omega_{II} - \omega_I)^2}{\omega_I \omega_{II}}$$

$e^{\cancel{+i\eta}}$ Similarly, the in going wave

$$+ \;=\; \cancel{\alpha_k}\ \beta_k \frac{\sqrt{2}\, e^{+i\omega_I \eta} e^{ikx}}{\sqrt{2\pi}\sqrt{2\omega_I}}$$

$-i$

$$= \frac{\alpha_k}{\sqrt{2\pi}\sqrt{2\omega_{II}}} e^{+i\omega_{II}\eta} e^{ikx} + \frac{\beta_k}{\sqrt{2\pi}\sqrt{2\omega_I}} e^{-i\omega_{II}\eta} e^{ikx}$$

so a wave

$$\frac{a_k e^{-i\omega_I \eta} e^{ikx}}{\sqrt{\omega_I}} + \frac{a_k^{+} e^{i\omega_I \eta} e^{-ikx}}{\sqrt{\omega_I}} \qquad x < 0.$$

$$x > 0 \quad \frac{\alpha_k a_k}{\sqrt{\omega_{II}}} e^{-i\omega_{II}\eta} e^{ikx} + \frac{\alpha_k^* a_k^{+}}{\sqrt{\omega_{II}}} e^{+i\omega_{II}\eta} e^{-ikx}$$

$$+ \frac{\beta_k a_k}{\sqrt{\omega_{II}}} e^{i\omega_{II}\eta} e^{ikx} + \frac{\beta_k^* a_k^{+}}{\sqrt{\omega_{II}}} e^{-i\omega_{II}\eta} e^{-ikx}$$

$$\text{Written as} \quad \frac{b_k e^{-i\omega_{II}\eta} e^{ikx}}{\sqrt{\omega_{II}}} + \frac{b_k^{+} e^{+i\omega_{II}\eta} e^{-ikx}}{\sqrt{\omega_{II}}}$$

making with

$$b_k = \alpha_k a_k + \beta_{-k}^* a_k^{+}$$

$$b_k^{+} = \alpha_k^* a_k^{+} + \beta_{-k} a_k$$

$$\langle m \,|\, b_k^{+} b_k | 0 \rangle = \beta_k^* \beta_k$$

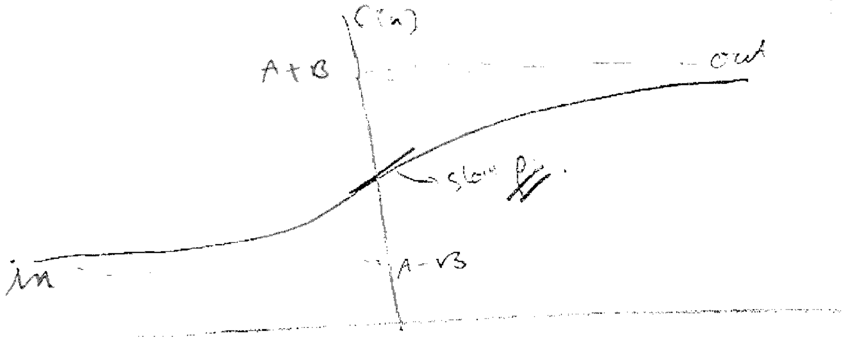
$$= \frac{(\omega_I - \omega_{II})^2}{4(\omega_I \omega_{II})}$$

Lets now do the 'more' realistic example

Where

$$a^2(\eta) = A + B\tanh\rho\eta$$

, A, B ρ constant (this is also exactly solvable)



Again one gets the same wave eq

$$\frac{d^2\chi_k(\eta)}{d\eta^2} + (k^2 + a^2(\eta)\, m^2)\,\chi_k(\eta) = 0$$

(exactly solved problem)

The soln with the + frequency for the in going wave

$$u_k^{in}(\eta,x) = \frac{1}{\sqrt{4\pi\omega_I}} \exp\left\{ikx - i\omega_+\eta - \frac{i\omega_-}{\rho}\ln[2\cosh(\rho\eta)]\right\}$$

$${}_2F_1\left[\left(1+\frac{i\omega_-}{\rho};\ \frac{i\omega_-}{\rho};\ 1-\frac{i\omega_I}{\rho};\ \frac{1}{2}(1+\tanh\rho\eta)\right)\right]$$

$$\omega_I = \left[k^2 + m^2(A-B)\right]^{1/2}$$

$$\omega_{II} = \left[k^2 + m^2(A+B)\right]^{1/2}$$

$$\omega_\pm = \frac{1}{2}\left(\omega_{II} \pm \omega_I\right)$$

$$\longrightarrow \frac{1}{\sqrt{4\pi\omega_I}}\, e^{ikx - i\omega_I\eta}$$

+ve frequency

as $\eta \to -\infty$

Another soln of this equation which is more easily dealt with ρ +ve frequency out soln is given by

(41)

$$u_k^{out}(\eta x) = \frac{1}{\sqrt{4\pi\omega_{out}}} \exp\{ikx - i\omega_+\eta - (\frac{i\omega_-}{\rho}) \ln[2\cosh(\rho\eta)]\}$$

$$\times {}_2F_1\left(1+\frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}; 1+\frac{i\omega_{out}}{\rho}; \frac{1}{2}(1-\tanh\rho\eta)\right)$$

$$\xrightarrow[\eta\to+\infty]{} \frac{1}{\sqrt{4\pi\omega_{out}}} e^{ikx - i\omega_{out}\eta}$$

We want to know how the in soln can be expressed as out soln as it is a linear eq such an expression should exist

$$u_k^{in}(\eta, x) = \alpha_k u_k^{out}(\eta, x) + \beta_k u_{-k}^{out*}(\eta, x)$$

$$\alpha_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1-\frac{i\omega_{in}}{\rho}\right)\Gamma\left(1-\frac{i\omega_{out}}{\rho}\right)}{\Gamma\left(1-\frac{i\omega_+}{\rho}\right)\Gamma\left(1-\frac{i\omega_+}{\rho}\right)}$$

$$\beta_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1-\frac{i\omega_{in}}{\rho}\right)\Gamma\left(\frac{i\omega_{out}}{\rho}\right)}{\Gamma\left(\frac{i\omega_-}{\rho}\right)\Gamma\left(1+\frac{i\omega_-}{\rho}\right)}$$

from here

$$|\beta_k|^2 = \frac{\sinh^2 \frac{\pi\omega_-}{\rho}}{\sinh\left(\frac{\pi\omega_{in}}{\rho}\right)\sinh\left(\frac{\pi\omega_{out}}{\rho}\right)}$$

Note it goes to the previous expression for $\rho\to\infty$

$$|\alpha_k|^2 = \frac{\sinh^2 \frac{\pi\omega_+}{\rho}}{\sinh\frac{\pi\omega_{in}}{\rho}\sinh\left(\frac{\pi\omega_{out}}{\rho}\right)}$$

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

K.G eq In the Schwarzschild geometry and the effect of high frequency (42)

$$\frac{1}{\sqrt{g}}\partial_\mu\left[g^{\mu\nu}\sqrt{g}\,\partial_\nu\varphi\right] = 0$$

for $ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2$

$$\sqrt{g} = r^2\sin\theta \qquad \text{call } 1-\frac{2m}{r} = a$$

$$\to -\frac{1}{r^2\sin\theta}\partial_t\left[\left(1-\frac{2m}{r}\right)^{-1} r^2\sin\theta\,\partial_t\Phi\right] + \frac{1}{r^2\sin\theta}\partial_r\left[\left(1-\frac{2m}{r}\right)r^2\sin\theta\,\partial_r\Phi\right]$$

$$+ \frac{1}{r^2\sin\theta}\partial_\theta\left[\frac{1}{r^2}r^2\sin\theta\,\partial_\theta\Phi\right] + \frac{1}{r^2\sin\theta}\partial_\phi\left[\frac{r^2\sin\theta}{r^2\sin^2\theta}\partial_\phi\Phi\right]$$

$= +\frac{\omega}{a}$ assume $\Phi = F(r)/_r\, Y_{lm}(\theta,\phi)\, e^{i\omega t}$

$$\frac{\omega^2}{a}\Phi + \frac{1}{r^2}\partial_r\left[a\,r^2\partial_r\Phi\right] - \frac{l(l+1)}{r^2}\Phi = 0$$

Now put in $\Phi = \frac{F}{r}$

$$(\partial_r a)\,\partial_r\Phi + \frac{a}{r^2}\partial_r\left[r^2\partial_r\Phi\right] \qquad \Phi = \frac{F}{r}$$

$$(\partial_r a)\frac{\partial_r F}{r} + \frac{a F''}{r} \qquad -\frac{F}{r^2} + \frac{F'}{r}$$

$$-(\partial_r a)\frac{F}{r^2} \qquad -F + rF'$$

$$\frac{dF}{dr} = \frac{dF}{dr^*}\frac{dr^*}{dr}$$

$$\frac{1}{a}\frac{d^2F}{dr^{*2}} \qquad \partial_r a = \frac{2m}{r^2} \qquad \frac{d^2F}{dr^2} = \frac{d^2F}{dr^{*2}}\left(\frac{dr^*}{dr}\right)^2 + \frac{dF}{dr^*}\frac{d^2r^*}{dr^2}$$

So

$$\frac{1}{a}\frac{d^2F}{dr^{*2}} + \frac{\omega^2}{a}F - \frac{2m}{r^3}F - \left(\frac{l(l+1)}{r^2}\right) \qquad \frac{d^2F}{dr^2} = \frac{d^2F}{dr^{*2}}\frac{1}{a^2} - \frac{1}{a^2}\frac{da}{dr}\frac{dF}{dr^*}$$

$$\frac{d^2F}{dr^{*2}a^2} - \frac{1}{a}\frac{da}{dr}\frac{dF}{dr} \qquad \frac{dr^*}{dr} = \frac{1}{a}$$

$$\frac{d^2r^*}{dr^2} = -\frac{1}{a^2}\frac{da}{dr}$$

$$a^{-2}\frac{d^2F}{dr^2} + a\frac{d}{dr} \ldots$$

$$\frac{d^2F}{dr^{*2}} + \left[\omega^2 - \left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3}\right)\right]\left[1 - \frac{2m}{r}\right]F = 0$$

So one sees here for $\omega \to \infty$.

$$f = e^{i\omega r^*}$$

so Soln are $e^{i\omega(t+r^*)}$, $e^{i\omega(t-r^*)}$

$\to$ The phase are u & v.
Surfaces of constant Phase are Null
are $\perp$ to geodesics
geometric optics [illegible]

The Collapsing ball selection

One had the metric

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\chi^2\right) + a^2(\eta)\left(\sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2\right)$$

$$= \cancel{-d\tau^2 + a^2 [d\chi^2} +$$

$$\frac{1}{\sqrt{g}}$$

$$\sqrt{g} = a^4 \sin^2\chi \sin\theta$$

$$\therefore \frac{1}{a^4\sin^2\chi\sin\theta}\partial_\eta\left[-a^2\sin^2\chi\sin\theta\,\partial_\eta\Phi\right]$$

$$+ \frac{1}{a^4\sin^2\chi\sin\theta}\partial_\chi\left[a^2\sin^2\chi\sin\theta\,\partial_\chi\Phi\right]$$

$$+ \frac{1}{a^4\sin^2\chi\sin\theta}\partial_\theta\left(a^2\sin\theta\,\partial_\theta\Phi\right)$$

$$+ \frac{1}{a^4\sin^2\chi\sin\theta}\partial_\phi\left(a^2\frac{1}{\sin\theta}\,\partial_\phi\Phi\right)$$

$$\Rightarrow \frac{-1}{a^4}\partial_\eta\left[a^2\,\partial_\eta\Phi\right] + \frac{\partial_\chi\left(\sin^2\chi\,\partial_\chi\Phi\right)}{a^2\sin^2\chi}$$

$$- \frac{1}{a^2\sin^2\chi}\ell(\ell+1)\Phi = 0$$

Consider the ansatz $\phi = e^{i\omega\cdot(n \pm \chi)} \cdot f$ (8) (44)

to the leading order in ω.

$$+ \frac{\dot a^2}{a^2}\omega^2 \cdot f \quad - \frac{i}{a^2} \, 2a\,\dot a \, \omega \cdot f \quad + O(1)$$

$$\rightarrow \frac{P(\ldots)}{a^2 \sin^2 \chi}$$

$$- \omega^2 f \quad + \frac{i \, 2 \sin\chi \cos\chi}{\sin^2\chi} \, \omega f$$

$$+ \, \omega f' \text{ term}$$

leading order

so at $\omega \to \infty$. $\underline{\underline{f = 1}}$

Thus again Soln are such that

The phase e^{iS} Y_{lm}

are such that

~~They are constant on~~

They surface of constant phases

are null & $\perp$ to geodesics.

$$e^{i\omega(n \pm \chi)} \; Y_{lm}(\theta, \phi)$$

for ω large.

Calculation by Ray tracing

V

$\longrightarrow \quad e^{-i v \omega} - e^{-i u \omega}$

$\longrightarrow$

Outside Soln $\quad ds^2 = C(r)\, du\, dv$

$u = t - r^* + R_0^*$

$v = t + r^* - R_0^*$

$$ds^2 = A(u,v)\, du\, dv$$

$u = \tau - r + R_0$

$v = \tau + r - R_0$

Transition $\quad$ ~~$u = V = \alpha(v)$~~

$v = \beta(V)$

$U = \alpha(u)$

$V = U - 2R_0$ $\qquad$ Geometry.

$$\underbrace{e^{-i\omega v} - e^{-i\omega \beta[\alpha(u) - 2R_0]}}$$

reflection condition