Dyons

bound state of monopoles and electric charges

Bosonic part of $N=2$ Hamiltonian

$$H = \frac{1}{2} \mathrm{tr} \left(E^2 + B^2 + (D_0 X)^2 + (DX)^2 + (D_0 Y)^2 + (DY)^2 + (i[X,Y])^2 \right]$$

Turn off Y.

$$H = \frac{1}{2} \mathrm{tr} \left(E^2 + B^2 + (D_0\phi)^2 + (D\phi)^2 \right] \qquad \phi = X.$$

BPS bounds by completing squares.

$$H = \frac{1}{2} \mathrm{tr} \int d\vec{x} \left[(E - \sin\theta\, D\phi)^2 + (B - \cos\theta\, D\phi)^2 + (D_0\phi)^2 \right.$$
$$\left. + 2\sin\theta\, E D\phi + 2\cos\theta\, B D\phi \right]$$

$D_i B_i = 0$ Bianchi identity

$D\cdot E \quad - i[\phi, D\phi] = 0$

$$\int d\vec{x}\ \mathrm{tr}\, B D\phi \qquad \langle\phi\rangle_{r\to\infty} = a\,\hat{r}^a t^a = a \hat{r}^a \frac{\sigma^a}{2}$$

$$= \int d\vec{x}\ \mathrm{tr}\, B_i D_i \phi = \int d\vec{x} \left[\mathrm{tr}\, \partial_i B_i \phi - \mathrm{tr} (D_i B_i) \phi \right]$$

$$= \int d\vec{S}\cdot \mathrm{tr}\, \vec{B}\phi = a \int d\vec{S}\cdot \mathrm{tr}\, \vec{B} \frac{\phi}{a}$$

$$= a g \qquad g = \int d\vec{S}\cdot \mathrm{tr}\, \vec{B} \frac{\phi}{a}$$

Similarly,

$$\int d\vec{x}\ \mathrm{tr}\, E\cdot D\phi = a \int d\vec{S}\cdot \mathrm{tr}\, \vec{E} \frac{\phi}{a}$$

$$q \equiv \int d\vec{S}\cdot \mathrm{tr}\, \vec{E} \frac{\phi}{a}$$

Therefore,

$$E \geq a(\sin\theta\, g + \cos\theta\, g)$$

The saturation occurs if

$$E = \sin\theta\, D\phi$$

$$B = \cos\theta\, D\phi$$

Take the gauge choice

$$A_0 = \phi \sin\theta \qquad E = -\frac{\partial A}{\partial t} + \vec{D}_0 A_0$$

$$\rightarrow \quad \frac{\partial \vec{A}}{\partial t} = 0$$

The equation is reduced to

$$B = \cos\theta \;\; D\phi$$

$\theta = 0$ corresponds to the pure monopole solution

$$B = D\phi$$

One monopole solution has the following simple form.

$$\bar{\Phi} = a\hat{r}\cdot\vec{t}\,\left(\coth ear - \frac{1}{ear}\right)$$

$$\bar{A}_{ia} = \varepsilon_{aij}\,\frac{\hat{r}_j}{er}\,\left[1 - \frac{ear}{\sinh ear}\right]$$

~~For~~ ~~$r\to\infty$~~ large r,

$$\phi \longrightarrow \hat{r}\cdot\vec{t}\,\left(a - \frac{1}{er}\right)$$

$$B \longrightarrow \hat{r}\cdot\vec{t}\,\frac{\hat{r}}{er^2}$$

$$g = \frac{4\pi}{e}$$

Simple dyon solution

$$\phi = \bar{\Phi}(r\cos\theta)$$

$$A = \cos\theta \, \bar{A}(r\cos\theta)$$

$$\vec{\xi} = \vec{r}\cos\theta$$

$$D\phi = \cos\theta \frac{\partial}{\partial \xi} \bar{\Phi}(\xi) - i\left[\cos\theta \bar{A}(\xi), \bar{\Phi}\right]$$

$$= \cos\theta \quad \bar{D}\bar{\Phi}(\xi)$$

Therefore

$$A_0 = \sin\theta \; \bar{\Phi}(r\cos\theta)$$

$$\longrightarrow \quad \sin\theta \; a \quad \left(1 - \frac{1}{e a r \cos\theta}\right)$$

Hence $\qquad q = \tan\theta \; g$

$$M = a(\sin\theta \, q + \cos\theta \, g)$$

$$= ag\left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right) \qquad \frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$= ag \frac{1}{\cos\theta} = a\sqrt{q^2 + g^2}$$

Later we shall show $\quad q = e\, n_e \qquad n_e \in \mathbb{Z}$

quantum mechanical

$$g = \frac{e}{4\pi} n_m \qquad n_m \in \mathbb{Z}$$

topological

$ae \to a$

$$M = \sqrt{(a n_e)^2 + \left(a\frac{4\pi}{e^2}\right)^2 n_m^2}$$

$$\tau \equiv \frac{4\pi}{e^2} i + \frac{\theta}{2\pi}$$

$$= \sqrt{|a n_e + a \tau n_m|^2}$$

$a\tau$ is related to the vev of the dual Higgs at weak coupling limit.

$$Z = a n_e + a_D n_m$$

$$a_D = a\tau \quad \text{at weak coupling limit.}$$

Particle properties of monopoles

The monopole behave like particles.

1. If one turns on the unbroken magnetic field, the monopole will be accelerated because following the Lorentz force law.

2. What is the way of describing their low energy dynamics of such solitons?

Moduli space approximation

Space of ~~fields~~ solutions, $A(\vec{r},\xi)$ $\phi(\vec{r},\xi)$, which satisfies the BPS equation with the given monopole charge.

moduli

space of the minimum energy

$$M = |Z|$$

$$\mathcal{A} / \mathcal{G} = \text{moduli space} \quad (\text{coordinate } \xi^A)$$

$A_0 = 0$ gauge

Low-energy effective action $\xi(t)$

Nonrelativistic approximation.

$$L_{eff} = \frac{1}{2}\int d^3x \; \dot{\vec{A}}\cdot\dot{\vec{A}} + \dot{\phi}\cdot\dot{\phi} \quad - M$$

$$= \frac{1}{2} g_{ab}(\xi)\,\dot{\xi}^a \dot{\xi}^b$$

$$g_{ab} = \int d\vec{x} \left(\partial_{\xi_a}\vec{A}\cdot\partial_{\xi_b}\vec{A} + \partial_{\xi_a}\phi\, \partial_{\xi_b}\phi \right)$$

$(\delta\vec{A}, \delta\phi)$ infinitesimal translation ~~tangent vectors~~ in the solution space

<u>infinitesimal version</u>

$A = \bar{A} + \delta A \qquad \phi = \bar{\phi} + \delta\phi$

Work in the first order. Then

$(\delta A, \delta\phi)$ satisfy the linearized BPS equation.

↳ zero-mode if normalizable. $\int d^3x\, \mathrm{tr}[\delta A \cdot \delta A + \delta\phi \cdot \delta\phi]$

$\delta B = \delta D\phi \qquad \delta D\phi = \bar{D}\delta\phi - i[\delta A, \bar{\Phi}]$

$\bar{D}_i \delta A_i - i[\bar{\Phi}, \delta\phi] = 0$ Gauge condition.

$\delta B_i = \frac{1}{2}\varepsilon_{ijk}(\bar{D}_j \delta A_k - \bar{D}_k \delta A_j - \cancel{i}[\delta A_j, \bar{A}_k] - i[\bar{A}_j, \delta A_k])$

One monopole $\qquad = \varepsilon_{ijk} \bar{D}_j \delta A_k$

translational zero mode $\quad \vec{r} \Rightarrow \vec{r} - \vec{x} \qquad 3$

$g = e^{i\frac{\varphi}{a}\phi} \qquad 1$

Large gauge transformation.

$0 \le \varphi \le 2\pi$ on S^1

Gauge sym
$\begin{cases} g = e^{i\theta^a t^a} \\ g \longrightarrow I \end{cases}$

$\delta A_i = \frac{1}{a} D_i \phi$

$\delta\phi = -\frac{i}{a}[\phi, \phi] = 0$

$\cancel{D_i \Lambda}$
$\cancel{-i[\phi, \Lambda]}$
$\begin{cases} D_i \Lambda \\ -i[\phi, \Lambda] \end{cases}$

$\Rightarrow$ 4 bosonic coordinates

4 (real) fermionic coordinates ($N=2$ case)

(8 for $N=4$) $\qquad (A_{g(\varphi)}, \phi_{g(\varphi)})(\vec{x} - x(t))$

$$L_{eff} = \frac{4\pi a}{e}\left[\frac{1}{2}\left(\dot{\vec{x}}^2 + \frac{\dot{\varphi}^2}{(ae)^2}\right) + \sum_{A=1}^{2} i\bar{\chi}_A D_t \chi_A\right]$$

Half of original SUSY is preserved with this action
(4 real SUSY remains.)

$P_\varphi = n_e \in \mathbb{Z} \qquad$ electric charge $e n_e$

$= \frac{4\pi a}{e}\frac{\dot{\varphi}}{(ae)^2} = 4\pi\frac{\dot{\varphi}}{ae^3}$

$$H = \frac{4\pi a}{e} + \frac{1}{2}\frac{e^3 a}{4\pi} n^2$$

$$= \frac{4\pi a}{e}\left[1 + \frac{1}{2}\left(\frac{e^2}{4\pi}\right)^2 n^2\right]$$

Note

$$H = a\sqrt{\left(\frac{4\pi}{e}\right)^2 + (e n_e)^2}$$

$$= a\left(\frac{4\pi}{e}\right)\sqrt{1 + \left(\frac{e^2}{4\pi} n_e\right)^2}$$

$$= a\left(\frac{4\pi}{e}\right)\left(1 + \frac{1}{2}\left(\frac{e^2}{4\pi} n_e\right)^2 + O(e^4)\right)$$

This expansion is valid at weak coupling. $e \to 0$ limit,
or order of (P_x^2).

spin structure of monopoles

$$\{\chi_1, \chi_1^+\} = 1 \qquad \{\chi_2, \chi_2^+\} = 1$$

$$\begin{cases} 2 \text{ spin } 0 \text{ state} & \Rightarrow \text{2 spin 0 monopole} \\ 1 \text{ spin } \frac{1}{2} \text{ state} & \quad \text{1 spin } \frac{1}{2} \text{ fermionic monopole} \end{cases}$$

there are two scalars and one fermionic monopoles.
spin : $\frac{1}{2}$ BPS multiplet.

Lecture 2

$N=2$ theories by the dimensional reduction of 6 dimensional SYM theories.

$(F_{\mu\nu}, \lambda, D)$
$(X+iY, \psi, f)$

$$\mathcal{L} = \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu}^2 + |D(X+iY)|^2 - \tfrac{1}{2}[X,Y]^2 - i\lambda\sigma^\mu D_\mu\bar\lambda - i\bar\psi\bar\sigma^\mu D_\mu\psi - i[\lambda,\psi](X-iY) - i[\bar\lambda,\bar\psi](X+iY)\right]$$

D term potential

$$\mathcal{L} = \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(DX)^2 + \tfrac{1}{2}(DY)^2 - \tfrac{1}{2}[X,Y]^2 + i\bar\chi\gamma^\mu D_\mu\chi + i\bar\chi(-i)[-X+i\gamma_5 Y, \chi]\right]$$

$$\chi = \begin{bmatrix}\lambda_\alpha \\ \bar\psi^{\dot\alpha}\end{bmatrix} \qquad \Phi = (X+iY)(y) + \theta\psi(y) + \theta^2 F(y)$$

χSF

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu \\ \bar\sigma^\mu & 0\end{pmatrix} \qquad \sigma^\mu{}_{\alpha\dot\alpha} \quad \bar\sigma^{\mu\dot\alpha\alpha} \qquad y = x + i\theta\sigma\bar\theta$$

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}.$$

This may be obtained from the dimensional reduction of 6D nonabelian (1,0) SYM theories

$$\Gamma^\mu = \begin{pmatrix}0 & \gamma^\mu \\ \gamma^\mu & 0\end{pmatrix} \qquad \Gamma^5 = \begin{pmatrix}0 & i\gamma_5 \\ i\gamma_5 & 0\end{pmatrix} \qquad \Gamma^6 = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$$

Pseudo — Majorana Condition

$$C = \begin{bmatrix} 0 & \begin{matrix}\sigma_2 & 0 \\ 0 & -\sigma_2\end{matrix} \\ \begin{matrix}\sigma_2 & 0 \\ 0 & -\sigma_2\end{matrix} & 0\end{bmatrix}$$

$$\lambda_a = \begin{bmatrix}\chi \\ 0\end{bmatrix} \qquad a = 1, 2$$

$$\lambda_a = \varepsilon_{ab} C^{-1} (\bar\lambda^b)^t \qquad \text{pseudo real condition}$$

$*\ \bar\lambda = \lambda^\dagger i\Gamma^0$

$*\ C\Gamma^A C^{-1} = \Gamma^{tA}$

pseudo real $(1,0)$

$$\mathcal{L}_D^{Y-M} = \mathrm{tr}\left[-\tfrac{1}{4}F_{AB}F^{AB} + \tfrac{1}{2}\bar{\lambda}_a \not{D} \lambda_a\right]$$

SUSY

$AB = 0, 1, 2, 3, 5, 6$

$$\delta A_A = -\bar{\lambda}^a \Gamma_A \varepsilon_a$$

$$\delta \lambda_a = -\tfrac{1}{2} F_{AB}\Gamma^{AB}\varepsilon_a$$

ε_a: pseudo real spinor

Reduction to 4 dimensions

$$\partial_5 = \partial_6 = 0$$

$$F_{\mu 5} = \partial_\mu X - \partial_5 A_\mu - i[A_\mu, X]$$
$$= D_\mu X$$

$$F_{\mu 6} = D_\mu Y$$

$$F_{56} = -i[X, Y] \qquad D_5 = -i[X, \qquad D_6 = -i[Y,$$

$\rightarrow$

$$\mathcal{L} = \mathrm{tr}\left[-\tfrac{1}{4}F_{\mu\nu}^2 + (D_\mu X)^2 + (D_\mu Y)^2 + (-i[X,Y])^2 + i\bar{\chi}\gamma^\mu D_\mu \chi + i\bar{\chi}(-D_5 + i\gamma_5 D_6)\chi\right]$$

There are a few comments

1. Electric component of central charge arises as a KK momentum.

$$P_5 = Q_E = \int d\vec{x}\; F_{0i}F_{i5}$$

$$= \int d\vec{x}\; \mathrm{tr} E_i D_i X$$

2. Dyon solution may be constructed from the monopole solution by boosting in 5 direction & scaling.

$$\{\bar{Q}, Q\} = 2[\gamma_\mu P_\mu + (-Q_E + i\gamma_5 Q_M)]$$

BPS conditions

gaugino variation

1. Consider the cases where solutions are bosonic. $\Rightarrow$ Hence the bosonic parts do not vary under infinitesimal SUSY.
   (fields)
2. If the gaugino is inv, ~~then~~ ~~the solution~~ is invariant under the SUSY transformation.

$$\delta\chi = \left[\frac{\gamma^{\mu\nu}}{2}F_{\mu\nu} + \gamma^\mu D_\mu(-X + i\gamma_5 Y) - i\gamma_5 F_{56}\right]\epsilon$$

$\uparrow$ Dirac spinor 8 real

Consider the following projection operator

$$\Omega = -i\gamma_0\gamma_5$$

$$\Omega^\dagger = \Omega \qquad \Omega^2 = 1 \qquad\qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\Omega = -i\gamma_0\gamma_5$$
$$= -\gamma^{123}$$

$$\gamma^{ij}\Omega = \varepsilon^{ijk}\gamma^k$$

Now consider the case

$$P_-\epsilon = 0$$

$$\epsilon = P_+\epsilon = \Omega P_+\epsilon \qquad\qquad \frac{\gamma^{ij}}{2}\Omega = \frac{1}{2}\varepsilon^{ijk}\gamma^k$$

$$\delta\chi = \Big[\gamma^{0i}F_{0i} + \gamma^0 D_0(-X + i\gamma^5 Y) - i\gamma_5 F_{56} + D_i Y\, i\gamma^i\gamma^5 + \left(\tfrac{1}{2}\varepsilon_{ijk}F_{ij} - D_k X\right)\gamma^k\Big]P_+\epsilon$$

BPS equation preserving half SUSY

$$B_k = D_k X$$

$$D_0 X = D_0 Y = 0 \qquad F_{56} = 0 \qquad D_i Y,\ F_{0i} = 0$$

$P_+\epsilon$ is preserved and $P_-\epsilon$ is broken.

$(\gamma^0)^2 = 1$ dyonic case

$$\tilde\Omega = (\sin\alpha\,\gamma^0 + \cos\alpha\,\Omega)$$

$\tilde\Omega^2 = 1$ $\{\gamma^0; \Omega\} = 0$

$$\tilde P_+ = \tilde\Omega\,\tilde P_+$$

$$= (\gamma^0 \sin\alpha + \cos\alpha\,\Omega)\,\tilde P_+$$

$$\delta\chi = \Big[\gamma^\mu D_\mu Y\, i\gamma_5 - i\gamma_5 F_{56} + \gamma^0 D_0 X + \gamma^{0i}\,(F_{0i} - D_i X \sin\alpha) + (\gamma^k \cancel{B_k} - \gamma^k \cos\alpha\, D_k X)\,\Omega\Big]\tilde P_+ \epsilon$$

$= 0$

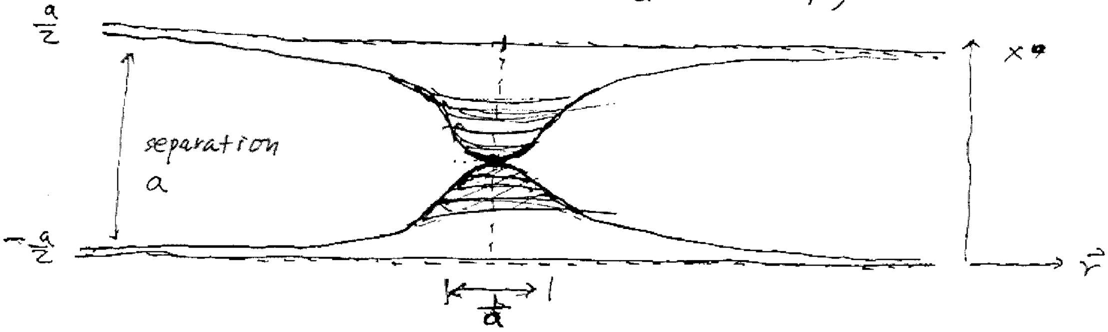
$$B = \cos\alpha\, D X$$
$$E = \sin\alpha\, D X$$

higher dimensional shape of the monopole

$$X = \hat r\cdot\frac{\vec\sigma}{2}\, a \left[\coth r - \frac{1}{r}\right]$$

Diagonalize

$$X = \begin{bmatrix} \frac{a}{2}(\coth r - \frac{1}{r}) & 0 \\ 0 & -\frac{a}{2}(\coth r - \frac{1}{r}) \end{bmatrix}$$



$$V_{WZ} = \theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D$$

$$W_\alpha = -\frac{1}{4}\bar D^2 e^{-2V} D_\alpha e^{2V}$$

$$\bar W_{\dot\alpha} = \frac{1}{4} D^2 e^{2V} \bar D_{\dot\alpha} e^{-2V}$$

$$W = -i\lambda + \theta D - \frac{i}{2}\sigma^\mu\bar\sigma^\nu\theta F_{\mu\nu} + \theta^2\sigma^\mu D_\mu\bar\lambda$$

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 f(y) \qquad y = x + i\theta\sigma\bar\theta$$

$$\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\,\tau\int d^2\theta\,\mathrm{Tr}\,W^2$$

$$= -\frac{1}{4}F^2 + \frac{\theta}{32\pi^2}F\tilde F + \frac{1}{2}D^2 - i\lambda\sigma^\mu D_\mu\bar\lambda$$

$N=2$ Lagrangian in $N=1$ formulation

$$\mathcal{L}_{YM} = \frac{1}{8\pi}\,\mathrm{Im}\,\tau\int d^2\theta\,\mathrm{Tr}\,W^2 + \int d^2\theta\, d\bar\theta\;\mathrm{Tr}\,\Phi^+ e^{2V}\Phi$$

XSF is in the adjoint representation

Full general $N=2$ Lagrangian without matter.

$$\mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\,\frac{1}{2}\int d^2\theta\;\mathcal{F}_{ab}(\Phi)\,W^aW^b + \int d^2\theta\, d\bar\theta\;(\Phi^+ e^{2V})^a\mathcal{F}_a(\Phi)$$

$$\Phi \longrightarrow e^{-2i\Lambda}\Phi$$

$$V \longrightarrow e^{-2i\Lambda^+}V e^{2i\Lambda}$$

$$W \longrightarrow e^{-2i\Lambda} W e^{2i\Lambda}$$

$N=2$ multiplet

$* \quad \bar D\Psi = \bar{\tilde D}\Psi = 0 \qquad$ chiral for $\bar D$ and $\tilde D$

$$\Psi = \Phi(\tilde y,\theta) + \sqrt{2}\,\tilde\theta^\alpha W_\alpha(\tilde y,\theta) + \tilde\theta^2 G(\tilde y,\theta)$$

$$\tilde y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\tilde\theta\sigma^\mu\bar{\tilde\theta}$$

$*$

$$D^2\Psi = \bar D^2\Psi^+ \quad , \quad D\tilde D\Psi = \bar D\bar{\tilde D}\Psi^+$$

$$\tilde D\tilde D\Psi = \bar{\tilde D}\bar{\tilde D}\Psi^+$$

$\Rightarrow$ $\quad \mathrm{Im}\ DW = 0 \qquad D^\alpha W_\alpha = \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha}$

$$G(\tilde{y},\theta) = \int d^2\bar\theta\ \Phi^+(\tilde{y} - i\theta\sigma\bar\theta, \bar\theta)\, e^{2V(\tilde{y}-i\theta\sigma\bar\theta, \theta, \bar\theta)}$$

for fixed $\tilde{y}$.

$$\Psi = \Phi + \tilde\theta\varphi_1 + \tilde\theta^2\varphi_2$$

$$\mathcal{F}_1(\Psi) = \mathcal{F}_1(\Phi) + \tilde\theta\varphi_1\mathcal{F}_1'(\Phi) \quad // \quad -\frac{\tilde\theta^2}{2}\tilde\varphi^2$$

$$+\tilde\theta^2\varphi_2\ \mathcal{F}_1'(\Phi) \quad \cancel{+}(\tilde\theta\varphi_1)^2\ \mathcal{F}_1''(\Phi)$$

$$\int d^2\tilde\theta\ \mathcal{F}_1(\Psi) = \int d^2\theta\ \varphi_2\mathcal{F}_1'(\Phi) + \int d^2\theta\ \frac{1}{2}\mathcal{F}_1''(\Phi)\, W^2.$$

$$\int d^4x\, \mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\int d^4x\left[\int d^2\theta\ \mathrm{tr}\frac{W^2}{2}\ \mathcal{F}_1'' + \int d^2\theta d^2\bar\theta\ \mathrm{tr}\ \Phi^+ e^{2V}\mathcal{F}_1'(\Phi)\right]$$

Proof of $SL(2,\mathbb{Z})$ of the $N=2$ theory

1. Consider $N=2$ $U(1)$ gauge theory starting from the $N=2$ $SU(2)$ theory by integrating out all the massive degrees (with nonvanishing vev)
2. ~~Inside~~ of singularities, our proof will be broken down.

$$S_{eff} = \frac{1}{4\pi}\int d^4x\, d^2\theta d^2\bar\theta\ \mathcal{F}_1(\Psi)\ \Big|_{\substack{\text{unbroken}\\ U(1)}}$$

Full quantum effective action.

$N=2$ $SU(2)$ gauge theory

$$\frac{1}{4\pi}\int d^4x \ \mathrm{Im}\left[\frac{1}{2}\int d^2\theta \ \mathrm{tr}\, \mathcal{F}''(\Phi)W^2 + \int d^2\theta d^2\bar\theta \ \mathrm{tr}\, \Phi^\dagger e^{2V}\mathcal{F}'(\Phi)\right]$$

$\mathcal{F}$: pre potential

Proof of $SL(2,\mathbb{Z})$ symmetry of $N=2$ $SU(2)$ gauge theor.

1. a Consider $N=2$ $SU(2)$ gauge theory in the broken phase.
   b Integrate out all the massive degrees.
   c Quantize the remaining $U(1)$ theory.
2. Due to the $N=2$ SUSY, the full quantum effective action is given by

$$S_{eff}^{U(1)} = \frac{1}{4\pi}\int d^4\theta d^2\tilde\theta \ \mathcal{F}_1(\Phi_{U(1)})$$

because the above is the fully general $N=2$ supersymmetric action.

3. In case of singularities, the $SL(2,\mathbb{Z})$ will be broken down in general.

singularity $\sim$ extra massless degrees

$$S_{eff} = \frac{1}{4\pi}\mathrm{Im}\int d^4x\left[\frac{1}{2}\int d^2\theta \ \mathcal{F}_1'' W^2 + \int d^2\theta d^2\bar\theta \ \Phi^\dagger \mathcal{F}_1'(\Phi)\right]$$

Before doing this proof, let us consider the following

$$\mathcal{L} = \frac{1}{32\pi}\mathrm{Im}\int \tau(a)\,(F+i\tilde F)^2$$

Use
$$\begin{cases} \tilde F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \\ \tilde{\tilde F} = -F \\ F^2 = \tilde F^2 \end{cases}$$

with the Bianchi identity $dF=0$

Introduce a Lagrange multiplier,

$$\frac{1}{8\pi}\int A_\mu^D \ \varepsilon^{\mu\nu\rho\lambda}\partial_\nu F_{\rho\sigma} = \frac{1}{8\pi}\int \tilde F_D F$$

$$\frac{1}{8\pi}\int \tilde{F}_D F = \frac{1}{16\pi}\,\mathrm{Im}\int (F_D + i\tilde{F}_D)(F + i\tilde{F})$$

Integrate $F + i\tilde{F}$

$$\mathcal{L}' = \frac{1}{32\pi}\,\mathrm{Im}\int \left(-\frac{1}{\tau}\right)(F_D + i\tilde{F}_D)^2$$

$$F_D + i\tilde{F}_D = -\tau(F + i\tilde{F}) \qquad \tau \to -\frac{1}{\tau}$$

$$\left[\begin{array}{l} F \to F_D \\ \tau \to -\frac{1}{\tau} \end{array}\right.$$

Action for the dual gauge field is obtained with

$$\tau_D = -\frac{1}{\tau}$$

We shall try the duality transformation for the effective $N=2$ Lagrangian.

Introduce the following Legendre transformation.

$$\phi_D \equiv \mathcal{F}'(\phi), \quad \mathcal{F}(\phi_D) = \mathcal{F}(\phi) - \phi\,\phi_D$$

$$\mathcal{F}_D'(\phi_D) = -\phi$$

$$\mathrm{Im}\, D^\alpha W_\alpha = 0 \qquad \text{Bianchi identity}$$

$$\mathrm{Im}\left[\int d^2\theta\, \frac{1}{2}\mathcal{F}''\, W^2 + \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\; V_D\, D^\alpha W_\alpha\right]$$

$$\frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\; V_D\, D^\alpha W_\alpha$$

$$= -\frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\,(D^\alpha V_D)\, W_\alpha$$

$$= \frac{1}{4}\int d^2\theta\,(\bar{D}^2 D^\alpha V_D)\, W_\alpha = -\int d^2\theta\; W_D W$$

$$\Rightarrow \quad \mathrm{Im} \int d^4x\, d^2\theta \left(-\frac{1}{2} \frac{1}{\mathcal{F}''} W_D^2 \right)$$

$$\mathcal{F}'' = \frac{d\phi_D}{d\phi} \qquad -\frac{1}{\mathcal{F}''} = -\frac{d\phi}{d\phi_D} = \mathcal{F}_D''(\phi_D)$$

$$\frac{1}{4} \mathrm{Im} \int d^4x\, d^2\theta \left(\left[\frac{d\phi_D}{d\phi}\right] W^2 + \left[\frac{d\phi}{d\phi_D}\right] W_D^2 \right]$$

$$+ \frac{1}{2i} \int d^4x\, d^2\theta\, d^2\bar{\theta} \left(\phi^\dagger \phi_D - \phi_D^\dagger \phi \right)$$

$$\begin{pmatrix} \phi_D' \\ \phi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \qquad \text{or} \quad \begin{matrix} \phi_D' = \phi \\ \phi' = -\phi_D \end{matrix}$$

$$\begin{pmatrix} W_D' \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} W_D \\ W \end{pmatrix}$$

Then the form of the action is clearly invariant under the transformation.

$$\begin{pmatrix} \phi_D' \\ \phi' \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} = \begin{pmatrix} \phi_D + n\phi \\ \phi \end{pmatrix}$$

$$\mathrm{Im}\, \phi^\dagger(\phi_D + n\phi) = \mathrm{Im}\, \phi^\dagger \phi_D$$

$$\frac{d\phi_D'}{d\phi'} = \frac{d\phi_D}{d\phi} + n$$

this corresponds to $\theta \to \theta + 2\pi n$, which is the symmetry because the action is exponentiated.

$$e^{2\pi n i} = 1 \qquad n \in \mathbb{Z}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ T generator $\qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ S generator

$$SL(2,\mathbb{Z}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, z \in \mathbb{Z}$$

$$ad - bc = 1$$

$$\tau \equiv \frac{d\Phi_D}{d\Phi}$$

$$\tau' = \frac{d\phi_D'}{d\phi'} = \frac{d(a\phi_D + b\phi)}{d(\cancel{e}c\phi_D + d\phi)} = \frac{a\tau + b}{c\tau + d}$$

The differential equation approach
to the solution of $(a_D(u), a(u))$

Some motivation

1. Consider the following second order differential equation

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\varphi(x) = 0$$

with $V(x+2\pi) = V(x)$.

Since there are only two independent solutions,

$$\begin{bmatrix}\varphi_1(x+2\pi)\\ \varphi_2(x+2\pi)\end{bmatrix} = M \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of the differential equation.

M is constant & Monodromy of the circle.

2. The similar situation arises for the holomorphic differential equation

$$\left[-\frac{d^2}{du^2} + V(u)\right]\varphi(u) = 0$$

$V(u)$: meromorphic potential.

$\cdot\, z_i$ singularity of the potential

$$\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix}_{z_i} = M_i \begin{pmatrix}\varphi_1(u)\\ \varphi_2(u)\end{pmatrix}$$

$\cdot\, z_i$

Logics

1. $a(u)$ & $a_D(u)$ are meromorphic function, whose singularity structures are given.

2. $a(u)$ & $a_D(u)$ will satisfy second order differential equations.

$$-\frac{d^2a}{du^2} + \frac{a''}{a}\, a = 0 \qquad V = \frac{a''}{a}$$

$$-\frac{d^2a_D}{du^2} + \frac{a_D''}{a_D}\, a_D = 0 \qquad V_D = \frac{a_D''}{a_D}$$

Now we shall argue

$$V(u) = V_D(u) = -\frac{1}{4}\left[\frac{1-\lambda_1^2}{(u+1)^2} + \frac{1-\lambda_2^2}{(u-1)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(u+1)(u-1)}\right]$$

double poles at -1, $+1$, ∞

& Terms of $\frac{1}{u\pm1}$ or $g(u)$ are absent.

entire ftn

3. $u \to \infty$

$$a_D \longrightarrow \frac{i}{\pi}\sqrt{2u}\left(\ln\frac{2u}{\Lambda^2} + 1\right)$$

$$a \longrightarrow \sqrt{2u}$$

$u \to 1$

$$a \longrightarrow a_0 + \frac{i}{\pi} c_0 (u-1)\ln(u-1)$$

$$a_D \longrightarrow c_0(u-1)$$

$\mathbb{Z}_2 \qquad u \leftrightarrow -u$

$$M_\infty = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad M_{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

4. $\frac{1}{u\pm 1}$, constant or any entire funtion $g(u)$ are not allowed into the potential.

To show this, consider for example

$$\frac{\alpha}{u-1}.$$

For $\quad z = \frac{1}{u} \quad , \quad \frac{d}{du} = -z^2\frac{d}{dz}$

Then the large u asymptotics are determined by

$$\left(-z^4\frac{d^2}{dz^2} + \alpha z\right)\psi = 0 \text{ or } \left(-\frac{d^2}{dz^2} + \frac{\alpha}{z^3}\right)\psi = 0$$

$$\psi \sim e^{\frac{\tilde{\alpha}}{\sqrt{z}}}$$

too singular

constant or any entire function

$\Rightarrow$ even more singular

5. $u \to \infty$

$$V(u) \sim -\frac{1}{4}\frac{1-\lambda_3^2}{u^2}$$

$$-\left(u^{\frac{1\pm\lambda_3}{2}}\right)'' = -u^{\frac{1\pm\lambda_3}{2}}\frac{1}{u^2}\frac{1\pm\lambda_3}{2}\frac{-1\pm\lambda_3}{2}$$

$$= +\frac{1-\lambda_3^2}{4}\frac{1}{u^2}\left(u^{\frac{1\pm\lambda_3}{2}}\right)$$

(compare this with the $u\to\infty$ behaviours of a and a_D

$$\lambda_3 = 0$$

$u \to 1$

$$V(u) \sim -\frac{1}{4}\frac{1-\lambda_2^2}{(u-1)^2}$$

$$\psi \sim (u-1)^{\frac{1\pm\lambda_2}{2}} \qquad \lambda_2 = 1$$

Use the $\mathbb{Z}_2$ symmetry under $u \leftrightarrow -u$

$$\lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 0$$

$$V(u) = V_D(u) = -\frac{1}{4}\frac{1}{(u-1)(u+1)}$$

$$\psi(u) = (u+1)^{\frac{1}{2}(1-\lambda_1)}(u-1)^{\frac{1}{2}(1-\lambda_2)} f\left(\frac{1+u}{2}\right)$$

$$x(1-x)f''(x) + [c-(a+b+1)x]f' - abf = 0$$

$$a = \frac{1}{2}(1-\lambda_1-\lambda_2+\lambda_3)$$

$$b = \frac{1}{2}(1-\lambda_1-\lambda_2-\lambda_3)$$

$$c = 1-\lambda_1 \qquad f = F(a,b,c;x)$$

$$a = -\frac{1}{2} \quad b = -\frac{1}{2} \quad c = 0$$

Two independent solutions

$$f_1(x) = (-x)^{-a} F\left(a, a+1-c, a+1-b; \frac{1}{x}\right)$$

$$f_2(x) = (1-x)^{c-a-b} F(c-a, c-b, c+1-a; 1-x)$$

$$a(u) = \frac{i(u-1)}{2} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right)$$

$$a_D(u) = \sqrt{2}\sqrt{u+1}\, F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1+u}\right)$$

- Use the integral representation

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dx\, x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{x^n}{n!}$$

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\,\frac{\sqrt{u-x}}{\sqrt{1-x^2}} = \frac{4}{\pi k}E(k) \qquad k^2 = \frac{2}{1+u}$$

$$a_D(u) = \frac{\sqrt{2}}{\pi}i\int_1^u dx\,\frac{\sqrt{u-x}}{\sqrt{x^2-1}} = \frac{4}{\pi i}\frac{E(k')-K(k')}{k} \qquad k'^2+k^2=1$$

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx$$

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}dx$$ complet elliptic function of the first kind.

- Monodromies can be checked explicitly.

Spectrum, curve of the marginal stability

1. We shall begin our discussion with the S-W Solution.

$$a_D(u) = \frac{i\sqrt{2}}{\pi}\int_1^u dx\,\frac{\sqrt{u-x}}{\sqrt{x^2-1}}$$

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx\,\frac{\sqrt{u-x}}{\sqrt{1-x^2}}$$

2. $$M = |n_e a - n_m a_D|$$

$$= \left|Z\left(\begin{pmatrix} n_e \\ n_m \end{pmatrix}, \begin{pmatrix} a_D \\ a \end{pmatrix}\right)\right|$$

$$M \in M^T = \in \qquad \in = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J$$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{matrix}$$

$$\begin{bmatrix} a_D' \\ a' \end{bmatrix} = M \begin{bmatrix} a_D \\ a \end{bmatrix} \qquad \begin{bmatrix} n_e' \\ n_m' \end{bmatrix} = M \begin{bmatrix} n_e \\ n_m \end{bmatrix}$$

$$[n_e' \; n_m'] \in \begin{bmatrix} a_D' \\ a' \end{bmatrix} = [n_e \, n_m] M^T \in M \begin{bmatrix} a_D \\ a \end{bmatrix} = [n_e \, n_m] \in \begin{bmatrix} a_D \\ a \end{bmatrix}$$

$$Z\left(M\begin{pmatrix} n_e \\ n_m \end{pmatrix}, M\begin{pmatrix} a_D \\ a \end{pmatrix}\right) = Z\left(\begin{pmatrix} n_e \\ n_m \end{pmatrix}, \begin{pmatrix} a_D \\ a \end{pmatrix}\right)$$

3. Curve of Marginal Stability C (MS)

stableness of the bound state

$$|n_e a - n_m a_D| \leq |n_e a| + |n_m a_D|$$

Equality may hold only if $a_D/a = r$ is real.

Example W to (1,-1) dyon and (0,1) monopole

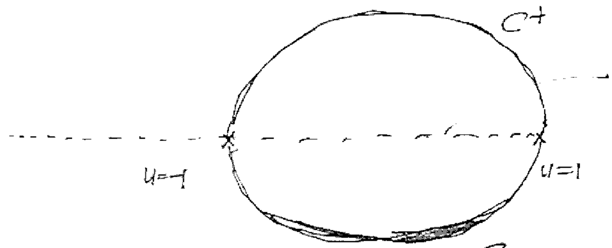
Assume $-1 < \frac{a_D}{a} = r < 0$.

$$|a+a_D| + |a_D|$$
$$= |a| \left(|1+r| + |r| \right) = |a|$$

$$(1,0) \Rightarrow (1,-1) + (0,1)$$

A W particle may decay into a (1,-1) dyon and a monopole and vice versa.

Shape of the curve $a_D(1)=0$, $a_D(1) = \frac{4}{\pi}$



curve for a and a_D

$$a_D(-1) = \pm \frac{4i}{\pi}$$
$$a(-1) = \pm \frac{4i}{\pi}$$
$$\frac{a_D(-1)}{a(-1)} = \pm 1$$

C^+ $r \in (0,1)$

C^- $r \in (-1,0)$

Large u $\tau = \frac{da_D}{da} = \frac{i}{\pi}\left(\ln \frac{a^2}{\Lambda^2} + 3\right)$

$$a_D \to \frac{i}{\pi} a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right)$$

$$a \to \sqrt{2u}$$

Weakly coupled region

P1 Massless states may only occur on the curve $C_\pm$

$$Z = n_e a - n_m a_D = 0$$

$$\frac{a_D}{a} = \frac{n_e}{n_m} = \text{real} = r$$

$u=1$ Monopole become massless

$r=0 \Rightarrow n_e = 0$

$u=-1$ The dyon $(\pm 1, 1)$ become massless

$r = \pm 1 \Rightarrow n_e = \pm n_m$

P2 When some state becomes massless somewhere, singularities of $u(a)$ description arises.

Thus ... massless state may only occur at $u=\pm 1$.

Weak coupling spectra (outside of CMS)

1. $(0,1)$ $(\pm 1, 1)$ exist. $(1,0)$

2. $|n_m| > |n_e|$ cannot exist.

On the CMS with $\frac{a_D}{a} = \frac{n_e}{n_m}$, the state become massless, which is in contradiction with P2.

3. $M_\infty^n \begin{pmatrix} n_e \\ n_m \end{pmatrix}$ should exist. $M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

$M_\infty^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $M_\infty^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ generate $\begin{bmatrix} n \\ 1 \end{bmatrix}$

$M_\infty^n \begin{bmatrix} \cancel{k} \\ m \end{bmatrix}$ with $|m| \geq 2 \rightarrow \left|\frac{n_e'}{n_m'}\right| < 1$ contradiction!

$$S_W = \{ \pm(1,0) , \pm(n,1) \quad n \in \mathbb{Z} \}$$

$Z_2 \qquad u \longleftrightarrow -u$

$$M\left(\begin{bmatrix}\tilde{n}_e\\ \tilde{n}_m\end{bmatrix}, \begin{bmatrix}a_D(-u)\\ a(-u)\end{bmatrix}\right) = M\left(\begin{bmatrix}n_e\\ m_n\end{bmatrix}, \begin{bmatrix}a_D(u)\\ a(u)\end{bmatrix}\right)$$

$$\begin{pmatrix}a_D(-u)\\ a(-u)\end{pmatrix} = e^{-\frac{i\pi\epsilon}{2}}\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{bmatrix}a_D(u)\\ a(u)\end{bmatrix}$$

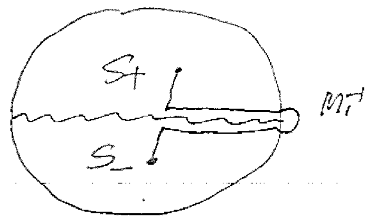
$\epsilon = \pm 1 \qquad$ UHP / LHP

$$\Rightarrow \begin{bmatrix}\tilde{n}_e\\ \tilde{n}_m\end{bmatrix} = \begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}n_e\\ n_m\end{pmatrix}$$

$$\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}n\\ 1\end{pmatrix} = \begin{pmatrix}n+\epsilon\\ 1\end{pmatrix} \qquad \begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$M_\infty S_W = S_W \qquad G\, S_W = S_W$$

Strongly coupled region



$S_+, \; S_-$

$M_1^{-1} S_+ = S_-$

$M_1^{-1} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$

$M_1 = \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$

$$M_1 G_+ = \begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix} \qquad (M_1 G_+)^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$\begin{bmatrix}n_e\\ n_m\end{bmatrix} \longrightarrow \begin{bmatrix}-n_e\\ -n_m\end{bmatrix} \qquad \frac{n_e}{n_m} \in r \in [0,-1]$$

Using PZ,

$$S_{S+} = \{\pm(0,1), \pm(-1,1)\}$$

$$S_{S-} = \{\pm(0,1), \pm(1,1)\}$$

$$\pm\begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}n_e\\ +n_m\end{bmatrix} = \pm\begin{bmatrix}n_e+m_n\\ -2n_e-n_m\end{bmatrix}$$

For instance $\quad (1,0) \longrightarrow (1,-1) + (0,1) \qquad r' = -\left[\frac{1+r}{1+2r}\right] \in [-1,0]$

W boson decay into the two when crossing COM.

$Z_2 \quad u \longleftrightarrow -u$

$$M\left(\begin{bmatrix}\tilde{n}_e\\ \tilde{n}_m\end{bmatrix}, \begin{bmatrix}a_D(-u)\\ a(-u)\end{bmatrix}\right) = M\left(\begin{bmatrix}n_e\\ m_n\end{bmatrix}, \begin{bmatrix}a_D(u)\\ a(u)\end{bmatrix}\right)$$

$$\begin{pmatrix}a_D(-u)\\ a(-u)\end{pmatrix} = e^{-\frac{i\pi\epsilon}{2}}\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{bmatrix}a_D(u)\\ a(u)\end{bmatrix}$$

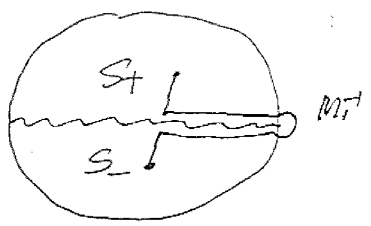
$\epsilon = \pm 1 \quad$ UHP / LHP

$$\Rightarrow \begin{bmatrix}\tilde{n}_e\\ \tilde{n}_m\end{bmatrix} = \begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}n_e\\ n_m\end{pmatrix}$$

$$\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}n\\ 1\end{pmatrix} = \begin{pmatrix}n+\epsilon\\ 1\end{pmatrix} \qquad \begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$M_\infty S_W = S_W \qquad G\, S_W = S_W$$

Strongly coupled region



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Using PZ,

$$S_{S_+} = \{\pm(0,1), \pm(-1,1)\}$$

$$S_{S_-} = \{\pm(0,1), \pm(1,1)\}$$

$$\pm\begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}n_e\\ +n_m\end{bmatrix} = \pm\begin{bmatrix}n_e+m_m\\ -2n_e-n_m\end{bmatrix}$$

$$r' = -\left[\frac{1+r}{1+2r}\right] \in [-1, 0]$$

For instance $\quad (1,0) \longrightarrow (1,-1) + (0,1)$

W boson decay into the two when crossing COM.