LECTURE I

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Some useful formulae

$$ {}_dC_0 + {}_dC_4 + \cdots = 2^{d-2} + 2^{\frac{d}{2}-1} \cos\frac{\pi d}{4} $$

$$ {}_dC_1 + {}_dC_5 + \cdots = 2^{d-2} + 2^{\frac{d}{2}-1} \sin\frac{\pi d}{4} $$

$$ {}_dC_2 + {}_dC_6 + \cdots = 2^{d-2} - 2^{\frac{d}{2}-1} \cos\frac{\pi d}{4} $$

$$ {}_dC_3 + {}_dC_7 + \cdots = 2^{d-2} - 2^{\frac{d}{2}-1} \sin\frac{\pi d}{4} . $$

pf) . $(x+y)^d = \sum_{k=0}^{d} x^{d-k} y^k {}_dC_k$

① $2^d = (1+1)^d = \sum_{k=0} {}_dC_k$

② $(\sqrt{2} e^{\frac{i\pi}{4}})^d = (1+i)^d = (1+e^{\frac{i\pi}{2}})^d$

$= \sum_{k=0} {}_dC_k \, e^{\frac{i\pi k}{2}}$

③ $0 = (1+e^{i\pi})^d = \sum_{k=0} {}_dC_k \, e^{i\pi k}$

④ $(\sqrt{2} e^{-\frac{i\pi}{4}})^d = (1+e^{\frac{i3\pi}{2}})^d = \sum_{k=0} {}_dC_k \, e^{\frac{3i\pi}{2}k}$

① + ③ ; $2^d = 2({}_dC_0 + {}_dC_2 + {}_dC_4 + \cdots)$ ⑤

① − ③ ; $2^d = 2({}_dC_1 + {}_dC_3 + {}_dC_5 + \cdots)$ ⑥

② + ④ ; $2^{\frac{d}{2}}(e^{\frac{id\pi}{4}} + e^{-\frac{id\pi}{4}})$

$= 2({}_dC_0 - {}_dC_2 + {}_dC_4 - \cdots)$ ⑦

② − ④ ; $2^{\frac{d}{2}}(e^{\frac{id\pi}{4}} - e^{-\frac{id\pi}{4}})$

$= 2i({}_dC_1 - {}_dC_3 + {}_dC_5 - \cdots)$ ⑧

From 5 & 6,

- $2^{d-1} + 2^{\frac{d}{2}} \cos\frac{d\pi}{4} = 2({}_dC_0 + {}_dC_4 + {}_dC_8 + \cdots)$

$\Rightarrow 2^{d-2} + 2^{\frac{d}{2}-1} \cos\frac{d\pi}{4} = {}_dC_0 + {}_dC_4 + {}_dC_8 + \cdots$

- $2^{d-1} - 2^{\frac{d}{2}} \cos\frac{d\pi}{4} = 2({}_dC_2 + {}_dC_6 + {}_dC_8 + \cdots)$

$\Rightarrow 2^{d-2} - 2^{\frac{d}{2}-1} \cos\frac{d\pi}{4} = {}_dC_2 + {}_dC_6 + {}_dC_8 + \cdots$

From 6 & 8,

- $2^{d-1} + 2^{\frac{d}{2}} \sin\frac{d\pi}{4} = 2({}_dC_1 + {}_dC_5 + {}_dC_9 + \cdots)$

$\Rightarrow 2^{d-2} + 2^{\frac{d}{2}-1} \sin\frac{d\pi}{4} = {}_dC_1 + {}_dC_5 + {}_dC_9 + \cdots$

- $2^{d-1} - 2^{\frac{d}{2}} \sin\frac{d\pi}{4} = 2({}_dC_3 + {}_dC_7 + {}_dC_{11} + \cdots)$

$\Rightarrow 2^{d-2} - 2^{\frac{d}{2}-1} \sin\frac{d\pi}{4} = {}_dC_3 + {}_dC_7 + {}_dC_{11} + \cdots$

$\{C\Gamma^{(n)}\}$ as the basis for $2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$ matrices

$(n = 0, 1, 2, \cdots, d.)$

$$* \quad (C\Gamma^{(n)})^T = \epsilon\,\eta^n\,(-1)^{\frac{1}{2}n(n-1)}\, C\Gamma^{(n)}$$

$$\text{pf)} \quad (C\Gamma^{(n)})^T = \Gamma^{(n)T} C^T \overset{C^T = \epsilon C}{=} \Gamma^{(n)T}(\epsilon C)$$

$$= (-1)^{\frac{1}{2}n(n-1)}\,\epsilon\,(\Gamma^T)^{(n)}\,C$$

$$= (-1)^{\frac{1}{2}n(n-1)}\,\epsilon\,\eta^n\,C\Gamma^{(n)}$$

$$* \quad 2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$$

$$= \frac{1}{2}\,2^{[\frac{d}{2}]}\left(2^{[\frac{d}{2}]} + 1\right) \quad ;\ \text{symm.}$$

$$+ \frac{1}{2}\,2^{[\frac{d}{2}]}\left(2^{[\frac{d}{2}]} - 1\right). \quad ;\ \text{anti-symm.}$$

N.B.

$$\frac{1}{2}\,2^{[\frac{d}{2}]}\left(2^{[\frac{d}{2}]} + 1\right) - \frac{1}{2}\,2^{[\frac{d}{2}]}\left(2^{[\frac{d}{2}]} - 1\right)$$

$$= 2^{[\frac{d}{2}]} > 0.$$

Determination of η & ϵ.

i) $n = 4l$;

$$(C\Gamma^{(4l)})^T = \epsilon\, \eta^{4l} (-1)^{4l(4l-1)/2}\, C\Gamma^{(4l)}$$

$$= \epsilon\, C\Gamma^{(4l)}$$

ii) $n = 4l + 1$

$$(C\Gamma^{(4l+1)})^T = \epsilon\, \eta^{4l+1} (-1)^{(4l+1)4l/2}\, C\Gamma^{(4l+1)}$$

$$= \epsilon\, \eta\, C\Gamma^{(4l+1)}$$

iii) $n = 4l + 2$

$$(C\Gamma^{(4l+2)})^T = \epsilon\, \eta^{4l+2} (-1)^{(4l+2)(4l+1)/2}\, C\Gamma^{(4l+2)}$$

$$= \epsilon\, (-1)\, C\Gamma^{(4l+2)}$$

iv) $n = 4l + 3$

$$(C\Gamma^{(4l+3)})^T = \epsilon\, \eta^{4l+3} (-1)^{(4l+3)(4l+2)/2}\, C\Gamma^{(4l+3)}$$

$$= \epsilon\, \eta\, (-1)\, C\Gamma^{(4l+3)}$$

When $\eta = 1$,

$$(C\Gamma^{(n)})^T = \epsilon C\Gamma^{(n)} \quad \text{for} \quad n = 4l,\ 4l+1$$

$$(C\Gamma^{(n)})^T = -\epsilon C\Gamma^{(n)} \quad \text{for} \quad n = 4l+2,\ 4l+3$$

When $\eta = -1$,

$$(C\Gamma^{(n)})^T = \epsilon C\Gamma^{(n)} \quad \text{for} \quad n = 4l,\ 4l+3$$

$$(C\Gamma^{(n)})^T = -\epsilon C\Gamma^{(n)} \quad \text{for} \quad n = 4l+1,\ 4l+2.$$

$*$ The number of $C\Gamma^{(n)}$ satisfying

$$(C\Gamma^{(n)})^T = \epsilon\, C\Gamma^{(n)} \quad \text{is}$$

$(\eta = 1)$ $\quad \sum_{l=0} {}_dC_{4l} + \sum_{l=0} {}_dC_{4l+1}$

$$= 2^{d-1} + 2^{\frac{d}{2}-1}\left(\cos\frac{\pi d}{4} + \sin\frac{\pi d}{4}\right)$$

$(\eta = -1)$ $\quad \sum_{l=0} {}_dC_{4l} + \sum_{l=0} {}_dC_{4l+3}$

$$= 2^{d-1} + 2^{\frac{d}{2}-1}\left(\cos\frac{\pi d}{4} - \sin\frac{\pi d}{4}\right)$$

$*$ The number of $C\Gamma^{(n)}$ satisfying

$$(C\Gamma^{(n)})^T = -\epsilon\, C\Gamma^{(n)} \quad \text{is}$$

$(\eta = 1)$ $\quad \sum_{l=0} {}_dC_{4l+2} + \sum_{l=0} {}_dC_{4l+3}$

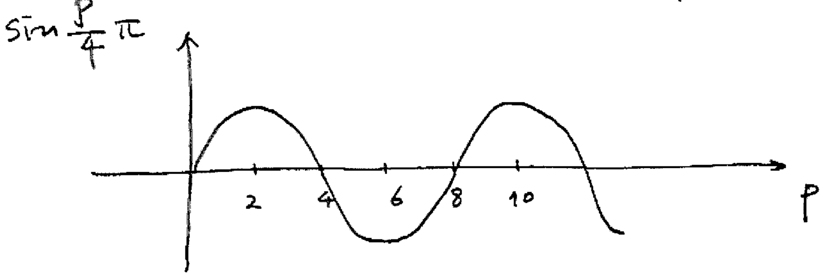
$$= 2^{d-1} - 2^{\frac{d}{2}-1}\left(\cos\frac{\pi d}{4} + \sin\frac{\pi d}{4}\right)$$

$(\eta = -1)$ $\quad \sum_{l=0} {}_dC_{4l+1} + \sum_{l=0} {}_dC_{4l+2}$

$$= 2^{d-1} - 2^{\frac{d}{2}-1}\left(\cos\frac{\pi d}{4} - \sin\frac{\pi d}{4}\right)$$

$$* \quad \cos\frac{\pi d}{4} \pm \sin\frac{\pi d}{4} = \pm\sqrt{2}\,\sin\frac{(d\pm 1)\pi}{4}$$

$$\sin\frac{p}{4}\pi \begin{cases} > 0 & p = 1,\ 2,\ 3,\ 9,\ 10,\ 11, \cdots \\ < 0 & p = 5,\ 6,\ 7,\ 13,\ 14,\ 15, \cdots \end{cases}$$



$$\cos\frac{\pi d}{4} + \sin\frac{\pi d}{4} \begin{cases} > 0 & d = 0,\ 1,\ 2,\ 8,\ 9,\ 10, \cdots \\ < 0 & d = 4,\ 5,\ 6,\ 12,\ 13,\ 14, \cdots \end{cases}$$

positive when $d = 8l,\ 8l+1,\ 8l+2$

negative when $d = 8l+4,\ 8l+5,\ 8l+6$

$$\cos\frac{\pi d}{4} - \sin\frac{\pi d}{4} \begin{cases} > 0 & d = 6,\ 7,\ 8,\ 14,\ 15,\ 16, \cdots \\ < 0 & d = 2,\ 3,\ 4,\ 10,\ 11,\ 12, \cdots \end{cases}$$

positive when $d = 8l+6,\ 8l+7,\ 8l$

negative when $d = 8l+2,\ 8l+3,\ 8l+4$

$* \quad \eta = 1.$

i) $d = 8l,\ 8l+1,\ 8l+2 \Rightarrow \epsilon = 1.$

$$n = 4k,\ 4k+1 \qquad (C\Gamma^{(n)})^T = C\Gamma^{(n)}$$

$$n = 4k+2,\ 4k+3 \qquad (C\Gamma^{(n)})^T = -C\Gamma^{(n)}$$

ii) $d = 8l+4,\ 8l+5,\ 8l+6 \Rightarrow \epsilon = -1$

$$n = 4k+2,\ 4k+3 \qquad (C\Gamma^{(n)})^T = C\Gamma^{(n)}$$

$$n = 4k,\ 4k+1 \qquad (C\Gamma^{(n)})^T = -C\Gamma^{(n)}$$

$* \quad \eta = -1$

i) $d = 8l+6,\ 8l+7,\ 8l \Rightarrow \epsilon = 1.$

$$n = 4k,\ 4k+3 \qquad (C\Gamma^{(n)})^T = C\Gamma^{(n)}$$

$$n = 4k+1,\ 4k+2 \qquad (C\Gamma^{(n)})^T = -C\Gamma^{(n)}$$

ii) $d = 8l+2,\ 8l+3,\ 8l+4 \Rightarrow \epsilon = -1$

$$n = 4k+1,\ 4k+2 \qquad (C\Gamma^{(n)})^T = C\Gamma^{(n)}$$

$$n = 4k,\ 4k+3 \qquad (C\Gamma^{(n)})^T = -C\Gamma^{(n)}$$

$$B^* B = \epsilon \, \eta^{d_-} (-1)^{\frac{1}{2} d_- (d_- - 1)}$$

* A, B, C are unitary $\Leftarrow$ $(\gamma'^a)^\dagger = \gamma'_a$

* $B^* B = B^{-T} B = (C^{-1} A^T)^T (A^{-T} C)$

$$= A C^{-T} A^{-T} C .$$

N. B. $A = \Gamma^1 \Gamma^2 \cdots \Gamma^{d_-}$

$$A^{-1} = \Gamma^{-d_-} \cdots \Gamma^{-1}$$

$$= (-1)^{d_-} \Gamma^{d_-} \cdots \Gamma^1$$

$$\Rightarrow A^{-T} = (-1)^{d_-} \Gamma^{1T} \cdots \Gamma^{d_- T}$$

$$= (-1)^{d_-} \eta^{d_-} C \Gamma^1 \cdots \Gamma^{d_-} C^{-1}$$

$$= (-1)^{d_-} \eta^{d_-} C A C^{-1}$$

$$\Rightarrow B^* B = A C^{-T} (-1)^{d_-} \eta^{d_-} C A C^{-1} C$$

$$= (-1)^{d_-} \eta^{d_-} A \underbrace{C^{-T} C}_{\epsilon} A$$

$$= (-1)^{d_-} \eta^{d_-} \epsilon \, A^2$$

$$= (-1)^{\frac{d_-}{2}(d_- - 1)} \eta^{d_-} \epsilon$$

$$\left\{ \begin{aligned} A^2 &= \Gamma^1 \cdots \Gamma^{d_-} \Gamma^1 \cdots \Gamma^{d_-} \\ &= (-1)^{\frac{d_-}{2}(d_- - 1) + d_-} \\ &= (-1)^{\frac{d_-}{2}(d_- + 1)} \end{aligned} \right.$$

$B^* B = 1.$

$\ast$ Let $\psi^* = X\psi$. Then $X = \alpha B$.

pf) $\delta\psi = \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi$.

$\Rightarrow \delta\psi^* = \frac{1}{4}\omega_{ab}(\Gamma^{ab})^*\psi^*$

Require ;
This transf. preserves the rel. $\psi^* = X\psi$.

$$\frac{1}{4}\omega_{ab}(\Gamma^{ab})^*\psi^* = X\frac{1}{4}\omega_{ab}\Gamma^{ab}\psi.$$

$$\Rightarrow (\Gamma^{ab})^*\underbrace{\psi^*}_{X\psi .} = X\Gamma^{ab}\psi$$

$$\Rightarrow (\Gamma^{ab})^* X = X\Gamma^{ab}$$

$$B\Gamma^{ab}\underbrace{B^{-1}X} = X\Gamma^{ab}$$

Therefore $B^{-1}X = \alpha I$.

$\ast$ $B^* B = 1$.

$X = \alpha B$, $X^* X = 1$ $(\psi^{**} = X^*\psi^* = X^* X\psi)$

$$\Rightarrow \alpha^* B^* B\alpha = |\alpha|^2 B^* B = |\alpha|^2 \underbrace{\epsilon\eta^{d}(-1)^{\frac{d}{2}(d-1)}}_{\pm 1 .} = 1$$

$$\Rightarrow |\alpha|^2 = 1, \ \epsilon\eta^{d}(-1)^{\frac{d}{2}(d-1)} = 1.$$

$$\Rightarrow B^* B = 1.$$

The conditions for $B^*B = 1$

$*$ $B^*B = 1$ when $\Delta = 0, 1, 7 \pmod 8$

$\Delta = 2 \pmod 8$ w/ $\eta(-1)^{\frac{d}{2}} = -1$

$\Delta = 6 \pmod 8$ w/ $\eta(-1)^{\frac{d}{2}} = 1$

pf) Let $\Delta = 8l + k = d_+ - d_-$

$\Rightarrow \quad d = d_+ - d_- + 2d_- = \Delta + 2d_-$

$= 8l + k + 2d_-$

$d_- = 0$; $B^*B = \epsilon$ $\quad d = 8l + k$

$d_- = 1$; $B^*B = \epsilon\eta$ $\quad d = 8l + k + 2$

$d_- = 2$; $B^*B = -\epsilon$ $\quad d = 8l + k + 4$

$d_- = 3$; $B^*B = -\epsilon\eta$ $\quad d = 8l + k + 6$.

$d_- = 4$; $B^*B = \epsilon$ $\quad d = 8l + k + 8$.

.... repeated ....

i) $d_- = 0$; $d = k \pmod 8$

$\epsilon = 1$ when $k = 0, 1, 2^+, 6^-, 7$

ii) $d_- = 1$; $d = k + 2$

$\epsilon\eta = 1$ when $k = 0, 1, 2^-, 6^+, 7$

iii) $d_- = 2$; $d = k + 4$

$\epsilon = -1$ when $k = 0, 1, 2^+, 6^-, 7$

iv) $d_- = 3$; $d = k + 6$

$\epsilon\eta = -1$ when $k = 0, 1, 2^-, 6^+, 7$.

Therefore $B^* B = 1$ when

$$\Delta = 0, 1, 7 \pmod 8$$

$$\Delta = 2 \quad w/ \quad \eta = (-1)^{\frac{d}{2}-1}$$

$$\Delta = 6 \quad w/ \quad \eta = (-1)^{\frac{d}{2}}$$

Symplectic Majorana

Even when $B^*B = -1$, there could be some relations among multi-component spinors.

— Let $\psi^{i*} = X\Omega_{ij}\psi^j$. $(i,j = 1,\cdots,2n)$

Under the Lorentz transf.,

$$\frac{1}{4}\omega_{ab}(\Gamma^{ab})^*\psi^{i*} = X\Omega_{ij}\left(\frac{1}{4}\omega_{ab}\Gamma^{ab}\psi^j\right)$$

$$\Rightarrow (\Gamma^{ab})^* X\Omega_{ij}\psi^j = X\Omega_{ij}\Gamma^{ab}\psi^j$$

$$\Rightarrow B\Gamma^{ab}B^{-1}X = X\Gamma^{ab}$$

$$\Rightarrow B^{-1}X \sim 1.$$

— Without loss of generality, we may set

$$\psi^{i*} = B\Omega_{ij}\psi^j \quad (i,j = 1,\cdots,2n)$$

By consistency,

$$\psi^i = (\psi^{i*})^* = B^*\Omega_{ij}^*\psi^{j*}$$

$$= \Omega_{ij}^* B^*(B\Omega_{jk}\psi^k)$$

$$= (\Omega^*\Omega)_{ik}B^*B\psi^k$$

$B^*B=-1$

$$= -(\Omega^*\Omega)_{ik}\psi^k$$

$$\Rightarrow \Omega^*\Omega = -1.$$

Majorana Weyl spinor

$$\psi_\pm^* = X \psi_\pm \quad ?$$

$$\psi_\pm^* = \frac{1}{2}\left(1 \pm \bar{\Gamma}^* / \sqrt{\beta^*}\right)\psi^*$$

$$= \frac{1}{2}\left(1 \pm \bar{\Gamma}^* / \sqrt{\beta^*}\right) X \psi .$$

N.B.

$$\bar{\Gamma}^* = \Gamma^{1*}\Gamma^{2*}\cdots\Gamma^{d*} = \eta^d (-1)^{dd_-} B \bar{\Gamma} B^{-1}$$

$$\underset{\uparrow\ d\,;\ \text{even}.}{=} B \bar{\Gamma} B^{-1}$$

$$\Rightarrow \psi_\pm^* \underset{X = B.}{=} X \frac{1}{2}\left(1 \pm \bar{\Gamma} / \sqrt{\beta^*}\right)\psi .$$

Therefore the reality cond. is satisfied if $\sqrt{\beta^*} = \sqrt{\beta}$ i.e. $(-i)^{-\frac{\Delta}{2}} = (i)^{-\frac{\Delta}{2}}$.

$$\Rightarrow (-1)^{-\frac{\Delta}{2}} = 1 .$$

$$\Delta = 0 \mod 4 .$$

The condition is consistent $(\psi^{**} = \psi)$ when $\Delta = 0 \mod 8$

The other case of $\Delta = 4 \mod 8$, gives the symplectic Majorana-Weyl.

$AdS_p \times S^q$ in the global coordinates.

* geometry (global coord.)

$$ds^2 = R^2\left(-ch^2\rho\, dt^2 + d\rho^2 + sh^2\rho\, d\Omega_{p-2}^2\right) + R^2\left(c^2\theta\, d\psi^2 + d\theta^2 + s^2\theta\, d\Omega_{q-2}^2\right)$$

$$0 \le \theta < \frac{\pi}{2}, \qquad 0 \le \psi < 2\pi$$

* k-sphere

$$d\Omega_k^2 = d\theta_1^2 + s^2\theta_1\, d\theta_2^2 + s^2\theta_1\, s^2\theta_2\, d\theta_3^2 + \cdots + s^2\theta_1\, s^2\theta_2 \cdots s^2\theta_{k-1}\, d\theta_k^2$$

$$0 \le \theta_1, \theta_2, \cdots, \theta_{k-1} < \pi$$

$$0 \le \theta_k < 2\pi.$$

* Orthonormal frame

$$e^0 = R_1\, ch\rho\, dt$$
$$e^1 = R_1\, d\rho$$
$$e^2 = R_1\, sh\rho\, d\varphi_2$$
$$e^3 = R_1\, sh\rho\, s\varphi_2\, d\varphi_3$$
$$\vdots$$
$$e^{p-1} = R_1\, sh\rho\, s\varphi_2\, s\varphi_3 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$
$$e^p = R_2\, c\theta\, d\psi$$
$$e^{p+1} = R_2\, d\theta$$
$$e^{p+2} = R_2\, s\theta\, d\chi_2$$
$$e^{p+3} = R_2\, s\theta\, s\chi_2\, d\chi_3$$
$$\vdots$$
$$e^{p+q} = R_2\, s\theta\, s\chi_2\, s\chi_3 \cdots s\chi_{q-2}\, d\chi_{q-1}$$

Calculating the spin connections

STEP 1.

* $C^a{}_{bc}$

$$de^a = \frac{1}{2} C^a{}_{bc}\, e^b \wedge e^c$$

$\cdot$ $$de^0 = R_1 sh\rho\, d\rho \wedge dt = \frac{th\rho}{R} e^1 \wedge e^0$$

$\cdot$ $$de^1 = 0$$

$\cdot$ $$de^2 = R_1 ch\rho\, d\rho \wedge d\varphi_2 = \frac{cth\rho}{R} e^1 \wedge e^2$$

$\cdot$ $$\begin{aligned} de^3 &= R_1 ch\rho\, d\rho \wedge s\varphi_2 d\varphi_3 + R_1 sh\rho\, c\varphi_2\, d\varphi_2 \wedge d\varphi_3 \\ &= \frac{cth\rho}{R} e^1 \wedge e^3 + \frac{ct\varphi_2}{R sh\rho} e^2 \wedge e^3 \end{aligned}$$

$\cdot$ $$\begin{aligned} de^{p-1} &= R_1 ch\rho\, d\rho \wedge s\varphi_2 s\varphi_3 \cdots s\varphi_{p-2}\, d\varphi_{p-1} \\ &+ R_1 sh\rho\, c\varphi_2\, d\varphi_2 \wedge s\varphi_3 \cdots s\varphi_{p-2}\, d\varphi_{p-1} \\ &+ R_1 sh\rho\, s\varphi_2\, c\varphi_3\, d\varphi_3 \wedge s\varphi_4 \cdots s\varphi_{p-2}\, d\varphi_{p-1} \\ &\cdots + R_1 sh\rho\, s\varphi_2 s\varphi_3 \cdots c\varphi_{p-2}\, d\varphi_{p-2} \wedge d\varphi_{p-1} \\ &= \frac{cth\rho}{R_1} e^1 \wedge e^{p-1} + \frac{ct\varphi_2}{R_1 sh\rho} e^2 \wedge e^{p-1} \\ &+ \frac{ct\varphi_3}{R_1 sh\rho\, s\varphi_2} e^3 \wedge e^{p-1} + \cdots \\ &\cdots + \frac{ct\varphi_{p-2}}{R_1 sh\rho\, s\varphi_2 s\varphi_3 \cdots s\varphi_{p-3}} e^{p-2} \wedge e^{p-1} \end{aligned}$$

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$$\cdot\ de^{p} = -R_2 s\theta\ d\theta \wedge d\psi = -\frac{t\theta}{R_2}\, e^{p+1} \wedge e^{p}$$

$$\cdot\ de^{p+1} = 0$$

$$\cdot\ de^{p+2} = R_2 c\theta\ d\theta \wedge d\chi_2 = \frac{ct\theta}{R_2}\, e^{p+1} \wedge e^{p+2}$$

$$\cdot\ de^{p+3} = R_2 c\theta\ d\theta \wedge s\chi_2\, d\chi_3 + R_2 s\theta\, c\chi_2\, d\chi_2 \wedge d\chi_3$$

$$= \frac{ct\theta}{R_2}\, e^{p+1} \wedge e^{p+3} + \frac{ct\chi_2}{R_2 s\theta}\, e^{p+2} \wedge e^{p+3}$$

$$\begin{aligned}\cdot\ de^{p+q} = {} & R_2 c\theta\ d\theta \wedge s\chi_2\, s\chi_3 \cdots s\chi_{q-2}\ d\chi_{q-1} \\ & + R_2 s\theta\ c\chi_2\, d\chi_2 \wedge s\chi_3 \cdots s\chi_{q-2}\ d\chi_{q-1} \\ & + R_2 s\theta\ s\chi_2\, c\chi_3\ d\chi_3 \wedge s\chi_4 \cdots s\chi_{q-2}\, d\chi_{q-1} \\ & + \cdots \\ & + R_2 s\theta\ s\chi_2 \cdots s\chi_{q-3}\ c\chi_{q-2}\ d\chi_{q-2} \wedge d\chi_{q-1}\end{aligned}$$

$$= \frac{ct\theta}{R_2}\, e^{p+1} \wedge e^{p+q} + \frac{ct\chi_2}{R_2 s\theta}\, e^{p+2} \wedge e^{p+q}$$

$$+ \frac{ct\chi_3}{R_2 s\theta\, s\chi_2}\, e^{p+3} \wedge e^{p+q} + \cdots$$

$$+ \frac{ct\chi_{q-2}}{R_2 s\theta\, s\chi_2 \cdots s\chi_{q-3}}\, e^{p+q-1} \wedge e^{p+q}\ .$$

$$\Rightarrow \quad C^{0}{}_{10} = -C^{0}{}_{01} = \frac{th\rho}{R_1}$$

$$C^{2}{}_{12} = -C^{2}{}_{21} = \frac{cth\rho}{R_1}$$

$$C^{3}{}_{13} = -C^{3}{}_{31} = \frac{cth\rho}{R_1}$$

$$C^{3}{}_{23} = -C^{3}{}_{32} = \frac{ct\,\varphi_2}{R_1 sh\rho}$$

$$\vdots$$

$$C^{p-1}{}_{1\,p-1} = -C^{p-1}{}_{p-1\,1} = \frac{cth\rho}{R_1}$$

$$C^{p-1}{}_{2\,p-1} = -C^{p-1}{}_{p-1\,2} = \frac{ct\,\varphi_2}{R_1 sh\rho}$$

$$C^{p-1}{}_{3\,p-1} = -C^{p-1}{}_{p-1\,3} = \frac{ct\,\varphi_3}{R_1 sh\rho\, s\varphi_2}$$

$$\vdots$$

$$C^{p-1}{}_{p-2\,p-1} = -C^{p-1}{}_{p-1\,p-2} = \frac{ct\,\varphi_{p-2}}{R_1 sh\rho\, s\varphi_2 s\varphi_3 \cdots s\varphi_{p-3}}$$

$$C^{p}{}_{p+1\,p} = -C^{p}{}_{p\,p+1} = -\frac{t\theta}{R_2}$$

$$C^{p+2}{}_{p+1\,p+2} = -C^{p+2}{}_{p+2\,p+1} = \frac{ct\theta}{R_2}$$

$$C^{p+3}{}_{p+1\,p+3} = -C^{p+3}{}_{p+3\,p+1} = \frac{ct\theta}{R_2}$$

$$C^{p+3}{}_{p+2\,p+3} = -C^{p+3}{}_{p+3\,p+2} = \frac{ct\chi_2}{R_2 s\theta}$$

$$\vdots$$

$$C^{p+q}{}_{p+1\,p+q} = -C^{p+q}{}_{p+q\,p+1} = \frac{ct\,\theta}{R_2}$$

$$C^{p+q}{}_{p+2\,p+q} = -C^{p+q}{}_{p+q\,p+2} = \frac{ct\chi_2}{R_2 s\theta}$$

$$C^{p+q}{}_{p+3\,p+q} = -C^{p+q}{}_{p+q\,p+3} = \frac{ct\chi_3}{R_2 s\theta s\chi_2}$$

$$\vdots$$

$$C^{p+q}{}_{p+q-1\,p+q} = -C^{p+q}{}_{p+q\,p+q-1} = \frac{ct\,\chi_{q-2}}{R_2 s\theta s\chi_2 \cdots s\chi_{q-3}}$$

STEP 2.

* Spin connection

$$\omega^a{}_{bc} = \frac{1}{2}\left(C^a{}_{bc} - \eta^{aa'}\eta_{bb'}C^{b'}{}_{a'c} - \eta^{aa'}\eta_{cc'}C^{c'}{}_{a'b} \right)$$

$$\cdot\ \omega^0{}_{10} = \frac{1}{2}\left(C^0{}_{10} + \cancelto{0}{C^1{}_{00}} - C^0{}_{01} \right) = C^0{}_{10}$$

$$\cdot\ \omega^0{}_{11} = \frac{1}{2}\left(\cancelto{0}{C^0{}_{11}} + C^1{}_{01} + C^1{}_{01} \right) = 0.$$

$$\cdot\ \omega^2{}_{12} = \frac{1}{2}\left(C^2{}_{12} - \cancelto{0}{C^1{}_{22}} - C^2{}_{21} \right) = C^2{}_{12}$$

$$\cdot\ \omega^3{}_{13} = C^3{}_{13}$$

$$\cdot\ \omega^3{}_{23} = C^3{}_{23}$$

$$\cdot\ \omega^{p-1}{}_{1\,p-1} = \frac{1}{2}\left(C^{p-1}{}_{1\,p-1} - \cancelto{0}{C^1{}_{p-1\,p-1}} - C^{p-1}{}_{p-1\,1} \right) = C^{p-1}{}_{1\,p-1}$$

$$\cdot\ \omega^{p-1}{}_{2\,p-1} = C^{p-1}{}_{2\,p-1}$$

$$\cdot\ \omega^{p-1}{}_{3\,p-1} = C^{p-1}{}_{3\,p-1}$$

$$\cdot\ \omega^{p-1}{}_{p-2\,p-1} = C^{p-1}{}_{p-2\,p-1}.$$

In the same way, one can show $\omega^{\cdot}{}_{\cdot\cdot} = C^{\cdot}{}_{\cdot\cdot}$ for the remaining cases.

$$\Rightarrow \quad \omega^0{}_1 = \omega^0{}_{10}\, e^0 = \frac{th\rho}{R_1} \cdot R_1 ch\rho \; dt$$

$$= sh\rho \; dt .$$

$$\omega^2{}_1 = \omega^2{}_{12}\, e^2 = \frac{cth\rho}{R_1} R_1 sh\rho \; d\varphi_2$$

$$= ch\rho \; d\varphi_2$$

$$\omega^3{}_1 = \omega^3{}_{13}\, e^3 = \frac{cth\rho}{R_1} R_1 sh\rho \; s\varphi_2 \, d\varphi_3$$

$$= ch\rho \; s\varphi_2 \; d\varphi_3$$

$$\omega^3{}_2 = \omega^3{}_{23}\, e^3 = \frac{ct\varphi_2}{R_1 sh\rho} R_1 sh\rho \; s\varphi_2 \, d\varphi_3$$

$$= c\varphi_2 \; d\varphi_3$$

$$\omega^{p-1}{}_1 = \omega^{p-1}{}_{1\,p-1}\, e^{p-1} = \frac{cth\rho}{R_1} R_1 sh\rho \; s\varphi_2 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$= ch\rho \; s\varphi_2 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$\omega^{p-1}{}_2 = \omega^{p-1}{}_{2\,p-1}\, e^{p-1} = \frac{ct\varphi_2}{R_1 sh\rho} R_1 sh\rho \; s\varphi_2 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$= c\varphi_2 \; s\varphi_3 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$\omega^{p-1}{}_3 = \omega^{p-1}{}_{3\,p-1}\, e^{p-1} = \frac{ct\varphi_3}{R_1 sh\rho \; s\varphi_2} R_1 sh\rho \; s\varphi_2 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$= c\varphi_3 \; s\varphi_4 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$\omega^{p-1}{}_{p-2} = \omega^{p-1}{}_{p-2\,p-1}\, e^{p-1} = \frac{ct\varphi_{p-2}\; R_1 sh\rho}{R_1 sh\rho \; s\varphi_2 \cdots s\varphi_{p-3}} s\varphi_2 \cdots s\varphi_{p-2}\, d\varphi_{p-1}$$

$$= c\varphi_{p-2}\; d\varphi_{p-1}$$

$$\omega^{p}{}_{p+1} = \omega^{p}{}_{p+1\,p}\, e^{p} = -\frac{t\theta}{R_2} R_2 c\theta\, d\psi$$

$$= -s\theta\, d\psi .$$

$$\omega^{p+2}{}_{p+1} = \omega^{p+2}{}_{p+1\,p+2}\, e^{p+2} = \frac{ct\theta}{R_2} R_2 s\theta\, d\chi_2$$

$$= c\theta\, d\chi_2$$

$$\omega^{p+3}{}_{p+1} = c\theta\, s\chi_2\, d\chi_3$$

$$\omega^{p+3}{}_{p+2} = c\chi_2\, d\chi_3$$

$$\omega^{p+q}{}_{p+1} = c\theta\, s\chi_2 \cdots s\chi_{p+q-1}\, d\chi_{p+q}$$

$$\omega^{p+q}{}_{p+2} = c\chi_2\, s\chi_3 \cdots s\chi_{p+q-1}\, d\chi_{p+q}$$

$$\omega^{p+q}{}_{p+3} = c\chi_3\, s\chi_4 \cdots s\chi_{p+q-1}\, d\chi_{p+q}$$

$$\omega^{p+q}{}_{p+q-1} = c\chi_{p+q-1}\, d\chi_{p+q} .$$

Covariant derivatives onto spinors

· Covariant Derivatives.

$$D_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} .$$

$$D_t = \partial_t + \frac{1}{2} \omega^{01}{}_t \Gamma_{01} = \partial_t + \frac{1}{2} sh\rho\, \Gamma_{01}$$

$$D_\rho = \partial_\rho$$

$$D_{\varphi_2} = \partial_{\varphi_2} + \frac{1}{2} \omega^{21}{}_{\varphi_2} \Gamma_{21} = \partial_{\varphi_2} + \frac{1}{2} ch\rho\, \Gamma_{21}$$

$$D_{\varphi_3} = \partial_{\varphi_3} + \frac{1}{2} \omega^{31}{}_{\varphi_3} \Gamma_{31} + \frac{1}{2} \omega^{32}{}_{\varphi_3} \Gamma_{32}$$

$$= \partial_{\varphi_3} + \frac{1}{2} ch\rho\, s\varphi_2 \Gamma_{31} + \frac{1}{2} c\varphi_2 \Gamma_{32}$$

$$\vdots$$

$$D_{\varphi_{p-1}} = \partial_{\varphi_{p-1}} + \frac{1}{2} ch\rho\, s\varphi_2 \cdots s\varphi_{p-2} \Gamma_{p-1\, 1}$$

$$+ \frac{1}{2} c\varphi_2\, s\varphi_3 \cdots s\varphi_{p-2} \Gamma_{p-1\, 2}$$

$$+ \frac{1}{2} c\varphi_3\, s\varphi_4 \cdots s\varphi_{p-2} \Gamma_{p-1\, 3} + \cdots$$

$$\cdots + \frac{1}{2} c\varphi_{p-1} \Gamma_{p-1\, p-2}$$

$$D_\psi = \partial_\psi + \frac{1}{2}\omega^{p\,p+1}{}_{\psi}\,\Gamma_{p\,p+1} = \partial_\psi - \frac{1}{2} s\theta\, \Gamma_{p\,p+1}$$

$$D_\theta = \partial_\theta$$

$$D_{\chi_2} = \partial_{\chi_2} + \frac{1}{2}\omega^{p+2\,p+1}{}_{\chi_2}\,\Gamma_{p+2\,p+1} = \partial_{\chi_2} + \frac{1}{2} c\theta\, \Gamma_{p+2\,p+1}$$

$$D_{\chi_3} = \partial_{\chi_3} + \frac{1}{2}\omega^{p+3\,p+1}{}_{\chi_3}\,\Gamma_{p+3\,p+1} + \frac{1}{2}\omega^{p+3\,p+2}{}_{\chi_3}\,\Gamma_{p+3\,p+2}$$

$$= \partial_{\chi_3} + \frac{1}{2} c\theta\, s\chi_2\, \Gamma_{p+3\,p+1} + \frac{1}{2} c\chi_2\, \Gamma_{p+3\,p+2}$$

$$\vdots$$

$$D_{\chi_{p+q}} = \partial_{\chi_{p+q}} + \frac{1}{2} c\theta\, s\chi_2 \cdots s\chi_{p+q-1}\, \Gamma_{p+q\,p+1}$$

$$+ \frac{1}{2} c\chi_2\, s\chi_3 \cdots s\chi_{p+q-1}\, \Gamma_{p+q\,p+2}$$

$$+ \frac{1}{2} c\chi_3\, s\chi_4 \cdots s\chi_{p+q-1}\, \Gamma_{p+q\,p+3} + \cdots$$

$$\cdots + \frac{1}{2} c\chi_{p+q-1}\, \Gamma_{p+q\,p+q-1}\,.$$

M5 brane

$*$ $$ds^2 = f^{-\frac{1}{3}}(-dt^2 + dx^a dx^a) + f^{\frac{2}{3}} dy^a dy^a$$

$$f = \frac{\pi N l_p^3}{r^3} + 1 \qquad r^2 = \sum_{a=1}^{5} y^a y^a$$

$*$ $r \ll 1$. ; $f \approx \frac{\pi N l_p^3}{r^3}$

$$ds^2 \approx \frac{r}{(\pi N l_p^3)^{\frac{1}{3}}}(-dt^2 + dx^a dx^a) + \frac{(\pi N l_p^3)^{\frac{2}{3}}}{r^2}(dr^2 + r^2 d\Omega_4^2)$$

$$\underbrace{R_{S^4}}_{\equiv R_2} = \frac{1}{2}\underbrace{R_{AdS_7}}_{\equiv R_1} = (\pi N l_p^3)^{\frac{1}{3}}$$

$*$ $AdS_7 \times S^4$ (Global coord.)

$$ds^2 = R_1^2(-ch^2\rho\, dt^2 + d\rho^2 + sh^2\rho\, d\Omega_5^2) + R_2^2(c^2\theta\, d\psi^2 + d\theta^2 + s^2\theta\, d\Omega_2^2)$$

$$F^{(4)} = Q\, c\theta\, s^2\theta\, s\chi_2\, d\psi \wedge d\theta \wedge d\chi_2 \wedge d\chi_3 .$$

$$Q = 3R_2$$

Killing spinor eqn's in the global coord

* Killing spinor eqn.

32 comp. Majorana

$$0 = \delta\psi_\mu = D_\mu \epsilon + \frac{1}{288}\left(\Gamma_\mu{}^{\nu\rho\sigma\kappa} - 8\,\delta_\mu^\nu \Gamma^{\rho\sigma\kappa}\right)\epsilon F_{\nu\rho\sigma\kappa}$$

$$= D_\mu \epsilon +$$

$$+\frac{1}{288}\Big(24\,\Gamma_\mu{}^{\psi\theta\chi_2\chi_3} - 6\cdot 8\,\delta_\mu^\psi \Gamma^{\theta\chi_2\chi_3}$$

$$+ 6\cdot 8\,\delta_\mu^\theta \Gamma^{\psi\chi_2\chi_3} - 6\cdot 8\,\delta_\mu^{\chi_2}\Gamma^{\psi\theta\chi_3}$$

$$+ 6\cdot 8\,\delta_\mu^{\chi_3}\Gamma^{\psi\theta\chi_2}\Big)\epsilon\, F_{\psi\theta\chi_2\chi_3}$$

$$= D_\mu \epsilon + \frac{1}{12}\Big(\Gamma_\mu{}^{\psi\theta\chi_2\chi_3} - 2\,\delta_\mu^\psi \Gamma^{\theta\chi_2\chi_3}$$

$$+ 2\,\delta_\mu^\theta \Gamma^{\psi\chi_2\chi_3} - 2\,\delta_\mu^{\chi_2}\Gamma^{\psi\theta\chi_3}$$

$$+ 2\,\delta_\mu^{\chi_3}\Gamma^{\psi\theta\chi_2}\Big)\epsilon\; Q\, c\theta\, s^2\theta\, s\chi_2$$

$$= D_\mu\epsilon + \frac{1}{12}\Big(\Gamma_{\mu\psi\theta\chi_2\chi_3}\cdot\frac{1}{R_2^8 c^2\theta\, s^4\theta\, s^2\chi_2}$$

$$- 2\,\delta_\mu^\psi\, \Gamma_{\theta\chi_2\chi_3}\cdot\frac{1}{R_2^6 s^4\theta\, s^2\chi_2}$$

$$+ 2\,\delta_\mu^\theta\, \Gamma_{\psi\chi_2\chi_3}\cdot\frac{1}{R_2^6 c^2\theta\, s^4\theta\, s^2\chi_2}$$

$$- 2\,\delta_\mu^{\chi_2}\, \Gamma_{\psi\theta\chi_3}\cdot\frac{1}{R_2^6 c^2\theta\, s^2\theta\, s^2\chi_2}$$

$$+ 2\,\delta_\mu^{\chi_3}\, \Gamma_{\psi\theta\chi_2}\cdot\frac{1}{R_2^6 c^2\theta\, s^2\theta}\Big)\epsilon\, Q\, c\theta\, s^2\theta\, s\chi_2$$

* Γ- matrices in the orthonormal frame.

$$\Gamma_\mu dx^\mu = \Gamma_a e^a$$

$$= \Gamma_0 R_1 ch\rho\, dt$$
$$+ \Gamma_1 R_1 d\rho$$
$$+ \Gamma_2 R_1 sh\rho\, d\varphi_2$$
$$+ \Gamma_3 R_1 sh\rho\, s\varphi_2\, d\varphi_3$$
$$+ \Gamma_4 R_1 sh\rho\, s\varphi_2\, s\varphi_3\, d\varphi_4$$
$$+ \Gamma_5 R_1 sh\rho\, s\varphi_2\, s\varphi_3\, s\varphi_4\, d\varphi_5$$
$$+ \Gamma_6 R_1 sh\rho\, s\varphi_2\, s\varphi_3\, s\varphi_4\, s\varphi_5\, d\varphi_6$$
$$+ \Gamma_7 R_2 c\theta\, d\psi$$
$$+ \Gamma_8 R_2 d\theta$$
$$+ \Gamma_9 R_2 s\theta\, d\chi_2$$
$$+ \Gamma_{10} R_2 s\theta\, s\chi_2\, d\chi_3 .$$

$$\Rightarrow \; D_\mu \epsilon + \frac{1}{12}\Big(\Gamma_{\mu 78910} \frac{1}{R_2^4}$$

$$- 2\,\delta_\mu^{\psi}\, \Gamma_{8910} \frac{c\theta}{R_2^3} + 2\,\delta_\mu^{\theta}\, \Gamma_{7910} \frac{1}{R_2^3}$$

$$- 2\,\delta_\mu^{\chi_2}\, \Gamma_{7810} \frac{s\theta}{R_2^3} + 2\,\delta_\mu^{\chi_3}\, \Gamma_{789} \frac{s\theta\, s\chi_2}{R_2^3} \Big) \epsilon\, Q$$

$$= 0 .$$

① $$\partial_t \epsilon + \frac{1}{2} sh\rho\, \Gamma_{01} \epsilon + \frac{Q}{12} \Gamma_{078910} \frac{R_1\, ch\rho}{R_2^4} \epsilon = 0$$

② $$\partial_\rho \epsilon + \frac{Q}{12} \Gamma_{178910} \epsilon \frac{R_1}{R_2^4} = 0$$

③ $$\partial_{\varphi_2} \epsilon + \frac{1}{2} ch\rho\, \Gamma_{21} \epsilon + \frac{Q}{12} \Gamma_{278910} \frac{R_1\, sh\rho}{R_2^4} \epsilon = 0 .$$

④ $$\partial_{\varphi_3} \epsilon + \frac{1}{2} ch\rho\, s\varphi_2\, \Gamma_{31} \epsilon + \frac{1}{2} c\varphi_2\, \Gamma_{32} \epsilon$$

$$+ \frac{Q}{12} \frac{R_1\, sh\rho\, s\varphi_2}{R_2^4} \Gamma_{378910} \epsilon = 0$$

⑤ $$\partial_{\varphi_4} \epsilon + \frac{1}{2} ch\rho\, s\varphi_2\, s\varphi_3\, \Gamma_{41} \epsilon + \frac{1}{2} c\varphi_2\, s\varphi_3\, \Gamma_{42} \epsilon$$

$$+ \frac{1}{2} c\varphi_3\, \Gamma_{43} \epsilon + \frac{Q}{12} \frac{R_1\, sh\rho\, s\varphi_2\, s\varphi_3}{R_2^4} \Gamma_{478910} \epsilon = 0$$

⑥ $$\partial_{\varphi_5} \epsilon + \frac{1}{2} ch\rho\, s\varphi_2\, s\varphi_3\, s\varphi_4\, \Gamma_{51} \epsilon + \frac{1}{2} c\varphi_2\, s\varphi_3\, s\varphi_4\, \Gamma_{52} \epsilon$$

$$+ \frac{1}{2} c\varphi_3\, s\varphi_4\, \Gamma_{53} \epsilon + \frac{1}{2} c\varphi_4\, \Gamma_{54} \epsilon$$

$$+ \frac{Q}{12} \frac{R_1\, sh\rho\, s\varphi_2\, s\varphi_2\, s\varphi_4}{R_2^4} \Gamma_{578910} \epsilon = 0 .$$

⑦
$$\partial_{\varphi_6}\epsilon + \frac{1}{2}\mathrm{ch}\rho\, s\varphi_2\, s\varphi_3\, s\varphi_4\, s\varphi_5\, \Gamma_{61}\,\epsilon$$
$$+ \frac{1}{2} c\varphi_2\, s\varphi_3\, s\varphi_4\, s\varphi_5\, \Gamma_{62}\,\epsilon$$
$$+ \frac{1}{2} c\varphi_3\, s\varphi_4\, s\varphi_5\, \Gamma_{63}\,\epsilon$$
$$+ \frac{1}{2} c\varphi_4\, s\varphi_5\, \Gamma_{64}\,\epsilon + \frac{1}{2} c\varphi_5\, \Gamma_{65}\,\epsilon$$
$$+ \frac{Q}{12}\, \frac{R_1\, \mathrm{sh}\rho\, s\varphi_2\, s\varphi_3\, s\varphi_4\, s\varphi_5}{R_2^4}\, \Gamma_{678910}\,\epsilon = 0$$

⑧
$$\partial_\psi \epsilon - \frac{1}{2} s\theta\, \Gamma_{78}\,\epsilon - \frac{Q}{6}\, \frac{c\theta}{R_2^3}\, \Gamma_{8910}\,\epsilon = 0$$

⑨
$$\partial_\theta \epsilon + \frac{Q}{6}\, \frac{1}{R_2^3}\, \Gamma_{7910}\,\epsilon = 0$$

⑩
$$\partial_{\chi_2}\epsilon + \frac{1}{2} c\theta\, \Gamma_{98}\,\epsilon - \frac{Q}{6}\, \frac{s\theta}{R_2^3}\, \Gamma_{7810}\,\epsilon = 0$$

⑪
$$\partial_{\chi_3}\epsilon + \frac{1}{2} c\theta\, s\chi_2\, \Gamma_{10,8}\,\epsilon + \frac{1}{2} c\chi_2\, \Gamma_{10,9}\,\epsilon$$
$$+ \frac{Q}{6}\, \frac{s\theta\, s\chi_2}{R_2^3}\, \Gamma_{789}\,\epsilon = 0.$$

* M_5 - brane

$$\frac{1}{2}R_1 = R_2 = (\pi N \ell_p^3)^{\frac{1}{3}}$$

$$Q = 3R_2 .$$

$\Rightarrow$ All the Killing spinor eqn's reduce to the form

$$\partial_\mu \epsilon + \frac{1}{2}\Omega_\mu \epsilon = 0 .$$

- $\Omega_t = sh\rho\, \Gamma_{01} - ch\rho\, \Gamma_{078910}$

$$\Omega_t^2 = sh^2\rho - ch^2\rho = -1 .$$

- $\Omega_\rho = \Gamma_{178910}$

$$\Omega_\rho^2 = (-1)^{10} = 1 .$$

- $\Omega_{\varphi_2} = ch\rho\, \Gamma_{21} + sh\rho\, \Gamma_{278\,9,10}$

$$\Omega_{\varphi_2}^2 = -ch^2\rho + sh^2\rho = -1 .$$

- $\Omega_{\varphi_3} = ch\rho\, s\varphi_2\, \Gamma_{31} + c\varphi_2\, \Gamma_{32} + sh\rho\, s\varphi_2\, \Gamma_{378910}$

$$\Omega_{\varphi_3}^2 = -ch^2\rho\, s^2\varphi_2 - c^2\varphi_2 + sh^2\rho\, s^2\varphi_2$$

$$= -(ch^2\rho - sh^2\rho)\, s^2\varphi_2 - c^2\varphi_2 = -1$$

$$\cdot\ \Omega_{\varphi_4} = ch\rho\, s\varphi_2 s\varphi_3 \Gamma_{41} + c\varphi_2 s\varphi_3 \Gamma_{42}$$
$$+ c\varphi_3 \Gamma_{43} + sh\rho\, s\varphi_2 s\varphi_3 \Gamma_{478910}$$
$$\Omega_{\varphi_4}^2 = -ch^2\rho\, s^2\varphi_2 s^2\varphi_3 - c^2\varphi_2 s^2\varphi_3$$
$$- c^2\varphi_3 + sh^2\rho\, s^2\varphi_2 s^2\varphi_3$$
$$= -1 .$$

$$\cdot\ \Omega_{\varphi_5} = ch\rho\, s\varphi_2 s\varphi_3 s\varphi_4 \Gamma_{51} + c\varphi_2 s\varphi_3 s\varphi_4 \Gamma_{52}$$
$$+ c\varphi_3 s\varphi_4 \Gamma_{53} + c\varphi_4 \Gamma_{54}$$
$$+ sh\rho\, s\varphi_2 s\varphi_3 s\varphi_4 \Gamma_{578910}$$
$$\Omega_{\varphi_5}^2 = -ch^2\rho\, s^2\varphi_2 s^2\varphi_3 s^2\varphi_4 - c^2\varphi_2 s^2\varphi_3 s^2\varphi_4$$
$$- c^2\varphi_3 s^2\varphi_4 - c^2\varphi_4 + sh^2\rho\, s^2\varphi_2 s^2\varphi_3 s^2\varphi_4$$
$$= -1 .$$

$$\cdot\ \Omega_{\varphi_6} = ch\rho\, s\varphi_2 s\varphi_3 s\varphi_4 s\varphi_5 \Gamma_{61}$$
$$+ c\varphi_2 s\varphi_3 s\varphi_4 s\varphi_5 \Gamma_{62}$$
$$+ c\varphi_3 s\varphi_4 s\varphi_5 \Gamma_{63} + c\varphi_4 s\varphi_5 \Gamma_{64}$$
$$+ c\varphi_5 \Gamma_{65} + sh\rho\, s\varphi_2 s\varphi_3 s\varphi_4 s\varphi_5 \Gamma_{678910}$$
$$\Omega_{\varphi_6}^2 = -1 .$$

- $\Omega_\psi = -s\theta\,\Gamma_{78} - c\theta\,\Gamma_{8\,9\,10}$

$$\Omega_\psi^2 = -s^2\theta - c^2\theta = -1 .$$

- $\Omega_\theta = \Gamma_{7\,9\,10}$

$$\Omega_\theta^2 = -1$$

- $\Omega_{\chi_2} = c\theta\,\Gamma_{98} - s\theta\,\Gamma_{7\,8\,10}$

$$\Omega_{\chi_2}^2 = -c^2\theta - s^2\theta = -1 .$$

- $\Omega_{\chi_3} = c\theta\, s\chi_2\,\Gamma_{10,8} + c\chi_2\,\Gamma_{10,9} + s\theta\, s\chi_2\,\Gamma_{789}$

$$\Omega_{\chi_3}^2 = -c^2\theta\, s^2\chi_2 - c^2\chi_2 - s^2\theta\, s^2\chi_2$$

$$= -1 .$$

Solving the eqn's.

$*\quad \partial_\rho \epsilon + \frac{1}{2}\Omega_\rho \epsilon = 0$.

$$\Rightarrow \epsilon = e^{-\frac{\rho}{2}\Omega_\rho}\eta \qquad (\partial_\rho \eta = 0)$$

$*\quad \partial_t \epsilon = e^{-\frac{\rho}{2}\Omega_\rho}\partial_t \eta = -\frac{1}{2}\Omega_t e^{-\frac{\rho}{2}\Omega_\rho}\eta$

$$\Rightarrow \partial_t \eta = -\frac{1}{2} e^{\frac{\rho}{2}\Omega_\rho}\Omega_t e^{-\frac{\rho}{2}\Omega_\rho}\eta .$$

$*\quad e^{\alpha\Omega_\rho} = \sum_{n=0}^{\infty}\frac{\alpha^n \Omega_\rho^n}{n!} = ch\alpha + sh\alpha\,\Omega_\rho$

$$\Rightarrow e^{\frac{\rho}{2}\Omega_\rho}\Omega_t e^{-\frac{\rho}{2}\Omega_\rho} = \left(ch\tfrac{\rho}{2}\,\Omega_t + sh\tfrac{\rho}{2}\,\Omega_\rho\Omega_t\right)\times$$

$$\times\left(ch\tfrac{\rho}{2} - sh\tfrac{\rho}{2}\,\Omega_\rho\right) .$$

$$= ch^2\tfrac{\rho}{2}\,\Omega_t - sh^2\tfrac{\rho}{2}\,\underbrace{\Omega_\rho\Omega_t\Omega_\rho}_{=-\Omega_t}$$

$$- ch\tfrac{\rho}{2}\, sh\tfrac{\rho}{2}\,\underbrace{\Omega_t\Omega_\rho}_{-\Omega_\rho\Omega_t} + sh\tfrac{\rho}{2}\, ch\tfrac{\rho}{2}\,\Omega_\rho\Omega_t .$$

$$= ch\rho\,\Omega_t + sh\rho\,\Omega_\rho\Omega_t$$

$$= ch\rho\left(sh\rho\,\Gamma_{01} + ch\rho\,\Gamma_{078910}\right)$$

$$+ sh\rho\left(\Gamma_{178910}\right)\left(sh\rho\,\Gamma_{01} + ch\rho\,\Gamma_{078910}\right)$$

$$= ch\rho\, sh\rho\,\Gamma_{01} + ch^2\rho\,\Gamma_{078910}$$

$$- sh^2\rho\,\Gamma_{078910} - sh\rho\, ch\rho\,\Gamma_{01}$$

$$= \Gamma_{078910} .$$

$$\Rightarrow \eta = e^{-\frac{t}{2}\Gamma_{078910}}\zeta \qquad (\partial_\rho\zeta = \partial_t\zeta = 0)$$

$$* \quad \text{So far} \quad \epsilon = e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} \zeta$$

$$\partial_{\varphi_2} \epsilon = e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} \partial_{\varphi_2} \zeta$$

$$= -\frac{1}{2} \Omega_{\varphi_2} e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} \zeta$$

$$\Rightarrow \partial_{\varphi_2} \zeta = -\frac{1}{2} e^{\frac{t}{2}\Gamma_{078910}} e^{\frac{\rho}{2}\Omega_\rho} \Omega_{\varphi_2} e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} \zeta$$

$$* \quad e^{\frac{\rho}{2}\Omega_\rho} \Omega_{\varphi_2} e^{-\frac{\rho}{2}\Omega_\rho} = \left(\mathrm{ch}\frac{\rho}{2}\Omega_{\varphi_2} + \mathrm{sh}\frac{\rho}{2}\Omega_\rho \Omega_{\varphi_2}\right) \times$$

$$\times \left(\mathrm{ch}\frac{\rho}{2} - \mathrm{sh}\frac{\rho}{2}\Omega_\rho\right)$$

$$= \mathrm{ch}^2\frac{\rho}{2} \Omega_{\varphi_2} + \mathrm{sh}^2\frac{\rho}{2}\Omega_{\varphi_2}$$

$$- \mathrm{ch}\frac{\rho}{2}\,\mathrm{sh}\frac{\rho}{2}\Omega_{\varphi_2}\Omega_\rho + \mathrm{ch}\frac{\rho}{2}\,\mathrm{sh}\frac{\rho}{2}\Omega_\rho\Omega_{\varphi_2}$$

$$= \mathrm{ch}\rho\,\Omega_{\varphi_2} + \mathrm{sh}\rho\,\Omega_\rho\Omega_{\varphi_2} = -\Gamma_{12}$$

$$* \quad e^{\frac{t}{2}\Gamma_{078910}} \left(\mathrm{ch}\rho\,\Omega_{\varphi_2} + \mathrm{sh}\rho\,\Omega_\rho\Omega_{\varphi_2}\right) e^{-\frac{t}{2}\Gamma_{078910}}$$

$$= \left(c\frac{t}{2} + s\frac{t}{2}\Gamma_{078910}\right)(-\Gamma_{12})\left(c\frac{t}{2} - s\frac{t}{2}\Gamma_{078910}\right)$$

$$= -c^2\frac{t}{2}\Gamma_{12} + s^2\frac{t}{2}\Gamma_{0789,10}\Gamma_{12}\Gamma_{078910}$$

$$+ c\frac{t}{2}\, s\frac{t}{2}\Gamma_{12}\Gamma_{0789,10} - s\frac{t}{2}\, c\frac{t}{2}\Gamma_{0789,10}\Gamma_{12}$$

$$= -\Gamma_{12}.$$

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$\star \quad \partial_{\varphi_2} \zeta = + \frac{1}{2} \Gamma_{12} \zeta$

$\Rightarrow \quad \zeta = e^{\frac{\varphi_2}{2}\Gamma_{12}} \xi \qquad (\partial_\rho \xi = \partial_t \xi = \partial_{\varphi_2} \xi = 0)$

$\Rightarrow \quad \epsilon = e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} e^{\frac{\varphi_2}{2}\Gamma_{12}} \xi$

One can continue this process to get

$$\epsilon = \underbrace{e^{-\frac{\rho}{2}\Omega_\rho} e^{-\frac{t}{2}\Gamma_{078910}} e^{\frac{\varphi_2}{2}\Gamma_{12}} \cdots\cdots}_{11 \text{ factors}} \epsilon_0$$

w/ a constant spinor ϵ_0.

$\Rightarrow$ Since ϵ_0 is not constrained, we have $\underline{\underline{32}}$ indep. Killing spinors according to 32 indep. comp. of ϵ_0

Massive Type IIA. (L.J. Romans PLB 169/374 1986)

* String frame.

$$I = \frac{1}{2\kappa^2}\int d^{10}x\Big[\sqrt{-g}\,\Big\{e^{-2\phi}\big(R + 4|\nabla\phi|^2 - \tfrac{1}{2}|H_3|^2\big) - \tfrac{1}{2}|F_2|^2 - \tfrac{1}{2}|F_4'|^2 - \underline{\tfrac{1}{2}m^2}\Big\} - \tfrac{1}{2}B_2\wedge F_4\wedge F_4\Big]$$

$$F_2 = dC_1 + \underline{m B_2}$$

$$H_3 = dB_2$$

$$F_4' = dC_3 - C_1\wedge H_3 + \underline{\tfrac{1}{2}m B_2\wedge B_2}$$

$$\Big(F_4 = dC_3 + \underline{\tfrac{1}{2}m B_2\wedge B_2}.\Big)$$

* Einstein frame

$$I = \frac{1}{2\kappa^2}\int d^{10}x\Big[\sqrt{-\tilde{g}}\,\Big\{\tilde{R} - \tfrac{1}{2}|\tilde{\nabla}\phi|^2 - \tfrac{1}{2}e^{-\phi}|\tilde{H}_3|^2 - \tfrac{1}{2}e^{\frac{3\phi}{2}}|\tilde{F}_2|^2 - \tfrac{1}{2}e^{\frac{\phi}{2}}|\tilde{F}_4'|^2 - \underline{\tfrac{1}{2}e^{\frac{5\phi}{2}}m^2}\Big\} - \tfrac{1}{2}B_2\wedge F_4\wedge F_4\Big]$$

positive cosmological constant.

F_{10} generating the 'cosmological constant'

- $F_{10} = dC_9$; sourced by D8-brane

- eqn. of motion.

$$d * F_{10} = 0 \qquad \text{(source free region)}$$

$$\Rightarrow * F_{10} = \text{const.}$$

$$\Rightarrow F_{10} = m \, \mathrm{Vol}(\mathbb{R}^{10}) .$$

- In the action, this will contribute

$$\cdots - \frac{1}{2} m^2 \cdots \quad \Leftarrow \quad -\frac{1}{2} |F_{10}|^2$$

* Background : D8 - brane .

- massive IIA does NOT admit
  a Minkowski +
  a maximally supersymm. background.

- D8 - brane (half supersymmetric)

$$ds^2 = H^{-\frac{1}{2}} d\sigma^2_{8,1} + H^{\frac{1}{2}} dx^2$$

$$e^{\phi} = H^{\frac{3-p}{4}}\Big|_{p=8} = H^{-\frac{5}{4}}$$

zero form.

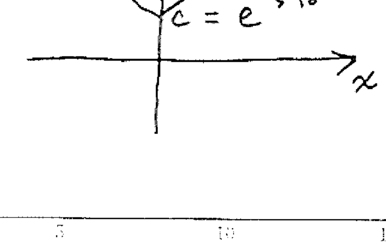
magnetic field $F = * dH$

↳ Hodge dual along x-coord.

$$\nabla^2 H = \frac{d^2}{dx^2} H = 0 .$$

$$\Rightarrow \quad H = \tilde{M}|x| + c .$$

$$\Rightarrow \quad F_{(10)} = \pm \tilde{M}\, \epsilon_{i_1 \cdots i_{10}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{10}}$$



SL(2,R) doublet

* Under $SL(2,R) \ni \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad - bc = 1)$,

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\lambda|^2 & \chi \\ \chi & 1 \end{pmatrix} \longrightarrow \Lambda \mathcal{M} \Lambda^T .$$

Then $\lambda \longrightarrow \lambda' = \dfrac{a\lambda + b}{c\lambda + d}$.

pf)

$$\mathcal{M}' = e^{\phi'} \begin{pmatrix} |\lambda'|^2 & \chi' \\ \chi' & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{\phi} \begin{pmatrix} |\lambda|^2 & \chi \\ \chi & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

$$= e^{\phi} \begin{pmatrix} (a\lambda + b)(a\lambda^* + b) & ac|\lambda|^2 + (ad+bc)\chi + bd \\ ac|\lambda|^2 + (ad+bc)\chi + bd & (c\lambda + d)(c\lambda^* + d) \end{pmatrix}$$

$$\Rightarrow \quad e^{\phi'} = e^{\phi} (c\lambda + d)(c\lambda^* + d)$$

$$e^{\phi'} \chi' = e^{\phi} [ac|\lambda|^2 + (ad+bc)\chi + bd]$$

$$\Rightarrow \quad \lambda' = \frac{ac|\lambda|^2 + (ad+bc)\chi + bd}{(c\lambda+d)(c\lambda^*+d)} + i \frac{e^{-\phi}}{(c\lambda+d)(c\lambda^*+d)}$$

$$= \frac{(a\lambda+b)(c\lambda^*+d) - bc\lambda^* - ad\lambda + (ad+bc)\chi - \chi + \lambda}{(c\lambda+d)(c\lambda^*+d)}$$

$$= \frac{(a\lambda+b)(c\lambda^*+d) - bc\lambda^* + (\underbrace{1-ad}_{-bc})\lambda + (\underbrace{ad+bc-1}_{2bc})\chi}{(c\lambda+d)(c\lambda^*+d)}$$

$$= \frac{a\lambda + b}{c\lambda + d} .$$

* Transformation of (F1, D1) charges.

Let us consider the minimal coupling term

$$\cdots \quad J^T \wedge \mathcal{B} = \cdots \quad \mathcal{B}_2^T \wedge J_8$$

Since $\mathcal{B} \to (\Lambda^T)^{-1}\mathcal{B}$ under $SL(2,R)$ (Λ),
the term is invariant if $J \to \Lambda J$.

— One can see this in the e.o.m. too.

$$\mathcal{L} \sim \frac{1}{2K^2}\left[-\frac{1}{12} H_3^T \wedge \mathcal{M} * H_3 \quad + \mathcal{B}_2^T \wedge J_8 \right]$$

J; current density.

$$\delta\mathcal{L} \sim \frac{1}{2K^2}\left[-\frac{1}{6} d\delta B_2^T \wedge \mathcal{M} * H_3 \quad + \delta B_2^T \wedge J_8 \right]$$

$$= \frac{1}{2K^2}\left[\frac{1}{6} \delta B_2^T \wedge \mathcal{M}\, d * H_3 \quad + \delta B_2^T \wedge J_8 \right]$$

$$\Rightarrow \frac{1}{6}\mathcal{M} d * H_3 \quad + J_8 \quad = 0 .$$

$$\Rightarrow \quad J_8 = -\frac{1}{6}\mathcal{M} d * H_3 \quad \longrightarrow \Lambda J_8$$

M origin of $SL(2,\mathbb{Z})$ in IIB (J.H. Schwarz 9508143)

*. (q_1, q_2) - string in $\lambda_0 = i$ background.

$$(\lambda_0 = \chi_0 + i e^{-\phi_0} \Rightarrow \chi_0 = 0, \ \phi_0 = 0 .)$$

→ 1 dim. object. w/ (q_1, q_2)-charge.

⇒ It generates the same geometry as that of the $(1,0)$-string.

Therefore,

T_q ; tension

$$A_q ; \ 1 + \frac{\alpha_q}{3r^6} \qquad \alpha_q = \frac{\kappa^2 T_q}{\omega_7}$$

$$\left.\begin{array}{l} B_{01} = e^{2\phi_q} = A_q^{-1} \\ \chi = 0 \end{array}\right] \Rightarrow \lambda = i A_q^{\frac{1}{2}} .$$

$$\lim_{r\to\infty} \lambda = i$$

$$ds^2 = A_q^{-\frac{3}{4}} [- dt^2 + (dx^1)^2] + A_q^{\frac{1}{4}} d\vec{x}\cdot d\vec{x}$$

$$= e^{\frac{\phi_q}{2}} \underbrace{\left(A_q^{-1} (-dt^2 + (dx^1)^2) + d\vec{x}\cdot d\vec{x} \right)}_{= ds^2_{\text{string}}}$$

in the Einstein frame.

— Hence (T_q, A_q, B_{01}, χ) solves the eqn's of motion in $\lambda_0 = i$ background.

BUT INCORRECT CHARGES!!

$$\left(\frac{\alpha_q}{Q} , 0 \right) \neq (q_1, q_2).$$

— In order to get correct charges (q_1, q_2), we make use of the $SL(2,\mathbb{R})$ transf.

- Consider

$$\Lambda = \frac{1}{\sqrt{q_1^2+q_2^2}} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \in SO(2) \subset SL(2,\mathbb{R})$$

which leaves λ_0 invariant!!

$$\lambda' = \frac{a\lambda + b}{c\lambda + d} = \frac{a\,i\,A_q^{\frac{1}{2}} + b}{c\,i\,A_q^{\frac{1}{2}} + d}$$

$$\lim_{r\to\infty} \lambda' = \frac{ai+b}{ci+d} = \frac{\overset{0 \ \because SO(2).}{\cancel{ac+bd}} + i(ad-bc)}{c^2+d^2} = i$$

- Since

$$\begin{pmatrix} H^{(1)\prime} \\ H^{(2)\prime} \end{pmatrix} = (\Lambda^T)^{-1} \begin{pmatrix} dA_q^{-1} \\ 0 \end{pmatrix} \overset{SO(2)}{=} \Lambda \begin{pmatrix} dA_q^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{q_1}{\sqrt{q_1^2+q_2^2}}\, dA_q^{-1} \\ \frac{q_2}{\sqrt{q_1^2+q_2^2}}\, dA_q^{-1} \end{pmatrix},$$

and the field dA_q^{-1} generates the charge α_q, $H^{(1)\prime}$ & $H^{(2)\prime}$ will give the charges

$$\left(\frac{q_1}{\sqrt{q_1^2+q_2^2}}\alpha_q,\ \frac{q_2}{\sqrt{q_1^2+q_2^2}}\alpha_q\right).$$

In order to get $(q_1 Q,\ q_2 Q)$ charges,

$$\alpha_q = \sqrt{q_1^2+q_2^2}\ Q.$$

N.B. How to get the charge from the field.

For example

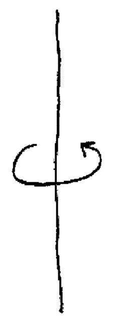
$$A_q = 1 + \frac{\alpha_q}{3r^6} \Rightarrow A_q^{-1} = \frac{3r^6}{3r^6 + \alpha_q}.$$

From $J \sim * \, d A_q^{-1}$,

$$Q \sim \lim_{r\to\infty} \int \frac{18 r^5 (3r^6 + \alpha_q) - 3r^6 \cdot 18 r^5}{(3r^6 + \alpha_q)^2} \underbrace{* \, d\Omega}_{r^7 d\Omega_7}$$

$$= \lim_{r\to\infty} \int_{S^7} \frac{18 r^{12} \alpha_q}{(3r^6 + \alpha_q)^2} \, d\Omega_7$$

$$= 2\alpha_q \, \omega_7 .$$



S^7

– The dilaton & axion fields:

$$e^{\phi'} = e^{\phi}(c\lambda + d)(c\lambda^* + d)$$

$$\chi' = \frac{ac|\lambda|^2 + \underbrace{\cancel{(ad+bc)\chi}}_{\to 0 \ \because SO(2)} + bd}{(c\lambda+d)(c\lambda^*+d)}$$

$$\Rightarrow e^{\phi'} = A_q^{-\frac{1}{2}}\left(\frac{q_2}{\sqrt{q_1^2+q_2^2}}\, iA_q^{\frac{1}{2}} + \frac{q_1}{\sqrt{q_1^2+q_2^2}}\right) \times$$

$$\times\left(-\frac{q_2}{\sqrt{q_1^2+q_2^2}}\, iA_q^{\frac{1}{2}} + \frac{q_1}{\sqrt{q_1^2+q_2^2}}\right).$$

$$= A_q^{-\frac{1}{2}}\left(\frac{q_2^2 A_q + q_1^2}{q_1^2 + q_2^2}\right)$$

$$\chi' = \frac{q_1^2 + q_2^2}{q_1^2 + q_2^2 A_q}\left[\frac{q_1 q_2 A_q}{q_1^2 + q_2^2} + \frac{-q_1 q_2}{q_1^2 + q_2^2}\right]$$

$$= \frac{(A_q - 1)\, q_1 q_2}{q_1^2 + q_2^2 A_q}$$

* (q_1', q_2') - string w/ a general vacuum modulus.

— Start w/ α_q , $\lambda_0 = i$, $\left(\begin{array}{l} B^{(1)} = c\theta \, A_q^{-1} \\ B^{(2)} = s\theta \, A_q^{-1} \end{array} \right.$

$$\lambda = \frac{i\, c\theta \, A_q^{\frac{1}{2}} - s\theta}{i\, s\theta \, A_q^{\frac{1}{2}} + c\theta} .$$

N.B. when $c\theta = \frac{q_1}{\sqrt{q_1^2 + q_2^2}}$, $s\theta = \frac{q_2}{\sqrt{q_1^2 + q_2^2}}$,

the sol. becomes that of (q_1, q_2) - string w/ $\lambda_0 = i$.

— The following Λ will make $\lambda_0 = i$ to $\lambda_0 = \chi_0 + i e^{-\phi_0}$.

• $\Lambda \in SL(2, \mathbb{R})$

$$\Lambda = \begin{pmatrix} e^{-\frac{\phi_0}{2}} & \chi_0 e^{\frac{\phi_0}{2}} \\ 0 & e^{\frac{\phi_0}{2}} \end{pmatrix} .$$

$$(\lambda_0 = i) \longrightarrow \frac{e^{-\frac{\phi_0}{2}} i + \chi_0 e^{\frac{\phi_0}{2}}}{e^{\frac{\phi_0}{2}}} = \chi_0 + i e^{-\phi_0}$$

Charge transf.

$$\begin{pmatrix} q^{1'} \\ q^{2'} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\phi_0}{2}} & \chi_0 e^{\frac{\phi_0}{2}} \\ 0 & e^{\frac{\phi_0}{2}} \end{pmatrix} \begin{pmatrix} c\theta\, \alpha_q \\ s\theta\, \alpha_q \end{pmatrix}.$$

$$\Rightarrow \quad \frac{q^{1'}}{q^{2'}} = \frac{1 + \chi_0 e^{\phi_0} t\theta}{e^{\phi_0} t\theta}$$

$$\Rightarrow \quad c\theta = \frac{e^{\phi_0}(q^{1'} - q^{2'}\chi_0)}{\sqrt{e^{2\phi_0}(q^{1'} - q^{2'}\chi_0)^2 + (q^{2'})^2}}$$

$$s\theta = \frac{q^{2'}}{\sqrt{e^{2\phi_0}(q^{1'} - q^{2'}\chi_0)^2 + (q^{2'})^2}}$$

$$\alpha_q = \frac{Q q^{2'}}{s\theta} e^{-\frac{\phi_0}{2}}$$

$$= Q \sqrt{e^{\phi_0}(q^{1'} - q^{2'}\chi_0)^2 + (q^{2'})^2 e^{-\phi_0}}$$

$$\equiv Q \Delta_q^{\frac{1}{2}}$$

$$\left(e^{i\theta} = \frac{e^{\phi_0}(q^{1'} - q^{2'}\chi_0) + i q^{2'}}{\sqrt{e^{2\phi_0}(q^{1'} - q^{2'}\chi_0)^2 + (q^{2'})^2}} \right.$$

$$= \frac{e^{\frac{\phi_0}{2}} q^{1'} - e^{\frac{\phi_0}{2}}(\chi_0 - i e^{-\phi_0}) q^{2'}}{\sqrt{e^{\phi_0}(q^{1'} - q^{2'}\chi_0)^2 + (q^{2'})^2 e^{-\phi_0}}}$$

$$\left. = e^{\frac{\phi_0}{2}} (q^{1'} - q^{2'}\bar{\lambda}_0) \Delta_q^{-\frac{1}{2}} \right)$$

* $\Delta_q = e^{\phi_0}(q^{1\prime} - q^{2\prime}\chi_0)^2 + (q^{2\prime})^2 e^{-\phi_0}$

$$= (q^{1\prime}, q^{2\prime})\, \mathcal{M}_0^{-1} \begin{pmatrix} q^{1\prime} \\ q^{2\prime} \end{pmatrix}.$$

pf) $\mathcal{M}_0 = e^{\phi_0}\begin{pmatrix} |\lambda_0|^2 & \chi_0 \\ \chi_0 & 1 \end{pmatrix}.$

$$\Rightarrow \mathcal{M}_0^{-1} = \frac{e^{-\phi_0}}{\underbrace{|\lambda_0|^2}_{\chi_0^2 + e^{-2\phi_0}} - \chi_0^2} \begin{pmatrix} 1 & -\chi_0 \\ -\chi_0 & |\lambda_0|^2 \end{pmatrix}$$

$$= e^{\phi_0} \begin{pmatrix} 1 & -\chi_0 \\ -\chi_0 & |\lambda_0|^2 \end{pmatrix}.$$

$$(q^{1\prime}, q^{2\prime})\, \mathcal{M}_0^{-1} \begin{pmatrix} q^{1\prime} \\ q^{2\prime} \end{pmatrix} = e^{\phi_0} (q^{1\prime}, q^{2\prime}) \begin{pmatrix} 1 & -\chi_0 \\ -\chi_0 & |\lambda_0|^2 \end{pmatrix} \begin{pmatrix} q^{1\prime} \\ q^{2\prime} \end{pmatrix}$$

$$= e^{\phi_0}\left[(q^{1\prime})^2 - 2\chi_0 q^{1\prime} q^{2\prime} + \underbrace{|\lambda_0|^2}_{\chi_0^2 + e^{-2\phi_0}} (q^{2\prime})^2\right]$$

$$= e^{\phi_0}\left[(q^{1\prime} - \chi_0 q^{2\prime})^2 - e^{-2\phi_0}(q^{2\prime})^2\right]$$

$$= \Delta_q^2.$$

* Tension -

$Q = \frac{\kappa^2 T}{\omega_7}$ $\Rightarrow$ $\alpha_q = \sqrt{q_1^2 + q_2^2}\, Q$, $T_q = \sqrt{q_1^2 + q_2^2}\, T$

$\Rightarrow$ $\alpha_q' = \Delta_q^{\frac{1}{2}} Q$, $T_q' = \Delta_q^{\frac{1}{2}} T$.

properties of $\Delta_q = e^{\phi_0}(q_1^2|\lambda_0|^2 + 2q_1q_2\chi_0 + q_2^2)$

i) when $|\lambda_0| = 1 = \sqrt{\chi_0^2 + e^{-2\phi_0}}$

$$\Rightarrow \quad \Delta_q = e^{\phi_0}(q_1^2 + 2q_1q_2\chi_0 + q_2^2)$$

$$\Rightarrow \quad \Delta_{q_1,q_2} = \Delta_{q_2,q_1}$$

ii) $\chi_0 = -\frac{1}{2}$

$$\Delta_q = e^{\phi_0}\left(q_1^2\left(\frac{1}{4} + e^{-2\phi_0}\right) - q_1q_2 + q_2^2\right)$$

$$= e^{\phi_0}\left[e^{-2\phi_0}q_1^2 + \frac{1}{4}(q_1 - 2q_2)^2\right]$$

$$= e^{\phi_0}\left[e^{-2\phi_0}q_1^2 + \frac{1}{4}(q_1 - 2(q_1 - q_2))^2\right]$$

$$\Rightarrow \quad T_{q_1 q_2} = T_{q_1 (q_1 - q_2)}$$

iii) $\chi_0 = -\frac{1}{2}$, $|\lambda_0| = \sqrt{\chi_0^2 + e^{-2\phi_0}} = 1.$

$$\Rightarrow \quad e^{-\phi_0} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \lambda_0 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = -e^{-\frac{i\pi}{3}}$$

$$= e^{\frac{2i\pi}{3}}$$

a 3-fold degeneracy : $T_{1,0} = T_{0,1} = T_{1,1}$

$= (T_{0,-1} = T_{-1,0} = T_{-1,-1})$

anti strings

Closed (p,q)-string

- $(1,0)$-string

$T=(2\pi\alpha')^{-1}$

$$X^\mu = x^\mu + \sqrt{\frac{\alpha'}{2}}\,\alpha^\mu_0 \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum \frac{\alpha^\mu_n}{n} e^{-in\sigma^-}$$

$$+\sqrt{\frac{\alpha'}{2}}\,\tilde{\alpha}^\mu_0 \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum \frac{\tilde{\alpha}^\mu_n}{n} e^{-in\sigma^+}$$

$$= x^\mu + \underbrace{\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0+\tilde{\alpha}^\mu_0)\tau}_{\alpha' p^\mu \tau \left(= \frac{\alpha' n}{R}\tau \text{ if compact}\right)} + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}^\mu_0-\alpha^\mu_0)\sigma + \cdots$$

under $\sigma \to \sigma + 2\pi$,

$$\Delta X^\mu = 2\pi\sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}^\mu_0-\alpha^\mu_0)$$

$$= 2\pi\,\omega R$$

$$\Rightarrow \quad \tilde{\alpha}^\mu_0 = \frac{1}{\sqrt{2\alpha'}}\left(\frac{\alpha' n}{R} + \omega R\right) \equiv \sqrt{\frac{\alpha'}{2}}\, p_L$$

$$\alpha^\mu_0 = \frac{1}{\sqrt{2\alpha'}}\left(\frac{\alpha' n}{R} - \omega R\right) \equiv \sqrt{\frac{\alpha'}{2}}\, p_R$$

N.B. $\quad L_0 - 1 = 0\ ,\quad \tilde{L}_0 - 1 = 0\ .$ (Virasoro const.)

$$\Rightarrow \quad \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n}\cdot\alpha_n - 1 = 0$$

$$\left(\frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n>0} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n - 1 = 0\right)$$

$$\alpha_0^2 = \sum_{\mu=0}^{D-1} \alpha^\mu_0 \alpha^\nu_0 \eta_{\mu\nu} + \alpha^D_0 \alpha^D_0$$

$$= \frac{\alpha'}{2} p^\mu p^\nu \eta_{\mu\nu} + \frac{\alpha'}{2} p_R^2$$

$$M^2 = -p^2 = \frac{4}{\alpha'}\left(\frac{\alpha'}{4}p_R^2 + N_R - 1\right)$$

$$= \frac{4}{\alpha'}\left(\frac{\alpha'}{4}p_L^2 + N_L - 1\right)$$

$$= \frac{1}{2}\cdot\frac{4}{\alpha'}\left(\frac{\alpha'}{4}(p_R^2 + p_L^2) + N_R + N_L - 2\right)$$

$$= 4\pi T\left(\frac{2\alpha'}{4}\left(\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2}\right) + N_R + N_L - 2\right)$$

$$= \frac{n^2}{R^2} + 4\pi^2 w^2 R^2 T^2$$

$$+ 4\pi T\,(N_R + N_L \underbrace{-\,2}_{})$$

zero pt energy.

→ disappear for the supersymm. case.

- Level matching condition.

$$0 = \left(\frac{\alpha'}{4}p_R^2 + N_R - 1\right) - \left(\frac{\alpha'}{4}p_L^2 + N_L - 1\right)$$

$$= \frac{\alpha'}{4}\,(p_R^2 - p_L^2) + N_R - N_L$$

$$= \frac{\alpha'}{4}\left(-\frac{4nw}{\alpha'}\right) + N_R - N_L.$$

$$\Rightarrow \quad N_R - N_L = nw.$$

As for the loopy (p, q) string,

$$M^2 = 4\pi T_q (N_L + N_R)$$

· All the different strings have the same lowest level ; IIB SUGRA.

· The excited states

the excited states of one string

→ non perturbative states in view of the other strings.

· (q_1, q_2) IIB string on a circle of radius R_B.

$$M_B^2 = \left(\frac{m}{R_B}\right)^2 + (2\pi R_B n T_q)^2 + 4\pi T_q (N_L + N_R)$$

$$N_R - N_L = n .$$

short multiplet ; $N_R = 0$ or $N_L = 0$.

Ultrashort " ; $N_R = N_L = 0$

* $N_L = 0$

$$M_B{}^2 = \left(\frac{m}{R_B}\right)^2 + (2\pi R_B n T_q)^2 + 4\pi T_q \underbrace{N_R}_{mn}$$

$$= \left(\frac{m}{R_B} + 2\pi R_B n T_q\right)^2$$

$$T_q = \Delta_q^{\frac{1}{2}} T$$

$$= [e^{\phi_0}(q_2\chi_0 - q_1)^2 + e^{-\phi_0} q_2^2]^{\frac{1}{2}} T$$

Let $l_1 = n q_1$, $l_2 = n q_2$

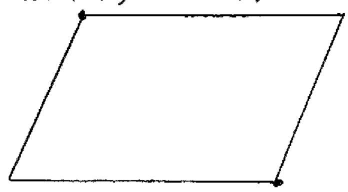
$$\Rightarrow n^2 T_q{}^2 = [e^{\phi_0}(l_2\chi_0 - l_1)^2 + e^{-\phi_0} l_2^2]^{\frac{1}{2}} T^2$$

As for the $(l_1 = nq_1,\ l_2 = nq_2)$-string its tension is n times that of the (q_1, q_2)-string. ; BPS.

11 D SUGRA on T^2

* $\tau = \tau_1 + i\tau_2$; moduli parameter of T^2.

$(2\pi R_{11}\tau_1 , 2\pi R_{11}\tau_2)$



$(2\pi R_{11} , 0)$

· $$z \equiv \frac{1}{2\pi R_{11}}(x + iy)$$

* Wave ftn. on T^2.

$$\psi(x,y) \sim e^{ik_x x + ik_y y}$$

· $\psi\left(z = \frac{1}{2\pi R_{11}}(x+iy)\right)$; inv. under

$$z \to z+1 \quad , \quad z \to z+\tau .$$

$$\begin{cases} ik_x \cdot 2\pi R_{11} = i2\pi \cdot l_2 \\ ik_x \cdot 2\pi R_{11}\tau_1 + ik_y \cdot 2\pi R_{11}\tau_2 = i2\pi \cdot l_1 \end{cases}$$

$$\Rightarrow \quad k_x = \frac{l_2}{R_{11}}$$

$$k_y = \frac{1}{R_{11}\tau_2}(l_1 - l_2\tau_1) .$$

$$l_1, \; l_2 \in \mathbb{Z}$$

* Kaluza - Klein mass in 9-dim.

$$p_x^2 + p_y^2 = -\partial_x^2 - \partial_y^2 .$$

$\hookrightarrow$ apply to $\psi_{\ell_1,\ell_2}(x,y) \sim e^{\frac{i}{R_{11}}[x\ell_2 + \frac{1}{\tau_2}y(\ell_1 - \ell_2\tau)]}$

$$\Rightarrow \left(\frac{\ell_2}{R_{11}}\right)^2 + \left(\frac{1}{\tau_2}(\ell_1 - \ell_2\tau)\right)^2 \frac{1}{R_{11}^2}$$

* a membrane wrapping m times over T^2 of area $A_{11} = (2\pi R_{11})\cdot(2\pi R_{11}\tau_2)$

$$= (2\pi R_{11})^2 \tau_2 .$$

$$M_{11}^2 = (m A_{11} T_{11})^2 + \frac{1}{R_{11}^2}\left(\ell_2^2 + \frac{1}{\tau_2^2}(\ell_1 - \ell_2\tau_1)^2\right)$$

$$+ \underbrace{\cdots\cdots}_{\text{membrane excitations.}}$$

* IIB on S^1 vs. 11D SUGRA on T^2.

- Scaling conversion ; $M_{11} = \beta M_B$.

- $$M_B^2 = \beta^{-2} M_{11}^2$$

$$= \beta^{-2}(m A_{11} T_{11})^2 + \frac{1}{R_{11}^2\beta^2}\left(l_2^2 + \frac{1}{T_2^2}(l_1 - l_2 T_1)^2\right) + \cdots$$

$$= \left(\frac{m}{R_B}\right)^2 + (2\pi R_B)^2\left[l_2^2 + e^{2\phi_0}(l_2\chi_0 - l_1)^2\right] e^{-\phi_0} T$$

$$+ 4\pi T_q (N_L + N_R)$$

$\Rightarrow$ $\lambda_0 = \chi_0 + i e^{-\phi_0}$
$\tau = \tau_1 + i\tau_2$ } identify.

① $R_B^{-2} = \beta^{-2}(A_{11}T_{11})^2$

② $\dfrac{1}{R_{11}^2\beta^2} = (2\pi R_B)^2 e^{-\phi_0} T^2$

From ① & ② ,

$$R_B^{-2} = R_{11}^2 (2\pi R_B)^2 e^{-\phi_0} T^2 (A_{11}T_{11})^2$$

$$= R_B^2 A_{11}^3 T^2 T_{11}^2$$

$$\Rightarrow \frac{1}{R_B^2} = A_{11}^{\frac{3}{2}} T T_{11} .$$

$$\beta^2 = R_B^2 (A_{11}T_{11})^2 = A_{11}^{-\frac{3}{2}} T^{-1} T_{11}^{-1} (A_{11}T_{11})^2$$

$$= A_{11}^{\frac{1}{2}} T^{-1} T_{11} .$$

IIA on S^1 (of radius R_A).

* Spectrum

$$M_A^2 = \left(\frac{l_1}{R_A}\right)^2 + (2\pi R_A m T_A)^2 + 4\pi T_A (N_L + N_R).$$

* Comparison w/ IIB on S^1 (R_B).

$$M_B^2 = \left(\frac{m}{R_B}\right)^2 + (2\pi R_B n T_q)^2 + 4\pi T_q (N_L + N_R).$$

$$n^2 T_q^2 = \left[l_2^2 + e^{2\phi_0}(l_2\chi_0 - l_1)^2\right] e^{-\phi_0} T^2.$$

* The aboves are dual to each other when $(q_1, q_2) = (1, 0)$, $\chi_0 = 0$.

Let $M_B = \gamma M_A$.

① $\dfrac{\gamma^2}{R_A^2} = (2\pi R_B)^2 e^{\phi_0} T^2$

② $2\pi R_A T_A \gamma = \dfrac{1}{R_B}$ $\Rightarrow$ $R_A R_B = \dfrac{1}{2\pi \gamma T_A}$

③ $T_A \gamma^2 = e^{\frac{\phi_0}{2}} T$.

② into ① ; $\gamma^2 = (\gamma^2 T_A^2)^{-1} e^{\phi_0} T^2$

$\Rightarrow$ $\gamma^2 T_A = e^{\frac{\phi_0}{2}} T$.

consistent w/ ③.

* Comparison w/ 11 D SUGRA on T^2.

· Conversion factor

$$M_{11} = \beta M_B = \beta \gamma M_A$$

$$\cdot\ M_{11}^2 = \beta^2\gamma^2\left(\frac{l_1}{R_A}\right)^2 + \beta^2\gamma^2(2\pi R_A m T_A)^2 + 4\pi\beta^2\gamma^2 T_A(N_L+N_R)$$

$$= (m A_{11} T_{11})^2 + \frac{1}{R_{11}^2}\left(l_2^2 + \frac{1}{\tau_2^2}(l_1 - l_2\tau_1)^2\right)\Bigg|_{\substack{l_2=0\\ \tau_1=0}} + \cdots$$

$$\Rightarrow \text{①}\quad \frac{\beta^2\gamma^2}{R_A^2} = \frac{1}{R_{11}^2\tau_2^2}$$

$$\text{②}\quad \beta\gamma\, 2\pi R_A T_A = A_{11} T_{11} = (2\pi R_{11})^2 \tau_2 T_{11}$$

From ①, $R_A^{(11)} = \dfrac{R_A}{\beta\gamma} = R_{11}\tau_2$.

· $\chi_0 = \tau_1 = 0$; the torus is a rectangle w/ sides $2\pi R_{11}$, $2\pi R_{11}\tau_2$.

$\Rightarrow$ $2\pi r = 2\pi R_{11}\tau_2$ along which 11 D SUGRA is compactified to IIA.