Zero Problems in Theory and Applications

Chee Yap
KIAS, and
Courant Institute of Mathematical Sciences
Department of Computer Science
New York University

TALK OVERVIEW

- Is There a Problem with Zero?
- Why Care About Nothing (Zero)?
- On Zero Bounds and Adaptivity
- On Foundations of Real Computation
- Conclusions

PART I.
Is There a Problem with Zero?

"The history of the zero recognition problem is somewhat confused by the fact that many people do not recognize it as a problem at all."

— Daniel Richardson (1996)

What Appears to be the Problem?

- Is a number equal to zero?
    - Decision Problem – YES/NO answers

- Why is any effort needed at all?
    - Numbers have canonical names
    - E.g., zero, one, two, half, negative ten, square-root two, pi, etc
    - In symbols, $0,\ 1,\ 2,\ \frac{1}{2},\ -10,\ \sqrt{2},\ \pi,\ \ldots$

- Numerical Expressions (non-canonical!)
    - $1 - 1 + 1 - 1,$
    - $2^2 + 5 - 3^2,$
    - $1 - \sum_{n=1}^{\infty} 2^{-n},$
    - $\sqrt{2} + \sqrt{3} - \sqrt{5 + 2\sqrt{6}}$
    - These are all 0

What Appears to be the Problem?

- Is a number equal to zero?
  - Decision Problem – YES/NO answers
- Why is any effort needed at all?
  - Numbers have canonical names
  - E.g., zero, one, two, half, negative ten, square-root two, pi, etc
  - In symbols, 0, 1, 2, $\frac{1}{2}$, -10, $\sqrt{2}$, π, $\ldots$
- Numerical Expressions (non-canonical!)
  - $1 - 1 + 1 - 1$,
  - $2^2 + 5 - 3^2$,
  - $1 - \sum_{n=1}^{\infty} 2^{-n}$,
  - $\sqrt{2} + \sqrt{3} - \sqrt{5 + 2\sqrt{6}}$
  - These are all 0

What Appears to be the Problem?

- Is a number equal to zero?
  - Decision Problem – YES/NO answers
- Why is any effort needed at all?
  - Numbers have canonical names
  - E.g., zero, one, two, half, negative ten, square-root two, pi, etc
  - In symbols, 0, 1, 2, $\frac{1}{2}$, -10, $\sqrt{2}$, π, $\ldots$
- Numerical Expressions (non-canonical!)
  - $1-1+1-1$,
  - 2^2+5-3^2,
  - $1-\sum_{n=1}^{\infty} 2^{-n}$,
  - $\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}}$
  - These are all 0

What Appears to be the Problem?

- Is a number equal to zero?
  - Decision Problem – YES/NO answers

- Why is any effort needed at all?
  - Numbers have canonical names
  - E.g., zero, one, two, half, negative ten, square-root two, pi, etc
  - In symbols, 0, 1, 2, $\frac{1}{2}$, -10, $\sqrt{2}$, π, $\ldots$

- Numerical Expressions (non-canonical!)
  - $1 - 1 + 1 - 1$,
  - $2^2 + 5 - 3^2$,
  - $1 - \sum_{n=1}^{\infty} 2^{-n}$,
  - $\sqrt{2} + \sqrt{3} - \sqrt{5 + 2\sqrt{6}}$
  - These are all 0

But is it REALLY zero?

- Two ways to decide zero:
  - (A) Algebraically. E.g., repeated squaring
  - (B) Numerically. E.g., by approximation

- We are interested in (B):
  -

$$\begin{aligned}
\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}} &= 1.4142+1.7320-\sqrt{5+2\times 2.4494} \\
&= 3.1462-\sqrt{9.8989} \\
&= 3.1462-3.1462 \\
&= 0 \quad ??
\end{aligned}$$

But is it REALLY zero?

- Two ways to decide zero:
  - ∗ (A) Algebraically. E.g., repeated squaring
  - ∗ (B) Numerically. E.g., by approximation

- We are interested in (B):
  - ∗

$$\begin{aligned}
\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}} &= 1.4142+1.7320-\sqrt{5+2\times 2.4494} \\
&= 3.1462-\sqrt{9.8989} \\
&= 3.1462-3.1462 \\
&= 0 \quad ??
\end{aligned}$$

But is it REALLY zero?

- Two ways to decide zero:
  - (A) Algebraically. E.g., repeated squaring
  - (B) Numerically. E.g., by approximation

- We are interested in (B):
  -

$$
\begin{aligned}
\sqrt{2}+\sqrt{3}-\sqrt{5+2\sqrt{6}} &= 1.4142+1.7320-\sqrt{5+2\times 2.4494} \\
&= 3.1462-\sqrt{9.8989} \\
&= 3.1462-3.1462 \\
&= 0 \quad ??
\end{aligned}
$$

Apparent Zeros

- Folklore:
  - $\left(3 \cdot \log(640320)/\sqrt{163}\right) - \pi \; < 10^{-15}$

- R.Graham:
  - $$\begin{aligned} &\left(\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087}\right. \\ &\qquad \left. + \sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198}\right) \\ - \quad &\left(\sqrt{1000002} + \sqrt{1000018} + \sqrt{1000042} + \sqrt{1000066} + \sqrt{1000113}\right. \\ &\qquad \left. + \sqrt{1000116} + \sqrt{1000169} + \sqrt{1000175} + \sqrt{1000199}\right) \\ < \quad & 10^{-36} \end{aligned}$$

- Many more...

- Richardson (2005)
  - Let $F(x) = (1+x)^{1/2} - 2(1+3x/4)^{1/3} + 1$
  - Then $F(F(10^{-126})) \; < 10^{-1141}$

Apparent Zeros

- Folklore:
  - $\left(3 \cdot \log(640320)/\sqrt{163}\right) - \pi \; < 10^{-15}$

- R.Graham:
  - $$\begin{aligned} &\left(\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087}\right. \\ &\quad \left. + \sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198}\right) \\ - &\quad \left(\sqrt{1000002} + \sqrt{1000018} + \sqrt{1000042} + \sqrt{1000066} + \sqrt{1000113}\right. \\ &\quad \left. + \sqrt{1000116} + \sqrt{1000169} + \sqrt{1000175} + \sqrt{1000199}\right) \\ < &\quad 10^{-36} \end{aligned}$$

- Many more...

- Richardson (2005)
  - Let $F(x) = (1+x)^{1/2} - 2(1+3x/4)^{1/3} + 1$
  - Then $F(F(10^{-126})) \; < 10^{-1141}$

Apparent Zeros

- Folklore:
  - $\left(3 \cdot \log(640320)/\sqrt{163}\right) - \pi \;< 10^{-15}$
- R.Graham:
  - $$\begin{aligned} &\left(\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087}\right. \\ &\quad \left. + \sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198}\right) \\ - \quad &\left(\sqrt{1000002} + \sqrt{1000018} + \sqrt{1000042} + \sqrt{1000066} + \sqrt{1000113}\right. \\ &\quad \left. + \sqrt{1000116} + \sqrt{1000169} + \sqrt{1000175} + \sqrt{1000199}\right) \\ < \quad & 10^{-36} \end{aligned}$$
- Many more...
- Richardson (2005)
  - Let $F(x) = (1+x)^{1/2} - 2(1+3x/4)^{1/3} + 1$
  - Then $F(F(10^{-126})) \;< 10^{-1141}$

Apparent Zeros

- Folklore:
  - $\left(3 \cdot \log(640320)/\sqrt{163}\right) - \pi \;< 10^{-15}$

- R.Graham:
  - $$\begin{aligned} & \left(\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087}\right. \\ & \quad \left. + \sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198}\right) \\ - & \quad \left(\sqrt{1000002} + \sqrt{1000018} + \sqrt{1000042} + \sqrt{1000066} + \sqrt{1000113}\right. \\ & \quad \left. + \sqrt{1000116} + \sqrt{1000169} + \sqrt{1000175} + \sqrt{1000199}\right) \\ < & \quad 10^{-36} \end{aligned}$$

- Many more...

- Richardson (2005)
  - Let $F(x) = (1+x)^{1/2} - 2(1+3x/4)^{1/3} + 1$
  - Then $F(F(10^{-126})) \;< 10^{-1141}$

Apparent Zeros

- Folklore:
  - $\left(3 \cdot \log(640320)/\sqrt{163}\right) - \pi \ < 10^{-15}$
- R.Graham:
  - $$\begin{aligned} &\left(\sqrt{1000001} + \sqrt{1000025} + \sqrt{1000031} + \sqrt{1000084} + \sqrt{1000087}\right. \\ &\quad \left. + \sqrt{1000134} + \sqrt{1000158} + \sqrt{1000182} + \sqrt{1000198}\right) \\ - \quad &\left(\sqrt{1000002} + \sqrt{1000018} + \sqrt{1000042} + \sqrt{1000066} + \sqrt{1000113}\right. \\ &\quad \left. + \sqrt{1000116} + \sqrt{1000169} + \sqrt{1000175} + \sqrt{1000199}\right) \\ < \quad & 10^{-36} \end{aligned}$$
- Many more...
- Richardson (2005)
  - Let $F(x) = (1+x)^{1/2} - 2(1+3x/4)^{1/3} + 1$
  - Then $F(F(10^{-126})) \ < 10^{-1141}$

The Zero Problems

- $\Omega =$ set of algebraic operators
  - E.g., $\Omega = \{\pm, \times, \div\} \cup \mathbb{Z}$
- Let $\mathrm{Expr}(\Omega)$ denote expressions over Ω
- Evaluation function:
  $\mathrm{Val} : \mathrm{Expr}(\Omega) \dashrightarrow \mathbb{C}$
  - Say e is invalid if $\mathrm{Val}(e) = \uparrow$
- Zero problem, $ZERO(\Omega)$:
  - Given $e \in \mathrm{Expr}(\Omega)$, is $\mathrm{Val}(e) = 0$?

The Zero Problems

- $\Omega =$ set of algebraic operators
  - ∗ E.g., $\Omega = \{\pm, \times, \div\} \cup \mathbb{Z}$
- Let $\mathrm{Expr}(\Omega)$ denote expressions over Ω
- Evaluation function:
  $\mathrm{Val} : \mathrm{Expr}(\Omega) \dashrightarrow \mathbb{C}$
  - ∗ Say e is invalid if $\mathrm{Val}(e) = \uparrow$
- Zero problem, $ZERO(\Omega)$:
  - ∗ Given $e \in \mathrm{Expr}(\Omega)$, is $\mathrm{Val}(e) = 0$?

The Zero Problems

- Ω = set of algebraic operators
  - E.g., $\Omega = \{\pm, \times, \div\} \cup \mathbb{Z}$
- Let $\mathrm{Expr}(\Omega)$ denote expressions over Ω
- Evaluation function:
  $\mathrm{Val} : \mathrm{Expr}(\Omega) \dashrightarrow \mathbb{C}$
  - Say e is invalid if $\mathrm{Val}(e) = \uparrow$
- Zero problem, $ZERO(\Omega)$:
  - Given $e \in \mathrm{Expr}(\Omega)$, is $\mathrm{Val}(e) = 0$?

The Zero Problems

- Ω = set of algebraic operators
  - E.g., $\Omega = \{\pm, \times, \div\} \cup \mathbb{Z}$
- Let $\mathrm{Expr}(\Omega)$ denote expressions over Ω
- Evaluation function:
  $\mathrm{Val} : \mathrm{Expr}(\Omega) \dashrightarrow \mathbb{C}$
  - Say e is invalid if $\mathrm{Val}(e) = \uparrow$
- Zero problem, $ZERO(\Omega)$:
  - Given $e \in \mathrm{Expr}(\Omega)$, is $\mathrm{Val}(e) = 0$?

The Zero Problems

- $\Omega =$ set of algebraic operators
  - E.g., $\Omega = \{\pm, \times, \div\} \cup \mathbb{Z}$
- Let $\mathrm{Expr}(\Omega)$ denote expressions over Ω
- Evaluation function:
  $\mathrm{Val} : \mathrm{Expr}(\Omega) \dashrightarrow \mathbb{C}$
  - Say e is invalid if $\mathrm{Val}(e) = \uparrow$
- Zero problem, $ZERO(\Omega)$:
  - Given $e \in \mathrm{Expr}(\Omega)$, is $\mathrm{Val}(e) = 0$?

Error Notation

- NOTATION: $\widetilde{x} = x \pm h$
  - means $|\widetilde{x} - x| \leq h$
- Absolute Error
  - $\widetilde{x}$ is a absolute p-bit approximation of x if $\widetilde{x} = x \pm 2^{-p}$
- Relative Error
  - $\widetilde{x}$ is a relative p-bit approximation of x if $\widetilde{x} = x(1 \pm 2^{-p}|x|)$
- Precision vs. Error
  - (1) "precision" is a priori error
  - (2) "error" is a posteriori error

Error Notation

- NOTATION: $\widetilde{x} = x \pm h$
  - means $|\widetilde{x} - x| \leq h$
- Absolute Error
  - $\widetilde{x}$ is a absolute p-bit approximation of x if $\widetilde{x} = x \pm 2^{-p}$
- Relative Error
  - $\widetilde{x}$ is a relative p-bit approximation of x if $\widetilde{x} = x(1 \pm 2^{-p}|x|)$
- Precision vs. Error
  - (1) "precision" is a priori error
  - (2) "error" is a posteriori error

Error Notation

- NOTATION: $\widetilde{x} = x \pm h$
  - means $|\widetilde{x} - x| \le h$
- Absolute Error
  - $\widetilde{x}$ is a absolute p-bit approximation of x if $\widetilde{x} = x \pm 2^{-p}$
- Relative Error
  - $\widetilde{x}$ is a relative p-bit approximation of x if $\widetilde{x} = x(1 \pm 2^{-p}|x|)$
- Precision vs. Error
  - (1) "precision" is a priori error
  - (2) "error" is a posteriori error

Error Notation

- NOTATION: $\widetilde{x} = x \pm h$
  - means $|\widetilde{x} - x| \leq h$
- Absolute Error
  - $\widetilde{x}$ is a absolute p-bit approximation of x if $\widetilde{x} = x \pm 2^{-p}$
- Relative Error
  - $\widetilde{x}$ is a relative p-bit approximation of x if $\widetilde{x} = x(1 \pm 2^{-p}|x|)$
- Precision vs. Error
  - (1) "precision" is a priori error
  - (2) "error" is a posteriori error

Error Notation

- NOTATION: $\widetilde{x} = x \pm h$
  - means $|\widetilde{x} - x| \leq h$
- Absolute Error
  - $\widetilde{x}$ is a absolute p-bit approximation of x if $\widetilde{x} = x \pm 2^{-p}$
- Relative Error
  - $\widetilde{x}$ is a relative p-bit approximation of x if $\widetilde{x} = x(1 \pm 2^{-p}|x|)$
- Precision vs. Error
  - (1) "precision" is a priori error
  - (2) "error" is a posteriori error

Partial Functions

- Partial function, $f : S \dashrightarrow T$
  - $*$ Nominal domain: S
  - $*$ Proper domain: $\mathrm{dom}(f) = \{w \in S : f(w) =\downarrow\}$
  - $*$ If $S = \mathrm{dom}(f)$, then f is total, written $f : S \to \Sigma^*$

- Two cases of interest:
  - $*$ (Discrete computation) Turing Machines for computing $f : S \subseteq \Sigma^* \dashrightarrow \Sigma^*$
  - $*$ (Continuous computation) Real functions $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$

- Why have both S <u>and</u> partial f?
  - $*$ In analysis: we often choose S to be nice
  - $*$ In algebra: we have no choice about S, and f may be partial (E.g., $\div$)

Partial Functions

- Partial function, $f : S \dashrightarrow T$
  - $*$ Nominal domain: S
  - $*$ Proper domain: $\mathrm{dom}(f) = \{w \in S : f(w) = \downarrow\}$
  - $*$ If $S = \mathrm{dom}(f)$, then f is total, written $f : S \to \Sigma^*$

- Two cases of interest:
  - $*$ (Discrete computation) Turing Machines for computing $f : S \subseteq \Sigma^* \dashrightarrow \Sigma^*$
  - $*$ (Continuous computation) Real functions $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$

- Why have both S <u>and</u> partial f?
  - $*$ In analysis: we often choose S to be nice
  - $*$ In algebra: we have no choice about S, and f may be partial (E.g., $\div$)

Partial Functions

- Partial function, $f : S \dashrightarrow T$
  - Nominal domain: S
  - Proper domain: $\mathrm{dom}(f) = \{w \in S : f(w) = \downarrow\}$
  - If $S = \mathrm{dom}(f)$, then f is total, written $f : S \to \Sigma^*$
- Two cases of interest:
  - (Discrete computation) Turing Machines for computing $f : S \subseteq \Sigma^* \dashrightarrow \Sigma^*$
  - (Continuous computation) Real functions $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$
- Why have both S <u>and</u> partial f?
  - In analysis: we often choose S to be nice
  - In algebra: we have no choice about S, and f may be partial (E.g., $\div$)

Partial Functions

- Partial function, $f : S \dashrightarrow T$
  - Nominal domain: S
  - Proper domain: $\mathrm{dom}(f) = \{w \in S : f(w) = \downarrow\}$
  - If $S = \mathrm{dom}(f)$, then f is total, written $f : S \to \Sigma^*$

- Two cases of interest:
  - (Discrete computation) Turing Machines for computing $f : S \subseteq \Sigma^* \dashrightarrow \Sigma^*$
  - (Continuous computation) Real functions $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$

- Why have both S and partial f?
  - In analysis: we often choose S to be nice
  - In algebra: we have no choice about S, and f may be partial (E.g., $\div$)

Classic (Discrete) Computability Theory

- f is partial recursive if there is a Turing machine M :
  - (1) For all $w \in \mathrm{dom}(f)$, M halts with outputs $f(w)$
  - (2) For all $w \notin \mathrm{dom}(f)$, M does not halt

- f is recursive if ...
  - (1) For all $w \in \mathrm{dom}(f)$, ...
  - (2) For all $w \notin \mathrm{dom}(f)$, M halts in a state $q_\uparrow$

- E.g., $\div : \mathbb{Q} \times \mathbb{Q} \dashrightarrow \mathbb{Q}$ a recursive partial (!) function

- $Rec =$ Recursive functions;
  $Prec =$ Partial Recursive functions

Classic (Discrete) Computability Theory

- f is partial recursive if there is a Turing machine M :
  - (1) For all $w \in \mathrm{dom}(f)$, M halts with outputs $f(w)$
  - (2) For all $w \notin \mathrm{dom}(f)$, M does not halt

- f is recursive if ...
  - (1) For all $w \in \mathrm{dom}(f)$, ...
  - (2) For all $w \notin \mathrm{dom}(f)$, M halts in a state $q_{\uparrow}$

- E.g., $\div : \mathbb{Q} \times \mathbb{Q} \dashrightarrow \mathbb{Q}$ a recursive partial (!) function

- Rec = Recursive functions;
  $Prec$ = Partial Recursive functions

Classic (Discrete) Computability Theory

- f is partial recursive if there is a Turing machine M :
  - (1) For all $w \in \mathrm{dom}(f)$, M halts with outputs $f(w)$
  - (2) For all $w \notin \mathrm{dom}(f)$, M does not halt

- f is recursive if ...
  - (1) For all $w \in \mathrm{dom}(f)$, ...
  - (2) For all $w \notin \mathrm{dom}(f)$, M halts in a state $q_{\uparrow}$

- E.g., $\div : \mathbb{Q} \times \mathbb{Q} \dashrightarrow \mathbb{Q}$ a recursive partial (!) function

- $Rec =$ Recursive functions;
  $Prec =$ Partial Recursive functions

Classic (Discrete) Computability Theory

- f is partial recursive if there is a Turing machine M :
  - (1) For all $w \in \mathrm{dom}(f)$, M halts with outputs $f(w)$
  - (2) For all $w \notin \mathrm{dom}(f)$, M does not halt

- f is recursive if ...
  - (1) For all $w \in \mathrm{dom}(f)$, ...
  - (2) For all $w \notin \mathrm{dom}(f)$, M halts in a state $q_{\uparrow}$

- E.g., $\div : \mathbb{Q} \times \mathbb{Q} \dashrightarrow \mathbb{Q}$ a recursive partial (!) function

- Rec = Recursive functions;
  $Prec$ = Partial Recursive functions

Classic (Discrete) Computability Theory

- f is partial recursive if there is a Turing machine M :
  - (1) For all $w \in \mathrm{dom}(f)$, M halts with outputs $f(w)$
  - (2) For all $w \notin \mathrm{dom}(f)$, M does not halt

- f is recursive if ...
  - (1) For all $w \in \mathrm{dom}(f)$, ...
  - (2) For all $w \notin \mathrm{dom}(f)$, M halts in a state $q_\uparrow$

- E.g., $\div : \mathbb{Q} \times \mathbb{Q} \dashrightarrow \mathbb{Q}$ a recursive partial (!) function

- Rec = Recursive functions;
  $Prec$ = Partial Recursive functions

Classic (Discrete) Computability Theory

- $Rec \subseteq Prec$
  - There exists a function $HALT$ in $Prec \setminus Rec$
  - Every $f \in Prec$ is reducible to $HALT$

- What happens to the Halting Problem in the continuous domain?

Classic (Discrete) Computability Theory

- $Rec \subseteq Prec$
  - $*$ There exists a function $HALT$ in $Prec \setminus Rec$
  - $*$ Every $f \in Prec$ is reducible to $HALT$

- What happens to the Halting Problem in the continuous domain?

Classic (Discrete) Computability Theory

- $Rec \subseteq Prec$
  - There exists a function $HALT$ in $Prec \setminus Rec$
  - Every $f \in Prec$ is reducible to $HALT$

- What happens to the Halting Problem in the continuous domain?

Real Approximations

- A set $\mathbb{F}$ of base reals is any subset $\mathbb{F} \subseteq \mathbb{R}$ such that
  - $*$ (1) $\mathbb{F}$ is a ring extension of $\mathbb{Z}$
  - $*$ (2) $\mathbb{F}$ is countably dense in $\mathbb{R}$
  - $*$ (3) Ring operations, $x \mapsto x/2$ and comparisons of $\mathbb{F}$ are efficient
- E.g., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{n2^m : n, m \in \mathbb{Z}\}$
- Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and $\widetilde{f} : \mathbb{F} \times \mathbb{Z} \dashrightarrow \mathbb{F}$
- $\widetilde{f}$ is an $\mathcal{A}$-approximation of f if:
  for all $x \in \mathbb{F}$, $\widetilde{f}(x; p) \equiv f(x) \pm 2^{-p}$.
  - $*$ Similarly for $\mathcal{R}$-approximable.

Real Approximations

- A set $\mathbb{F}$ of base reals is any subset $\mathbb{F} \subseteq \mathbb{R}$ such that
  - $*$ (1) $\mathbb{F}$ is a ring extension of $\mathbb{Z}$
  - $*$ (2) $\mathbb{F}$ is countably dense in $\mathbb{R}$
  - $*$ (3) Ring operations, $x \mapsto x/2$ and comparisons of $\mathbb{F}$ are efficient

- E.g., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{n2^m : n, m \in \mathbb{Z}\}$

- Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and $\widetilde{f} : \mathbb{F} \times \mathbb{Z} \dashrightarrow \mathbb{F}$

- $\widetilde{f}$ is an $\mathcal{A}$-approximation of f if:
  for all $x \in \mathbb{F}$, $\widetilde{f}(x;p) \equiv f(x) \pm 2^{-p}$.
  - $*$ Similarly for $\mathcal{R}$-approximable.

Real Approximations

- A set $\mathbb{F}$ of base reals is any subset $\mathbb{F} \subseteq \mathbb{R}$ such that
  - ∗ (1) $\mathbb{F}$ is a ring extension of $\mathbb{Z}$
  - ∗ (2) $\mathbb{F}$ is countably dense in $\mathbb{R}$
  - ∗ (3) Ring operations, $x \mapsto x/2$ and comparisons of $\mathbb{F}$ are efficient
- E.g., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{n2^m : n, m \in \mathbb{Z}\}$
- Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and $\widetilde{f} : \mathbb{F} \times \mathbb{Z} \dashrightarrow \mathbb{F}$
- $\widetilde{f}$ is an $\mathcal{A}$-approximation of f if:
  for all $x \in \mathbb{F}$, $\widetilde{f}(x; p) \equiv f(x) \pm 2^{-p}$.
  - ∗ Similarly for $\mathcal{R}$-approximable.

Real Approximations

- A set $\mathbb{F}$ of base reals is any subset $\mathbb{F} \subseteq \mathbb{R}$ such that
  - (1) $\mathbb{F}$ is a ring extension of $\mathbb{Z}$
  - (2) $\mathbb{F}$ is countably dense in $\mathbb{R}$
  - (3) Ring operations, $x \mapsto x/2$ and comparisons of $\mathbb{F}$ are efficient
- E.g., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{n2^m : n, m \in \mathbb{Z}\}$
- Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and $\widetilde{f} : \mathbb{F} \times \mathbb{Z} \dashrightarrow \mathbb{F}$
- $\widetilde{f}$ is an $\mathcal{A}$-approximation of f if:
  for all $x \in \mathbb{F}$, $\widetilde{f}(x; p) \equiv f(x) \pm 2^{-p}$.
  - Similarly for $\mathcal{R}$-approximable.

Real Approximations

- A set $\mathbb{F}$ of *base reals* is any subset $\mathbb{F} \subseteq \mathbb{R}$ such that
  - (1) $\mathbb{F}$ is a ring extension of $\mathbb{Z}$
  - (2) $\mathbb{F}$ is countably dense in $\mathbb{R}$
  - (3) Ring operations, $x \mapsto x/2$ and comparisons of $\mathbb{F}$ are efficient
- E.g., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \{n2^m : n, m \in \mathbb{Z}\}$
- Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and $\widetilde{f} : \mathbb{F} \times \mathbb{Z} \dashrightarrow \mathbb{F}$
- $\widetilde{f}$ is an $\mathcal{A}$-approximation of f if:
  for all $x \in \mathbb{F}$, $\widetilde{f}(x;p) \equiv f(x) \pm 2^{-p}$.
  - Similarly for $\mathcal{R}$-approximable.

- Let $\mathcal{A}_f$ denote (an arbitrary member of) the set of all $\mathcal{A}$-approximations of f.

- f is $\mathcal{A}$-approxible if some $f \in \mathcal{A}_f$ is computable by a halting Turing machine
  - $*$ Similarly for $\mathcal{R}$-approximable and the set $\mathcal{R}_f$.

- E.g., the field operations $\pm, \times, \div$ and $\exp, \log$ are all $\mathcal{R}$-approximable.

- Let $\mathcal{A}_f$ denote (an arbitrary member of) the set of all $\mathcal{A}$-approximations of f.

- f is $\mathcal{A}$-approxible if some $f \in \mathcal{A}_f$ is computable by a halting Turing machine
  * Similarly for $\mathcal{R}$-approximable and the set $\mathcal{R}_f$.

- E.g., the field operations $\pm, \times, \div$ and $\exp, \log$ are all $\mathcal{R}$-approximable.

- Let $\mathcal{A}_f$ denote (an arbitrary member of) the set of all $\mathcal{A}$-approximations of f.

- f is $\mathcal{A}$-approxible if some $f \in \mathcal{A}_f$ is computable by a halting Turing machine
  - $*$ Similarly for $\mathcal{R}$-approximable and the set $\mathcal{R}_f$.

- E.g., the field operations $\pm, \times, \div$ and $\exp, \log$ are all $\mathcal{R}$-approximable.

- Let $\mathcal{A}_f$ denote (an arbitrary member of) the set of all $\mathcal{A}$-approximations of f.

- f is $\mathcal{A}$-approxible if some $f \in \mathcal{A}_f$ is computable by a halting Turing machine
  - Similarly for $\mathcal{R}$-approximable and the set $\mathcal{R}_f$.

- E.g., the field operations $\pm, \times, \div$ and $\exp, \log$ are all $\mathcal{R}$-approximable.

Sign Lemma

- SIGN LEMMA:
  $\mathtt{sign}(f(x)) = \mathtt{sign}(\mathcal{R}_f(x;1))$
- Proof: $\mathcal{R}_f(x;1) = f(x)(1 \pm 2^{-1})$.

Zero Problem in Real Approximations

- $Zero(f)$ is the problem of deciding for $x \in \mathbb{F}$, whether $f(x) = 0$

- THEOREM: The following is equivalent:
  - ∗ f is $\mathcal{R}$-approximable
  - ∗ f is $\mathcal{A}$-approximable and $Zero(f)$ is decidable

- Proof: ($\Rightarrow$) Given x and p, we want to compute $\mathcal{A}_f(x;p)$.
  - ∗ (1) compute $c = \mathcal{R}_f(x;1)$
  - ∗ (2) $x \in Zero(f)$ iff $c = 0$ (by SIGN LEMMA)
  - ∗ (3) If $c \neq 0$, OUTPUT $\mathcal{R}_f(x, p + 1 + \lceil \lg |c| \rceil)$

- Proof: ($\Leftarrow$) Given x and p, we want to compute $\mathcal{R}_f(x;p)$.
  - ∗ (1) If $f(x) = 0$, OUTPUT 0
  - ∗ (2) Find the first n such that $|\mathcal{A}_f(x;n)| \geq 2^{1-n}$
  - ∗ (3) So $|f(x)| \geq 2^{-n}$. OUTPUT $\mathcal{A}_f(x; n + p)$

Zero Problem in Real Approximations

- $Zero(f)$ is the problem of deciding for $x \in \mathbb{F}$, whether $f(x) = 0$

- THEOREM: The following is equivalent:
  - f is $\mathcal{R}$-approximable
  - f is $\mathcal{A}$-approximable and $Zero(f)$ is decidable

- Proof: ($\Rightarrow$) Given x and p, we want to compute $\mathcal{A}_f(x; p)$.
  - (1) compute $c = \mathcal{R}_f(x; 1)$
  - (2) $x \in Zero(f)$ iff $c = 0$ (by SIGN LEMMA)
  - (3) If $c \neq 0$, OUTPUT $\mathcal{R}_f(x, p + 1 + \lceil \lg |c| \rceil)$

- Proof: ($\Leftarrow$) Given x and p, we want to compute $\mathcal{R}_f(x; p)$.
  - (1) If $f(x) = 0$, OUTPUT 0
  - (2) Find the first n such that $|\mathcal{A}_f(x; n)| \geq 2^{1-n}$
  - (3) So $|f(x)| \geq 2^{-n}$. OUTPUT $\mathcal{A}_f(x; n + p)$

Zero Problem in Real Approximations

- $Zero(f)$ is the problem of deciding for $x \in \mathbb{F}$, whether $f(x) = 0$

- THEOREM: The following is equivalent:
  - f is $\mathcal{R}$-approximable
  - f is $\mathcal{A}$-approximable and $Zero(f)$ is decidable

- Proof: ($\Rightarrow$) Given x and p, we want to compute $\mathcal{A}_f(x;p)$.
  - (1) compute $c = \mathcal{R}_f(x;1)$
  - (2) $x \in Zero(f)$ iff $c = 0$ (by SIGN LEMMA)
  - (3) If $c \neq 0$, OUTPUT $\mathcal{R}_f(x, p + 1 + \lceil \lg |c| \rceil)$

- Proof: ($\Leftarrow$) Given x and p, we want to compute $\mathcal{R}_f(x;p)$.
  - (1) If $f(x) = 0$, OUTPUT 0
  - (2) Find the first n such that $|\mathcal{A}_f(x;n)| \geq 2^{1-n}$
  - (3) So $|f(x)| \geq 2^{-n}$. OUTPUT $\mathcal{A}_f(x; n + p)$

Zero Problem in Real Approximations

- $Zero(f)$ is the problem of deciding for $x \in \mathbb{F}$, whether $f(x) = 0$

- THEOREM: The following is equivalent:
  - f is $\mathcal{R}$-approximable
  - f is $\mathcal{A}$-approximable and $Zero(f)$ is decidable

- Proof: ($\Rightarrow$) Given x and p, we want to compute $\mathcal{A}_f(x;p)$.
  - (1) compute $c = \mathcal{R}_f(x;1)$
  - (2) $x \in Zero(f)$ iff $c = 0$ (by SIGN LEMMA)
  - (3) If $c \neq 0$, OUTPUT $\mathcal{R}_f(x, p + 1 + \lceil \lg |c| \rceil)$

- Proof: ($\Leftarrow$) Given x and p, we want to compute $\mathcal{R}_f(x;p)$.
  - (1) If $f(x) = 0$, OUTPUT 0
  - (2) Find the first n such that $|\mathcal{A}_f(x;n)| \geq 2^{1-n}$
  - (3) So $|f(x)| \geq 2^{-n}$. OUTPUT $\mathcal{A}_f(x; n + p)$

Zero Problem in Real Approximations

- $Zero(f)$ is the problem of deciding for $x \in \mathbb{F}$, whether $f(x) = 0$

- THEOREM: The following is equivalent:
  - $*$ f is $\mathcal{R}$-approximable
  - $*$ f is $\mathcal{A}$-approximable and $Zero(f)$ is decidable

- Proof: ($\Rightarrow$) Given x and p, we want to compute $\mathcal{A}_f(x;p)$.
  - $*$ (1) compute $c = \mathcal{R}_f(x;1)$
  - $*$ (2) $x \in Zero(f)$ iff $c = 0$ (by SIGN LEMMA)
  - $*$ (3) If $c \neq 0$, OUTPUT $\mathcal{R}_f(x, p + 1 + \lceil \lg |c| \rceil)$

- Proof: ($\Leftarrow$) Given x and p, we want to compute $\mathcal{R}_f(x;p)$.
  - $*$ (1) If $f(x) = 0$, OUTPUT 0
  - $*$ (2) Find the first n such that $|\mathcal{A}_f(x;n)| \geq 2^{1-n}$
  - $*$ (3) So $|f(x)| \geq 2^{-n}$. OUTPUT $\mathcal{A}_f(x;n+p)$

Separation of Abs and Rel

- THEOREM: There is an $\mathcal{A}$-approximable function that is not $\mathcal{R}$-approximable.

- Proof: Let $H : \mathbb{N} \to \mathbb{F}$ where $H(i) = \begin{cases} 0 & \text{if } i\text{-th TM on 0 halts} \\ 2^{-k} & \text{if } i\text{-th TM on 0 halts in } k \text{ steps} \end{cases}$
  - $*$ H is not $\mathcal{R}$-approximable: otherwise $HALT$ is recursive.
  - $*$ H is $\mathcal{A}$-approximable: on i, j we can compute $\widetilde{H}(i;j)$

  such that $|\widetilde{H}(i;j) - H(i)| \leq 2^{-j}$

- OPEN PROBLEM: Find a Natural Example

Separation of Abs and Rel

- THEOREM: There is an $\mathcal{A}$-approximable function that is not $\mathcal{R}$-approximable.

- Proof: Let $H : \mathbb{N} \to \mathbb{F}$ where $H(i) = \begin{cases} 0 & \text{if } i\text{-th TM on 0 halts} \\ 2^{-k} & \text{if } i\text{-th TM on 0 halts in } k \text{ steps} \end{cases}$
  - H is not $\mathcal{R}$-approximable: otherwise $HALT$ is recursive.
  - H is $\mathcal{A}$-approximable: on i, j we can compute $\widetilde{H}(i;j)$

  such that $|\widetilde{H}(i;j) - H(i)| \leq 2^{-j}$

- OPEN PROBLEM: Find a Natural Example

Separation of Abs and Rel

- THEOREM: There is an $\mathcal{A}$-approximable function that is not $\mathcal{R}$-approximable.

- Proof: Let $H : \mathbb{N} \to \mathbb{F}$ where $H(i) = \begin{cases} 0 & \text{if } i\text{-th TM on 0 halts} \\ 2^{-k} & \text{if } i\text{-th TM on 0 halts in } k \text{ steps} \end{cases}$
    * H is not $\mathcal{R}$-approximable: otherwise $HALT$ is recursive.
    * H is $\mathcal{A}$-approximable: on i, j we can compute $\widetilde{H}(i;j)$

  such that $|\widetilde{H}(i;j) - H(i)| \leq 2^{-j}$

- OPEN PROBLEM: Find a Natural Example

Separation of Abs and Rel

- THEOREM: There is an $\mathcal{A}$-approximable function that is not $\mathcal{R}$-approximable.

- Proof: Let $H : \mathbb{N} \to \mathbb{F}$ where $H(i) = \begin{cases} 0 & \text{if } i\text{-th TM on 0 halts} \\ 2^{-k} & \text{if } i\text{-th TM on 0 halts in } k \text{ steps} \end{cases}$
  - $*$ H is not $\mathcal{R}$-approximable: otherwise $HALT$ is recursive.
  - $*$ H is $\mathcal{A}$-approximable: on i, j we can compute $\widetilde{H}(i;j)$

  such that $|\widetilde{H}(i;j) - H(i)| \leq 2^{-j}$

- OPEN PROBLEM: Find a Natural Example

Complexity of Zero Problems

- Useful complexity classification

- Zero Hierarchy
  * Polynomial: $\Omega_0 := \{+, -, \times\} \cup \mathbb{Z}$
  * Rational: $\Omega_1 := \Omega_0 \cup \{\div\}$
  * Radical: $\Omega_2 := \Omega_1 \cup \{\sqrt[k]{\cdot} : k \geq 2\}$
  * Algebraic: $\Omega_3 := \Omega_2 \cup \{\mathrm{RootOf}\}$
  * Elementary: $\Omega_4 = \Omega_3 \cup \{\exp, \log\}$

- Complexity
  * ZERO(Ω_1) is in $PSPACE$ and P-complete [Mehlhorn-Schmitt-Yap]
  * ZERO(Ω_3) is decidable in Single Exponential Time [Tarski,Grigoriev,etc]
  * ZERO(Ω_4) is decidable provided Schanuel's conjecture holds [Richardson]

- OPEN PROBLEM: Is $ZERO(\Omega_4)$ decidable?

Complexity of Zero Problems

- Useful complexity classification

- Zero Hierarchy
  - Polynomial: $\Omega_0 := \{+, -, \times\} \cup \mathbb{Z}$
  - Rational: $\Omega_1 := \Omega_0 \cup \{\div\}$
  - Radical: $\Omega_2 := \Omega_1 \cup \{\sqrt[k]{\cdot} : k \geq 2\}$
  - Algebraic: $\Omega_3 := \Omega_2 \cup \{\mathrm{RootOf}\}$
  - Elementary: $\Omega_4 = \Omega_3 \cup \{\exp, \log\}$

- Complexity
  - ZERO(Ω_1) is in $PSPACE$ and P-complete [Mehlhorn-Schmitt-Yap]
  - ZERO(Ω_3) is decidable in Single Exponential Time [Tarski,Grigoriev,etc]
  - ZERO(Ω_4) is decidable provided Schanuel's conjecture holds [Richardson]

- OPEN PROBLEM: Is $ZERO(\Omega_4)$ decidable?

Complexity of Zero Problems

- Useful complexity classification

- Zero Hierarchy
  - Polynomial: $\Omega_0 := \{+, -, \times\} \cup \mathbb{Z}$
  - Rational: $\Omega_1 := \Omega_0 \cup \{\div\}$
  - Radical: $\Omega_2 := \Omega_1 \cup \{\sqrt[k]{\cdot} : k \geq 2\}$
  - Algebraic: $\Omega_3 := \Omega_2 \cup \{\mathrm{RootOf}\}$
  - Elementary: $\Omega_4 = \Omega_3 \cup \{\exp, \log\}$

- Complexity
  - ZERO(Ω_1) is in $PSPACE$ and P-complete [Mehlhorn-Schmitt-Yap]
  - ZERO(Ω_3) is decidable in Single Exponential Time [Tarski,Grigoriev,etc]
  - ZERO(Ω_4) is decidable provided Schanuel's conjecture holds [Richardson]

- OPEN PROBLEM: Is $ZERO(\Omega_4)$ decidable?

Complexity of Zero Problems

- Useful complexity classification
- Zero Hierarchy
  * Polynomial: $\Omega_0 := \{+, -, \times\} \cup \mathbb{Z}$
  * Rational: $\Omega_1 := \Omega_0 \cup \{\div\}$
  * Radical: $\Omega_2 := \Omega_1 \cup \{\sqrt[k]{\cdot} : k \geq 2\}$
  * Algebraic: $\Omega_3 := \Omega_2 \cup \{\mathrm{RootOf}\}$
  * Elementary: $\Omega_4 = \Omega_3 \cup \{\exp, \log\}$
- Complexity
  * ZERO(Ω_1) is in $PSPACE$ and P-complete [Mehlhorn-Schmitt-Yap]
  * ZERO(Ω_3) is decidable in Single Exponential Time [Tarski,Grigoriev,etc]
  * ZERO(Ω_4) is decidable provided Schanuel's conjecture holds [Richardson]
- OPEN PROBLEM: Is $ZERO(\Omega_4)$ decidable?

Complexity of Zero Problems

- Useful complexity classification
- Zero Hierarchy
  - Polynomial: $\Omega_0 := \{+, -, \times\} \cup \mathbb{Z}$
  - Rational: $\Omega_1 := \Omega_0 \cup \{\div\}$
  - Radical: $\Omega_2 := \Omega_1 \cup \{\sqrt[k]{\cdot} : k \geq 2\}$
  - Algebraic: $\Omega_3 := \Omega_2 \cup \{\mathrm{RootOf}\}$
  - Elementary: $\Omega_4 = \Omega_3 \cup \{\exp, \log\}$
- Complexity
  - ZERO(Ω_1) is in $PSPACE$ and P-complete [Mehlhorn-Schmitt-Yap]
  - ZERO(Ω_3) is decidable in Single Exponential Time [Tarski,Grigoriev,etc]
  - ZERO(Ω_4) is decidable provided Schanuel's conjecture holds [Richardson]
- OPEN PROBLEM: Is $ZERO(\Omega_4)$ decidable?

PART II.
Why Care About Nothing (Zero)?

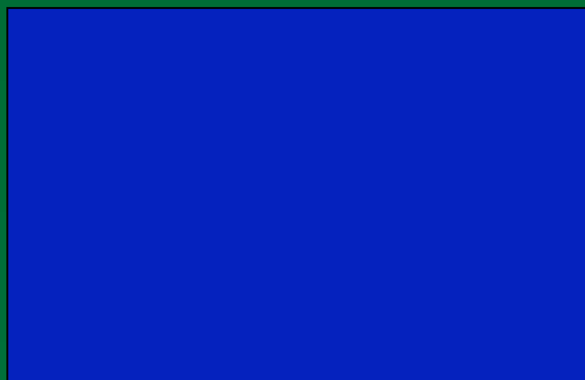
Much Ado About Nothing

– Shakespeare (1600)

Points and Lines

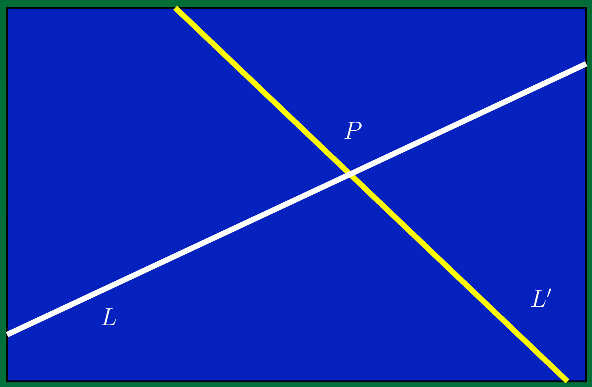
- Simple geometric test:
  - ∗ Is a point P on a line L?
- 
- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - ∗ $L: \quad aX + bY + c = 0$
  - ∗ $P: \quad (x_0, y_0)$
- In Meshing Applications
  - ∗ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - ∗ Sign determination
- Upshot
  - ∗ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - $*$ Is a point P on a line L?
- 
- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - $*$ $L: \quad aX + bY + c = 0$
  - $*$ $P: \quad (x_0, y_0)$
- In Meshing Applications
  - $*$ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - $*$ Sign determination
- Upshot
  - $*$ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - $*$ Is a point P on a line L?



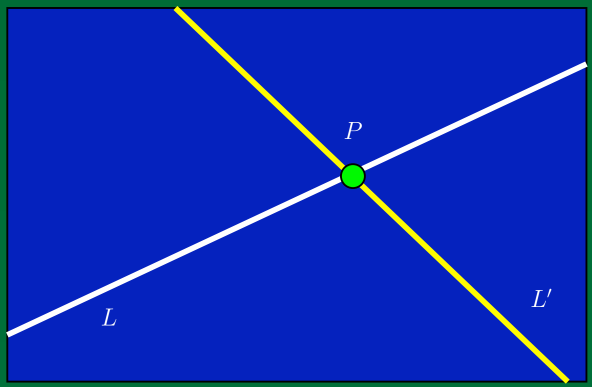
- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - $*$ $L: \quad aX + bY + c = 0$
  - $*$ $P: \quad (x_0, y_0)$

- In Meshing Applications
  - $*$ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - $*$ Sign determination

- Upshot
  - $*$ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - ∗ Is a point P on a line L?

- 

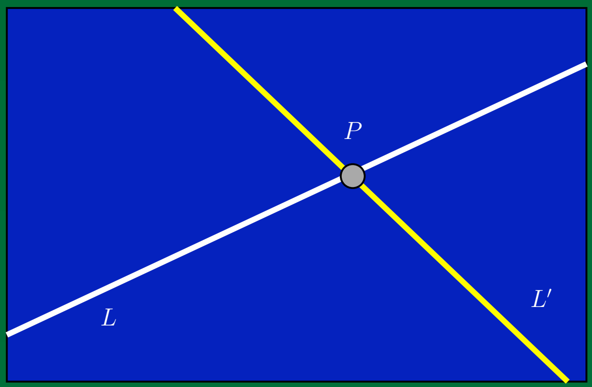
- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - ∗ $L: \quad aX + bY + c = 0$
  - ∗ $P: \quad (x_0, y_0)$

- In Meshing Applications
  - ∗ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - ∗ Sign determination

- Upshot
  - ∗ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - ∗ Is a point P on a line L?



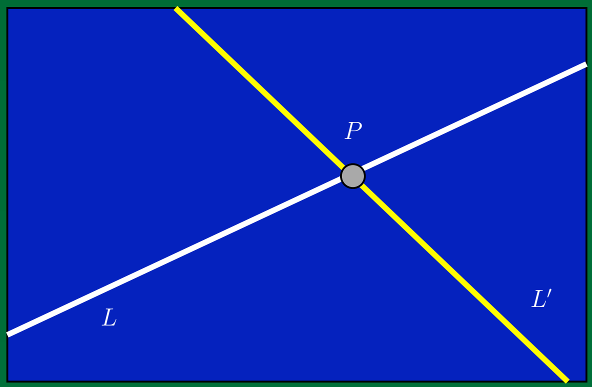
- P is on L $\quad\Leftrightarrow\quad ax_0 + by_0 + c = 0$
  - ∗ $L: \quad aX + bY + c = 0$
  - ∗ $P: \quad (x_0, y_0)$

- In Meshing Applications
  - ∗ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - ∗ Sign determination

- Upshot
  - ∗ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - $*$ Is a point P on a line L?



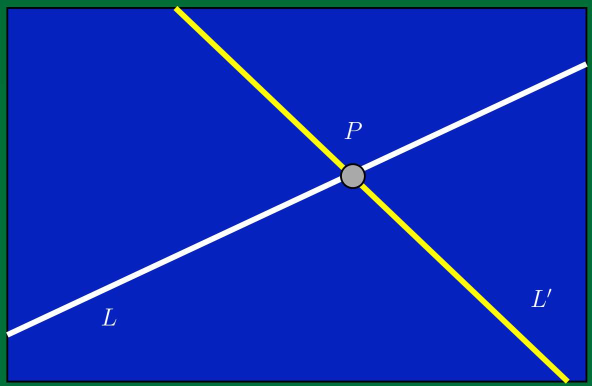
- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - $*$ $L: \quad aX + bY + c = 0$
  - $*$ $P: \quad (x_0, y_0)$

- In Meshing Applications
  - $*$ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - $*$ Sign determination

- Upshot
  - $*$ Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - Is a point P on a line L?

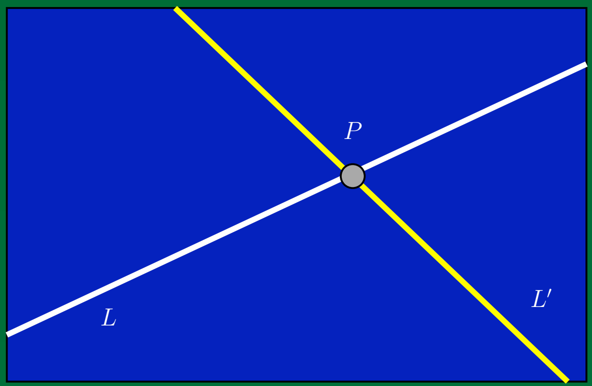


- P is on L $\quad\Leftrightarrow\quad ax_0 + by_0 + c = 0$
  - $L: \quad aX + bY + c = 0$
  - $P: \quad (x_0, y_0)$

- In Meshing Applications
  - Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - Sign determination

- Upshot
  - Knowing zero (and sign) is necessary for computing correct geometry

Points and Lines

- Simple geometric test:
  - ∗ Is a point P on a line L?
- 

- P is on $L \quad \Leftrightarrow \quad ax_0 + by_0 + c = 0$
  - ∗ $L: \quad aX + bY + c = 0$
  - ∗ $P: \quad (x_0, y_0)$
- In Meshing Applications
  - ∗ Point Classification Problem: Is P IN/OUT/ON a given triangle?
  - ∗ Sign determination
- Upshot
  - ∗ Knowing zero (and sign) is necessary for computing correct geometry

Again, What is Geometry?

- Geometric vein permeates all of Mathematics
  - ∗ Number theory, algebraic & differential geometry, topology, probability, ...

- What is Geometry?
  - ∗ Euclid: Axiomatic Approach
  - ∗ Descartes: Algebraization of Geometry
  - ∗ Klein: Transformation Groups
  - ∗ Hilbert: Logical Foundations
  - ∗ Tarski: Elementary Geometry and Algebra
  - ∗ Erdos: Combinatorial Vein

- Computational Perspective:
  - ∗ (1) Geometry is comprised of discrete relations among geometric objects
  - ∗ (2) Computational Geometry computes these relations, by deciding zero & sign of expressions

- Exact Geometric Computation (EGC):
  — an approach requiring error-free zero & sign computation

Again, What is Geometry?

- Geometric vein permeates all of Mathematics
  - ∗ Number theory, algebraic & differential geometry, topology, probability, ...
- What is Geometry?
  - ∗ Euclid: Axiomatic Approach
  - ∗ Descartes: Algebraization of Geometry
  - ∗ Klein: Transformation Groups
  - ∗ Hilbert: Logical Foundations
  - ∗ Tarski: Elementary Geometry and Algebra
  - ∗ Erdos: Combinatorial Vein
- Computational Perspective:
  - ∗ (1) Geometry is comprised of discrete relations among geometric objects
  - ∗ (2) Computational Geometry computes these relations, by deciding zero & sign of expressions
- Exact Geometric Computation (EGC):
  — an approach requiring error-free zero & sign computation

Again, What is Geometry?

- Geometric vein permeates all of Mathematics
  - ∗ Number theory, algebraic & differential geometry, topology, probability, ...

- What is Geometry?
  - ∗ Euclid: Axiomatic Approach
  - ∗ Descartes: Algebraization of Geometry
  - ∗ Klein: Transformation Groups
  - ∗ Hilbert: Logical Foundations
  - ∗ Tarski: Elementary Geometry and Algebra
  - ∗ Erdos: Combinatorial Vein

- Computational Perspective:
  - ∗ (1) Geometry is comprised of discrete relations among geometric objects
  - ∗ (2) Computational Geometry computes these relations, by deciding zero & sign of expressions

- Exact Geometric Computation (EGC):
  — an approach requiring error-free zero & sign computation

Again, What is Geometry?

- Geometric vein permeates all of Mathematics
  - Number theory, algebraic & differential geometry, topology, probability, ...
- What is Geometry?
  - Euclid: Axiomatic Approach
  - Descartes: Algebraization of Geometry
  - Klein: Transformation Groups
  - Hilbert: Logical Foundations
  - Tarski: Elementary Geometry and Algebra
  - Erdos: Combinatorial Vein
- Computational Perspective:
  - (1) Geometry is comprised of discrete relations among geometric objects
  - (2) Computational Geometry computes these relations, by deciding zero & sign of expressions
- Exact Geometric Computation (EGC):
  — an approach requiring error-free zero & sign computation

Again, What is Geometry?

- Geometric vein permeates all of Mathematics
  - ∗ Number theory, algebraic & differential geometry, topology, probability, ...
- What is Geometry?
  - ∗ Euclid: Axiomatic Approach
  - ∗ Descartes: Algebraization of Geometry
  - ∗ Klein: Transformation Groups
  - ∗ Hilbert: Logical Foundations
  - ∗ Tarski: Elementary Geometry and Algebra
  - ∗ Erdos: Combinatorial Vein
- Computational Perspective:
  - ∗ (1) Geometry is comprised of discrete relations among geometric objects
  - ∗ (2) Computational Geometry computes these relations, by deciding zero & sign of expressions
- Exact Geometric Computation (EGC):
  — an approach requiring error-free zero & sign computation

Robust Geometric Computation

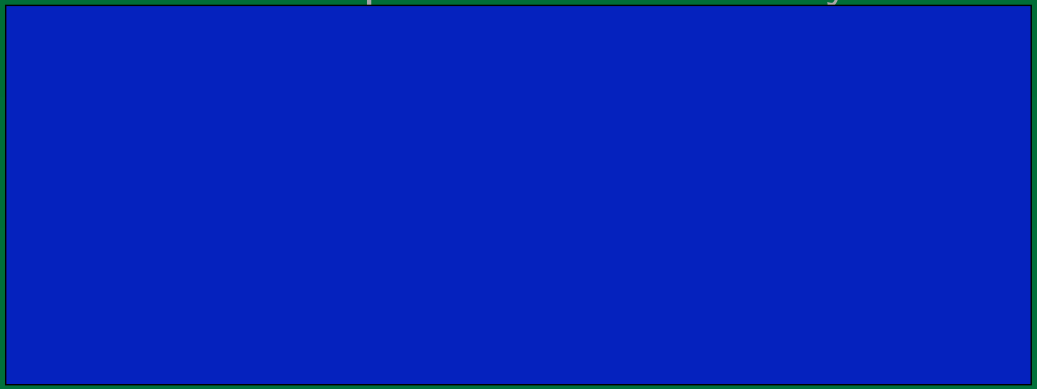
- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous
- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous
- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous



- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

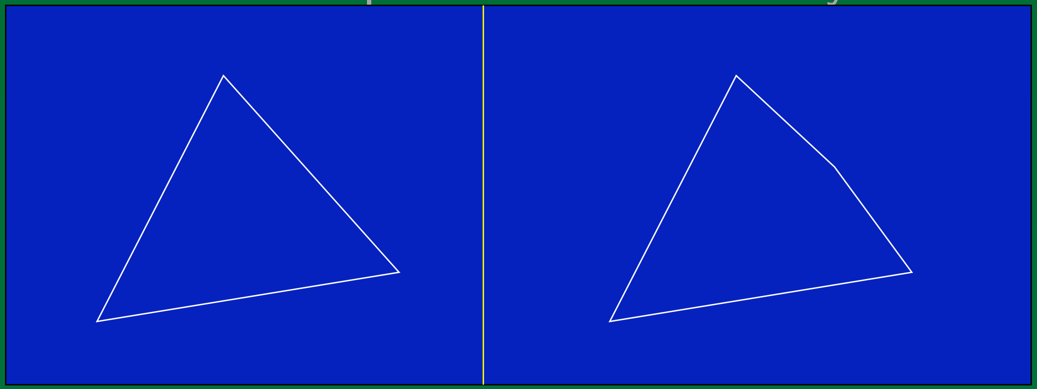
- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous

- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
    - ∗ Computer Science's dirty secret...

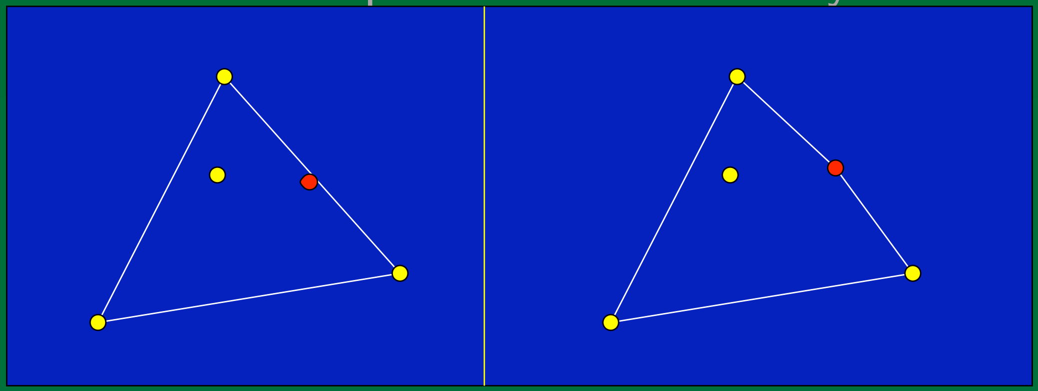
- Geometric Computation is inherently discontinuous



- Upshot: if we can efficiently solve the zero problem
    - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous

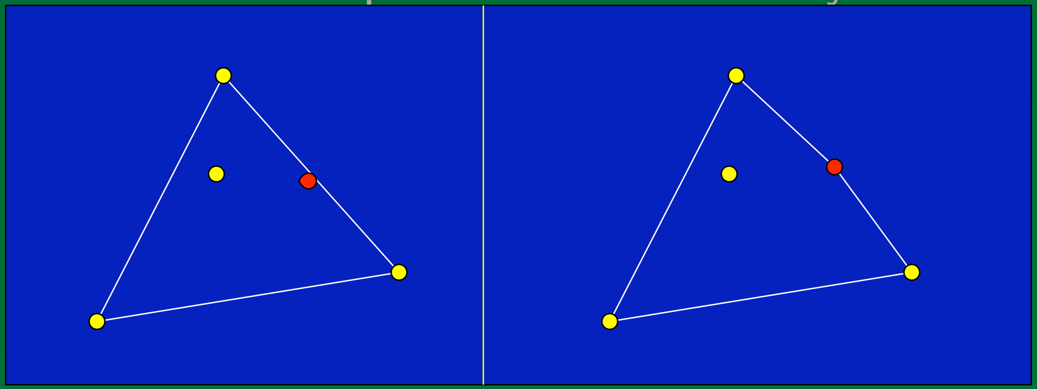


- Why EGC?
  - ∗ (1) Inconsistencies cause nonrobustness
  - ∗ (2) EGC ensures exact, hence consistent, geometry
  - ∗ (3) Most successful of known approaches
- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...

- Geometric Computation is inherently discontinuous

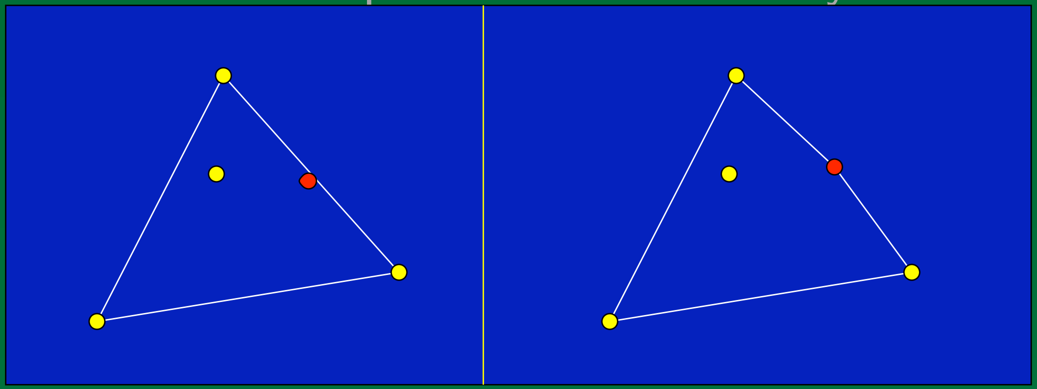


- Why EGC?
  - ∗ (1) Inconsistencies cause nonrobustness
  - ∗ (2) EGC ensures exact, hence consistent, geometry
  - ∗ (3) Most successful of known approaches

- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Robust Geometric Computation

- Widespread numerical nonrobustness issues in computational science and engineering
  - ∗ Computer Science's dirty secret...
- Geometric Computation is inherently discontinuous



- Why EGC?
  - ∗ (1) Inconsistencies cause nonrobustness
  - ∗ (2) EGC ensures exact, hence consistent, geometry
  - ∗ (3) Most successful of known approaches
- Upshot: if we can efficiently solve the zero problem
  - ∗ ... then we can produce robust software

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  - ∗ "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  - ∗ – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)
- Computer-Aided Theorem Proving
  - ∗ E.g., Kepler's conjecture (T. Hale)
- Automated Theorem Proving (D.M.Wang)
  - ∗ Randomized Testing, Proving by Example, etc
- Table Maker's Dilemma (J-M.Muller)
- Test Suites for Statistical Software (B.D.McCullough)
- Guaranteed Precision Arithmetic software
  - ∗ E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  - * "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  - * – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)
- Computer-Aided Theorem Proving
  - * E.g., Kepler's conjecture (T. Hale)
- Automated Theorem Proving (D.M.Wang)
  - * Randomized Testing, Proving by Example, etc
- Table Maker's Dilemma (J-M.Muller)
- Test Suites for Statistical Software (B.D.McCullough)
- Guaranteed Precision Arithmetic software
  - * E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)

  * "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".

  * – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)

- Computer-Aided Theorem Proving

  * E.g., Kepler's conjecture (T. Hale)

- Automated Theorem Proving (D.M.Wang)

  * Randomized Testing, Proving by Example, etc

- Table Maker's Dilemma (J-M.Muller)

- Test Suites for Statistical Software (B.D.McCullough)

- Guaranteed Precision Arithmetic software

  * E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  - ∗ "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  - ∗ – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)
- Computer-Aided Theorem Proving
  - ∗ E.g., Kepler's conjecture (T. Hale)
- Automated Theorem Proving (D.M.Wang)
  - ∗ Randomized Testing, Proving by Example, etc
- Table Maker's Dilemma (J-M.Muller)
- Test Suites for Statistical Software (B.D.McCullough)
- Guaranteed Precision Arithmetic software
  - ∗ E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  * "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  * – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)

- Computer-Aided Theorem Proving
  * E.g., Kepler's conjecture (T. Hale)

- Automated Theorem Proving (D.M.Wang)
  * Randomized Testing, Proving by Example, etc

- Table Maker's Dilemma (J-M.Muller)

- Test Suites for Statistical Software (B.D.McCullough)

- Guaranteed Precision Arithmetic software
  * E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  - ∗ "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  - ∗ – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)
- Computer-Aided Theorem Proving
  - ∗ E.g., Kepler's conjecture (T. Hale)
- Automated Theorem Proving (D.M.Wang)
  - ∗ Randomized Testing, Proving by Example, etc
- Table Maker's Dilemma (J-M.Muller)
- Test Suites for Statistical Software (B.D.McCullough)
- Guaranteed Precision Arithmetic software
  - ∗ E.g., Core Library, LEDA

Other Applications of Zero Problems

- Surface-Surface Intersection (SSI) (Farouki)
  - ∗ "The single greatest cause of poor reliability of CAD systems is lack of topologically consistent surface intersection algorithms".
  - ∗ – consensus, Workshop on Math. Foundations of CAD, MSRI, Berkeley (1999)
- Computer-Aided Theorem Proving
  - ∗ E.g., Kepler's conjecture (T. Hale)
- Automated Theorem Proving (D.M.Wang)
  - ∗ Randomized Testing, Proving by Example, etc
- Table Maker's Dilemma (J-M.Muller)
- Test Suites for Statistical Software (B.D.McCullough)
- Guaranteed Precision Arithmetic software
  - ∗ E.g., Core Library, LEDA

PART III.
On Zero Bounds and Adaptivity

"It can be of no practical use to know that π is irrational, but if we can know, it surely would be intolerable not to know."

— E.C. Titchmarsh

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
  - $e_1, e_2, e_3, \ldots$
  - where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
  - — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
    - $e_1, e_2, e_3, \ldots$
    - where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
    - — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
  - $* \ e_1, e_2, e_3, \ldots$
  - $*$ where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
  - $*$ — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
  - $e_1, e_2, e_3, \ldots$
  - where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
  - — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
  - $e_1, e_2, e_3, \ldots$
  - where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
  - — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

The Numerical Halting Problem

- To decide if $e = 0$, we compute approximations
  - $* \ e_1, e_2, e_3, \ldots$
  - $*$ where $e = e_n \pm 2^{-n}$
- OUTPUT "$e \neq 0$" when $|e_n| > 2^{-n}$
- What if $e = 0$?
- The ZERO PROBLEM is the continuous analogue of the HALTING problem
  - $*$ — like the halting problem, ZERO is "semi-decidable"
- The HALTING problem is complete for $Prec$
  We will show the continuous analogue.

Two Ways to to Use Zero Bounds

- Suppose we have $B(e) > 0$ such that:
  - if $\mathrm{Val}(e) \neq 0$, then $|\mathrm{Val}(e)| > B(e)$
  - B is a (conditional) zero bound function

- METHOD 1: Compute approximation $\widetilde{e}$ for e so that $|\widetilde{e} - e| < B(e)/2$
  - If $|\widetilde{e}| \geq B(e)$, then OUTPUT "$e \neq 0$"
  - Otherwise, OUTPUT "$e = 0$"

- METHOD 2: Compute sequence $(e_1, e_2, e_3, \ldots)$ as before
  - If $|e_n| > 2^{-n}$, OUTPUT "$e \neq 0$"
  - If $2^{-n} < \frac{B(e)}{2}$, OUTPUT "$e = 0$"

- Semi-adaptivity of METHOD 2
  - if $x \neq y$, the complexity depends on $-\log_2 |x - y|$

Two Ways to to Use Zero Bounds

- Suppose we have $B(e) > 0$ such that:
  - if $\mathrm{Val}(e) \neq 0$, then $|\mathrm{Val}(e)| > B(e)$
  - B is a (conditional) zero bound function

- METHOD 1: Compute approximation $\widetilde{e}$ for e so that $|\widetilde{e} - e| < B(e)/2$
  - If $|\widetilde{e}| \geq B(e)$, then OUTPUT "$e \neq 0$"
  - Otherwise, OUTPUT "$e = 0$"

- METHOD 2: Compute sequence $(e_1, e_2, e_3, \ldots)$ as before
  - If $|e_n| > 2^{-n}$, OUTPUT "$e \neq 0$"
  - If $2^{-n} < \frac{B(e)}{2}$, OUTPUT "$e = 0$"

- Semi-adaptivity of METHOD 2
  - if $x \neq y$, the complexity depends on $-\log_2 |x - y|$

Two Ways to to Use Zero Bounds

- Suppose we have $B(e) > 0$ such that:
  - if $\mathrm{Val}(e) \neq 0$, then $|\mathrm{Val}(e)| > B(e)$
  - B is a (conditional) zero bound function

- METHOD 1: Compute approximation $\widetilde{e}$ for e so that $|\widetilde{e} - e| < B(e)/2$
  - If $|\widetilde{e}| \geq B(e)$, then OUTPUT "$e \neq 0$"
  - Otherwise, OUTPUT "$e = 0$"

- METHOD 2: Compute sequence $(e_1, e_2, e_3, \ldots)$ as before
  - If $|e_n| > 2^{-n}$, OUTPUT "$e \neq 0$"
  - If $2^{-n} < \frac{B(e)}{2}$, OUTPUT "$e = 0$"

- Semi-adaptivity of METHOD 2
  - if $x \neq y$, the complexity depends on $-\log_2 |x - y|$

Two Ways to to Use Zero Bounds

- Suppose we have $B(e) > 0$ such that:
  - $*$ if $\mathrm{Val}(e) \neq 0$, then $|\mathrm{Val}(e)| > B(e)$
  - $*$ B is a (conditional) zero bound function

- METHOD 1: Compute approximation $\widetilde{e}$ for e so that $|\widetilde{e} - e| < B(e)/2$
  - $*$ If $|\widetilde{e}| \geq B(e)$, then OUTPUT "$e \neq 0$"
  - $*$ Otherwise, OUTPUT "$e = 0$"

- METHOD 2: Compute sequence $(e_1, e_2, e_3, \ldots)$ as before
  - $*$ If $|e_n| > 2^{-n}$, OUTPUT "$e \neq 0$"
  - $*$ If $2^{-n} < \frac{B(e)}{2}$, OUTPUT "$e = 0$"

- Semi-adaptivity of METHOD 2
  - $*$ if $x \neq y$, the complexity depends on $-\log_2 |x - y|$

Two Ways to to Use Zero Bounds

- Suppose we have $B(e) > 0$ such that:
  - ∗ if $\mathrm{Val}(e) \neq 0$, then $|\mathrm{Val}(e)| > B(e)$
  - ∗ B is a (conditional) zero bound function
- METHOD 1: Compute approximation $\widetilde{e}$ for e so that $|\widetilde{e} - e| < B(e)/2$
  - ∗ If $|\widetilde{e}| \geq B(e)$, then OUTPUT "$e \neq 0$"
  - ∗ Otherwise, OUTPUT "$e = 0$"
- METHOD 2: Compute sequence $(e_1, e_2, e_3, \ldots)$ as before
  - ∗ If $|e_n| > 2^{-n}$, OUTPUT "$e \neq 0$"
  - ∗ If $2^{-n} < \frac{B(e)}{2}$, OUTPUT "$e = 0$"
- Semi-adaptivity of METHOD 2
  - ∗ if $x \neq y$, the complexity depends on $-\log_2 |x - y|$

Why Adaptivity Algorithms

- Adaptive Complexity
  - (1) The numerical method (B) is adaptive...
  - (2) Algebraic methods are inherently non-adaptive, too inefficient
  - (3) Only algebraic information: Zero Bounds
- Other advantages:
  - simpler algorithms
  - exploits geometry
  - algorithms are independent of bounds
- General trend in computer algebra
  - (1) PROBLEM: most adaptive algorithms are incomplete
  - (2) E.g., no known adaptive complete algorithm for topological analysis of curve
  - (3) ONE SOLUTION: Hybrid algorithms
- OPEN PROBLEM:
  Construct Complete and Fully Adaptive Algorithms for basic problems
  - E.g., topological analysis of curves

Why Adaptivity Algorithms

- Adaptive Complexity
  - ∗ (1) The numerical method (B) is adaptive...
  - ∗ (2) Algebraic methods are inherently non-adaptive, too inefficient
  - ∗ (3) Only algebraic information: Zero Bounds
- Other advantages:
  - ∗ simpler algorithms
  - ∗ exploits geometry
  - ∗ algorithms are independent of bounds
- General trend in computer algebra
  - ∗ (1) PROBLEM: most adaptive algorithms are incomplete
  - ∗ (2) E.g., no known adaptive complete algorithm for topological analysis of curve
  - ∗ (3) ONE SOLUTION: Hybrid algorithms
- OPEN PROBLEM:
  Construct Complete and Fully Adaptive Algorithms for basic problems
  - ∗ E.g., topological analysis of curves

Why Adaptivity Algorithms

- Adaptive Complexity
  - (1) The numerical method (B) is adaptive...
  - (2) Algebraic methods are inherently non-adaptive, too inefficient
  - (3) Only algebraic information: Zero Bounds
- Other advantages:
  - simpler algorithms
  - exploits geometry
  - algorithms are independent of bounds
- General trend in computer algebra
  - (1) PROBLEM: most adaptive algorithms are incomplete
  - (2) E.g., no known adaptive complete algorithm for topological analysis of curve
  - (3) ONE SOLUTION: Hybrid algorithms
- OPEN PROBLEM:
  Construct Complete and Fully Adaptive Algorithms for basic problems
  - E.g., topological analysis of curves

Why Adaptivity Algorithms

- Adaptive Complexity
  - (1) The numerical method (B) is adaptive...
  - (2) Algebraic methods are inherently non-adaptive, too inefficient
  - (3) Only algebraic information: Zero Bounds
- Other advantages:
  - simpler algorithms
  - exploits geometry
  - algorithms are independent of bounds
- General trend in computer algebra
  - (1) PROBLEM: most adaptive algorithms are incomplete
  - (2) E.g., no known adaptive complete algorithm for topological analysis of curve
  - (3) ONE SOLUTION: Hybrid algorithms
- OPEN PROBLEM:
  Construct Complete and Fully Adaptive Algorithms for basic problems
  - E.g., topological analysis of curves

Why Adaptivity Algorithms

- Adaptive Complexity
  - (1) The numerical method (B) is adaptive...
  - (2) Algebraic methods are inherently non-adaptive, too inefficient
  - (3) Only algebraic information: Zero Bounds

- Other advantages:
  - simpler algorithms
  - exploits geometry
  - algorithms are independent of bounds

- General trend in computer algebra
  - (1) PROBLEM: most adaptive algorithms are incomplete
  - (2) E.g., no known adaptive complete algorithm for topological analysis of curve
  - (3) ONE SOLUTION: Hybrid algorithms

- OPEN PROBLEM:
  Construct Complete and Fully Adaptive Algorithms for basic problems
  - E.g., topological analysis of curves

Where do we get Zero Bounds?

- Classical Root Bounds
  - Constructive Root Bounds

- BFMS Bound
  - $B(e) = \frac{1}{L(e)U(e)^{D(e)^2-1}}$

	e	$U(e)$	$L(e)$
1.	rational a/b	a	b
2.	$e_1 \pm e_2$	$U(e_1)L(e_2) + L(e_1)U(e_2)$	$L(e_1)L(e_2)$
3.	$e_1 \times e_2$	$U(e_1)U(e_2)$	$L(e_1)L(e_2)$
4.	$e_1 \div e_2$	$U(e_1)L(e_2)$	$L(e_1)U(e_2)$
5.	$\sqrt[k]{e_1}$	$\sqrt[k]{U(e_1)}$	$\sqrt[k]{L(e_1)}$

- Other Constructive Bounds:
  BFMSS, Mahler Measure, Li-Yap, k-ary, etc

Where do we get Zero Bounds?

- Classical Root Bounds
  - Constructive Root Bounds

- BFMS Bound
  - $B(e) = \frac{1}{L(e)U(e)^{D(e)^2-1}}$

	e	$U(e)$	$L(e)$
1.	rational a/b	a	b
2.	$e_1 \pm e_2$	$U(e_1)L(e_2) + L(e_1)U(e_2)$	$L(e_1)L(e_2)$
3.	$e_1 \times e_2$	$U(e_1)U(e_2)$	$L(e_1)L(e_2)$
4.	$e_1 \div e_2$	$U(e_1)L(e_2)$	$L(e_1)U(e_2)$
5.	$\sqrt[k]{e_1}$	$\sqrt[k]{U(e_1)}$	$\sqrt[k]{L(e_1)}$

- Other Constructive Bounds:
  BFMSS, Mahler Measure, Li-Yap, k-ary, etc

Where do we get Zero Bounds?

- Classical Root Bounds
  - Constructive Root Bounds

- BFMS Bound
  - $B(e) = \frac{1}{L(e)U(e)^{D(e)^2-1}}$

	e	$U(e)$	$L(e)$
1.	rational a/b	a	b
2.	$e_1 \pm e_2$	$U(e_1)L(e_2) + L(e_1)U(e_2)$	$L(e_1)L(e_2)$
3.	$e_1 \times e_2$	$U(e_1)U(e_2)$	$L(e_1)L(e_2)$
4.	$e_1 \div e_2$	$U(e_1)L(e_2)$	$L(e_1)U(e_2)$
5.	$\sqrt[k]{e_1}$	$\sqrt[k]{U(e_1)}$	$\sqrt[k]{L(e_1)}$

- Other Constructive Bounds:
  BFMSS, Mahler Measure, Li-Yap, k-ary, etc

Where do we get Zero Bounds?

- Classical Root Bounds
  * Constructive Root Bounds

- BFMS Bound
  * $B(e) = \frac{1}{L(e)U(e)^{D(e)^2-1}}$

	e	$U(e)$	$L(e)$
1.	rational a/b	a	b
2.	$e_1 \pm e_2$	$U(e_1)L(e_2) + L(e_1)U(e_2)$	$L(e_1)L(e_2)$
3.	$e_1 \times e_2$	$U(e_1)U(e_2)$	$L(e_1)L(e_2)$
4.	$e_1 \div e_2$	$U(e_1)L(e_2)$	$L(e_1)U(e_2)$
5.	$\sqrt[k]{e_1}$	$\sqrt[k]{U(e_1)}$	$\sqrt[k]{L(e_1)}$

- Other Constructive Bounds:
  BFMSS, Mahler Measure, Li-Yap, k-ary, etc

The Central Algorithm

- Guaranteed precision computation
  - Viewed as a generalization of EGC

- Basic idea:
  Each Operator $f \in \Omega$ is computed by $\mathcal{R}_f$
  - Precision (a priori error) is driven down
  - Approximation (+ a posteriori error) is propagated up
  - $\mathcal{R}_f$ is applied to approximations, and propagated up (unless)

-

	Precision in x	Precision in y	Operation Precision
$z = x \pm y$	$p+2$	$p+2$	∞
$z = x \times y$	$p+2+\mu^+(y)$	$p+1+\mu^+(x)$	∞
$z = x/y$	$p+2-\mu^-(y)$	$\max\{1-\mu^-(y),$ $p+2-2\mu^-(y)+2\mu^+(x)\}$	$p+1$ $p+1$
$z = \sqrt{x}$	$\max\{p+1, 1-\mu^-(x)/2$		$p+1$
$z = \exp x$	$\max\{1, p+2+2^{\mu^+(x)+1}\}$		$p+1$
$z = \log x$	$\max\{1-\mu^-(x), p+2-\mu^-(x)\}$		$p+1$

The Central Algorithm

- Guaranteed precision computation
  - ∗ Viewed as a generalization of EGC

- Basic idea:
  Each Operator $f \in \Omega$ is computed by $\mathcal{R}_f$
  - ∗ Precision (a priori error) is driven down
  - ∗ Approximation (+ a posteriori error) is propagated up
  - ∗ $\mathcal{R}_f$ is applied to approximations, and propagated up (unless)

-

	Precision in x	Precision in y	Operation Precision
$z = x \pm y$	$p+2$	$p+2$	∞
$z = x \times y$	$p+2+\mu^+(y)$	$p+1+\mu^+(x)$	∞
$z = x/y$	$p+2-\mu^-(y)$	$\max\{1-\mu^-(y),$ $p+2-2\mu^-(y)+2\mu^+(x)\}$	$p+1$ $p+1$
$z = \sqrt{x}$	$\max\{p+1, 1-\mu^-(x)/2$		$p+1$
$z = \exp x$	$\max\{1, p+2+2^{\mu^+(x)+1}\}$		$p+1$
$z = \log x$	$\max\{1-\mu^-(x), p+2-\mu^-(x)\}$		$p+1$

The Central Algorithm

- Guaranteed precision computation
  - * Viewed as a generalization of EGC

- Basic idea:
  Each Operator $f \in \Omega$ is computed by $\mathcal{R}_f$
  - * Precision (a priori error) is driven down
  - * Approximation (+ a posteriori error) is propagated up
  - * $\mathcal{R}_f$ is applied to approximations, and propagated up (unless)

-

	Precision in x	Precision in y	Operation Precision
$z = x \pm y$	$p+2$	$p+2$	∞
$z = x \times y$	$p+2+\mu^{+}(y)$	$p+1+\mu^{+}(x)$	∞
$z = x/y$	$p+2-\mu^{-}(y)$	$\max\{1-\mu^{-}(y),$ $p+2-2\mu^{-}(y)+2\mu^{+}(x)\}$	$p+1$ $p+1$
$z = \sqrt{x}$	$\max\{p+1, 1-\mu^{-}(x)/2$		$p+1$
$z = \exp x$	$\max\{1, p+2+2^{\mu^{+}(x)+1}\}$		$p+1$
$z = \log x$	$\max\{1-\mu^{-}(x), p+2-\mu^{-}(x)\}$		$p+1$

The Central Algorithm

- Guaranteed precision computation
  - ∗ Viewed as a generalization of EGC

- Basic idea:
  Each Operator $f \in \Omega$ is computed by $\mathcal{R}_f$
  - ∗ Precision (a priori error) is driven down
  - ∗ Approximation (+ a posteriori error) is propagated up
  - ∗ $\mathcal{R}_f$ is applied to approximations, and propagated up (unless)

-

	Precision in x	Precision in y	Operation Precision
$z = x \pm y$	$p + 2$	$p + 2$	∞
$z = x \times y$	$p + 2 + \mu^+(y)$	$p + 1 + \mu^+(x)$	∞
$z = x/y$	$p + 2 - \mu^-(y)$	$\max\{1 - \mu^-(y),$ $p + 2 - 2\mu^-(y) + 2\mu^+(x)\}$	$p + 1$ $p + 1$
$z = \sqrt{x}$	$\max\{p + 1, 1 - \mu^-(x)/2$		$p + 1$
$z = \exp x$	$\max\{1, p + 2 + 2^{\mu^+(x)+1}\}$		$p + 1$
$z = \log x$	$\max\{1 - \mu^-(x), p + 2 - \mu^-(x)\}$		$p + 1$

- Cast of Characters
  Three Muskeeters and a friend:
  - (A) $\alpha(e;p)$ — absolute approximation
  - (M) $\mu^+(e)$ — upper bound on $\mu(e) = \lg|\mathrm{Val}(e)|$
  - (S) $\mathtt{sign}(e)$ — sign of $\mathrm{Val}(e)$
  - (B) $\beta(e)$ — root bound function

- OPEN PROBLEM: What is the optimal algorithm for evaluation?

- Cast of Characters
  Three Muskeeters and a friend:
  * (A) $\alpha(e;p)$ — absolute approximation
  * (M) $\mu^+(e)$ — upper bound on $\mu(e) = \lg|\mathrm{Val}(e)|$
  * (S) $\mathtt{sign}(e)$ — sign of $\mathrm{Val}(e)$
  * (B) $\beta(e)$ — root bound function

- OPEN PROBLEM: What is the optimal algorithm for evaluation?

- Cast of Characters
  Three Muskeeters and a friend:
  - (A) $\alpha(e;p)$ — absolute approximation
  - (M) $\mu^+(e)$ — upper bound on $\mu(e) = \lg|\mathrm{Val}(e)|$
  - (S) $\mathtt{sign}(e)$ — sign of $\mathrm{Val}(e)$
  - (B) $\beta(e)$ — root bound function

- OPEN PROBLEM: What is the optimal algorithm for evaluation?

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
  - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
  - (AP) Construction of Auxilliary Polynomial
  - (UB) Obtaining an upper bound
  - (LB) Obtaining a lower bound
  - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}$, $R_n = \beta - \beta_n$
  - (AP) Let $P(X,Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
  - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
  - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
  - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
  - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
  - (AP) Construction of Auxilliary Polynomial
  - (UB) Obtaining an upper bound
  - (LB) Obtaining a lower bound
  - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}$, $R_n = \beta - \beta_n$
  - (AP) Let $P(X, Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
  - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
  - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
  - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
  - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
  - (AP) Construction of Auxilliary Polynomial
  - (UB) Obtaining an upper bound
  - (LB) Obtaining a lower bound
  - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}$, $R_n = \beta - \beta_n$
  - (AP) Let $P(X, Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
  - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
  - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
  - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
  - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
  - (AP) Construction of Auxilliary Polynomial
  - (UB) Obtaining an upper bound
  - (LB) Obtaining a lower bound
  - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}$, $R_n = \beta - \beta_n$
  - (AP) Let $P(X,Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
  - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
  - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
  - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
    - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
    - (AP) Construction of Auxilliary Polynomial
    - (UB) Obtaining an upper bound
    - (LB) Obtaining a lower bound
    - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}$, $R_n = \beta - \beta_n$
    - (AP) Let $P(X, Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
    - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
    - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
    - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

Transcendental Number Theory

- Another conditional zero bound:
  $B(e) = 1$
  - (Fundamental Theorem of TNT)

- How do you prove that $\beta = \sum_{n=0}^{\infty} (2/3)^{2^n}$ is irrational?

- Outline of TNT (D.Masser)
  - (AP) Construction of Auxilliary Polynomial
  - (UB) Obtaining an upper bound
  - (LB) Obtaining a lower bound
  - (NV) Proving non-vanishing

- Let $\beta_n = \sum_{n=0}^{n} (2/3)^{2^n}, \qquad R_n = \beta - \beta_n$
  - (AP) Let $P(X, Y) = 2XY^2 + 4XY - 3Y^2 + X - Y$
  - (UB) $|\alpha_n| < (1/10)^{2^{n+1}}$ where $|P((2/3)^{2^{n+1}}, R_n)| < (1/10)^{2^{n+1}}$
  - (LB) If $\beta = r/s$ and $\alpha_n \neq 0$, then $|P((2/3)^{2^{n+1}}, R_n)| \geq s^{-2}(1/9)^{2^{n+1}}$
  - (NV) $\alpha_n \neq 0$

- From irrationality/transcendence to transcendence measures

PART IV
On Foundations of Real Computation

"There is a substantial conflict between theoretical computer science and numerical analysis. ... The conflict has at its roots another age-old conflict, that between the continuous and the discrete."

— Blum, Cucker, Shub, Smale (1996, Manifesto)

Two Approaches to Real Computation

- Analytic Approach
  - Turing(1936), Grzegorczyk(1955), Weihrauch, Ko, etc
  - Real numbers represented by rapidly converging Cauchy sequences
  - Extend Turing machines to compute with infinite input/output sequences

- Algebraic Approach
  - Blum-Shub-Smale (BSS) model, Real RAMs
  - Real numbers directly represented as atomic objects
  - Real numbers are compared without error
  - Algebraic operators are carried out without error

Two Approaches to Real Computation

- Analytic Approach
  - Turing(1936), Grzegorczyk(1955), Weihrauch, Ko, etc
  - Real numbers represented by rapidly converging Cauchy sequences
  - Extend Turing machines to compute with infinite input/output sequences

- Algebraic Approach
  - Blum-Shub-Smale (BSS) model, Real RAMs
  - Real numbers directly represented as atomic objects
  - Real numbers are compared without error
  - Algebraic operators are carried out without error

Two Approaches to Real Computation

- Analytic Approach
  - Turing(1936), Grzegorczyk(1955), Weihrauch, Ko, etc
  - Real numbers represented by rapidly converging Cauchy sequences
  - Extend Turing machines to compute with infinite input/output sequences

- Algebraic Approach
  - Blum-Shub-Smale (BSS) model, Real RAMs
  - Real numbers directly represented as atomic objects
  - Real numbers are compared without error
  - Algebraic operators are carried out without error

Where is the Zero Problem?

- Zero Problem is
  - undecidable in Analytic Approach
  - trivial in Algebraic Approach
  - more nuanced in our Approximation Approach
- Other Issues
  - (Analytic) Only continuous functions are computable (no geometry!)
  - (Algebraic) Exponential function is not computable
  - (Approximation) Composition is not automatic
- THEOREM: There exists $\mathcal{R}$-approximable f, g such that their composition $f \circ g$ is not $\mathcal{A}$-approximable

Where is the Zero Problem?

- Zero Problem is
  - undecidable in Analytic Approach
  - trivial in Algebraic Approach
  - more nuanced in our Approximation Approach

- Other Issues
  - (Analytic) Only continuous functions are computable (no geometry!)
  - (Algebraic) Exponential function is not computable
  - (Approximation) Composition is not automatic

- THEOREM: There exists $\mathcal{R}$-approximable f, g such that their composition $f \circ g$ is not $\mathcal{A}$-approximable

Where is the Zero Problem?

- Zero Problem is
  - undecidable in Analytic Approach
  - trivial in Algebraic Approach
  - more nuanced in our Approximation Approach
- Other Issues
  - (Analytic) Only continuous functions are computable (no geometry!)
  - (Algebraic) Exponential function is not computable
  - (Approximation) Composition is not automatic
- THEOREM: There exists $\mathcal{R}$-approximable f, g such that their composition $f \circ g$ is not $\mathcal{A}$-approximable

Where is the Zero Problem?

- Zero Problem is
  - undecidable in Analytic Approach
  - trivial in Algebraic Approach
  - more nuanced in our Approximation Approach

- Other Issues
  - (Analytic) Only continuous functions are computable (no geometry!)
  - (Algebraic) Exponential function is not computable
  - (Approximation) Composition is not automatic

- THEOREM: There exists $\mathcal{R}$-approximable f, g such that their composition $f \circ g$ is not $\mathcal{A}$-approximable

How We Solve Numerical Problems

- E.g., Solving a PDE model, A numerical optimization, etc
- STEP A:
  - ∗ Design an ideal Algorithm A
  - ∗ Assume certain operations such as $\pm, \times, \exp()$
  - ∗ Show that problem is solvable by Algorithm A
- STEP B:
  - ∗ Implement Algorithm A as a Numerical Program B
  - ∗ Account for numerical representation, errors, convergence, etc
  - ∗ Specify the "correctness" criteria

How We Solve Numerical Problems

- E.g., Solving a PDE model, A numerical optimization, etc
- STEP A:
  - ∗ Design an ideal Algorithm A
  - ∗ Assume certain operations such as $\pm, \times, \exp()$
  - ∗ Show that problem is solvable by Algorithm A
- STEP B:
  - ∗ Implement Algorithm A as a Numerical Program B
  - ∗ Account for numerical representation, errors, convergence, etc
  - ∗ Specify the "correctness" criteria

How We Solve Numerical Problems

- E.g., Solving a PDE model, A numerical optimization, etc

- STEP A:
  - ∗ Design an ideal Algorithm A
  - ∗ Assume certain operations such as $\pm, \times, \exp()$
  - ∗ Show that problem is solvable by Algorithm A

- STEP B:
  - ∗ Implement Algorithm A as a Numerical Program B
  - ∗ Account for numerical representation, errors, convergence, etc
  - ∗ Specify the "correctness" criteria

How We Solve Numerical Problems

- E.g., Solving a PDE model, A numerical optimization, etc
- STEP A:
  - ∗ Design an ideal Algorithm A
  - ∗ Assume certain operations such as $\pm, \times, \exp()$
  - ∗ Show that problem is solvable by Algorithm A
- STEP B:
  - ∗ Implement Algorithm A as a Numerical Program B
  - ∗ Account for numerical representation, errors, convergence, etc
  - ∗ Specify the "correctness" criteria

A New Synthesis

- Step A:
  - Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$
- Step B:
  - Program B belongs to some numerical model (Turing or Pointer Model)
- Critical Question:
  - Can Algorithm A be implemented by some Program B?
- Validity:
  Even numerical analysis books proceed to Step B via Step A
- The algebraic and numerical models play complementary roles!

A New Synthesis

- Step A:
  - ∗ Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - ∗ Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$
- Step B:
  - ∗ Program B belongs to some numerical model (Turing or Pointer Model)
- Critical Question:
  - ∗ Can Algorithm A be implemented by some Program B?
- Validity:
  Even numerical analysis books proceed to Step B via Step A
- The algebraic and numerical models play complementary roles!

A New Synthesis

- Step A:
  - Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$

- Step B:
  - Program B belongs to some numerical model (Turing or Pointer Model)

- Critical Question:
  - Can Algorithm A be implemented by some Program B?

- Validity:
  Even numerical analysis books proceed to Step B via Step A

- The algebraic and numerical models play complementary roles!

A New Synthesis

- Step A:
  - Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$
- Step B:
  - Program B belongs to some numerical model (Turing or Pointer Model)
- Critical Question:
  - Can Algorithm A be implemented by some Program B?
- Validity:
  Even numerical analysis books proceed to Step B via Step A
- The algebraic and numerical models play complementary roles!

A New Synthesis

- Step A:
  - ∗ Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - ∗ Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$
- Step B:
  - ∗ Program B belongs to some numerical model (Turing or Pointer Model)
- Critical Question:
  - ∗ Can Algorithm A be implemented by some Program B?
- Validity:
  Even numerical analysis books proceed to Step B via Step A
- The algebraic and numerical models play complementary roles!

A New Synthesis

- Step A:
  - Algorithm A belongs to an Algebraic Model (e.g., BSS Model)
  - Basis $\Omega = \{\pm, \times, \exp(), \ldots\}$
- Step B:
  - Program B belongs to some numerical model (Turing or Pointer Model)
- Critical Question:
  - Can Algorithm A be implemented by some Program B?
- Validity:
  Even numerical analysis books proceed to Step B via Step A
- The algebraic and numerical models play complementary roles!

Transfer Theorem

- Algebraic Pointer Model:
  - ∗ Schönhage's Pointer Machines, augmented with algebraic operations from Ω

- Numerical Pointer Model:
  - ∗ like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$

- Approximation in the EGC sense
  - ∗ Combinatorially exact, but numerically approximate

- THEOREM:

  The following are equivalent:
  - ∗ (I) Val_Ω is $\mathcal{R}$-approximable over Ω
  - ∗ (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).

- OPEN PROBLEM: Explore other transfer theorems

Transfer Theorem

- Algebraic Pointer Model:
  * Schönhage's Pointer Machines, augmented with algebraic operations from Ω
- Numerical Pointer Model:
  * like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$
- Approximation in the EGC sense
  * Combinatorially exact, but numerically approximate
- THEOREM:

  The following are equivalent:
  * (I) Val_Ω is $\mathcal{R}$-approximable over Ω
  * (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).
- OPEN PROBLEM: Explore other transfer theorems

Transfer Theorem

- Algebraic Pointer Model:
  - Schönhage's Pointer Machines, augmented with algebraic operations from Ω

- Numerical Pointer Model:
  - like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$

- Approximation in the EGC sense
  - Combinatorially exact, but numerically approximate

- THEOREM:
  The following are equivalent:
  - (I) Val_Ω is $\mathcal{R}$-approximable over Ω
  - (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).

- OPEN PROBLEM: Explore other transfer theorems

Transfer Theorem

- Algebraic Pointer Model:
  - Schönhage's Pointer Machines, augmented with algebraic operations from Ω
- Numerical Pointer Model:
  - like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$
- Approximation in the EGC sense
  - Combinatorially exact, but numerically approximate
- THEOREM:

  The following are equivalent:
  - (I) Val_Ω is $\mathcal{R}$-approximable over Ω
  - (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).
- OPEN PROBLEM: Explore other transfer theorems

Transfer Theorem

- Algebraic Pointer Model:
  - Schönhage's Pointer Machines, augmented with algebraic operations from Ω

- Numerical Pointer Model:
  - like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$

- Approximation in the EGC sense
  - Combinatorially exact, but numerically approximate

- THEOREM:

  The following are equivalent:
  - (I) Val_Ω is $\mathcal{R}$-approximable over Ω
  - (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).

- OPEN PROBLEM: Explore other transfer theorems

Transfer Theorem

- Algebraic Pointer Model:
    - $*$ Schönhage's Pointer Machines, augmented with algebraic operations from Ω

- Numerical Pointer Model:
    - $*$ like the Algebraic Pointer Model,

  but replace $f \in \Omega$ by $\mathcal{R}_f$

- Approximation in the EGC sense
    - $*$ Combinatorially exact, but numerically approximate

- THEOREM:
  The following are equivalent:
    - $*$ (I) Val_Ω is $\mathcal{R}$-approximable over Ω
    - $*$ (II) For all problems F, if F is Ω-computable (in algebraic model) then F is Ω-approximable (in numerical model).

- OPEN PROBLEM: Explore other transfer theorems

Connection to Analytic School

- THEOREM: Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and S regular.
  The following are equivalent:
  - $*$ f is computable (in analytic model)
  - $*$ f is partially $\mathcal{A}$-approximable and has a recursively enumerable modulus cover
- Thus, approximability of f gives up precisely one thing: continuity of f

Connection to Analytic School

- THEOREM: Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and S regular.
  The following are equivalent:
  - $*$ f is computable (in analytic model)
  - $*$ f is partially $\mathcal{A}$-approximable and has a recursively enumerable modulus cover

- Thus, approximability of f gives up precisely one thing: continuity of f

Connection to Analytic School

- THEOREM: Let $f : S \subseteq \mathbb{R} \dashrightarrow \mathbb{R}$ and S regular.
  The following are equivalent:
  - $*$ f is computable (in analytic model)
  - $*$ f is partially $\mathcal{A}$-approximable and has a recursively enumerable modulus cover

- Thus, approximability of f gives up precisely one thing: continuity of f

PART V
CONCLUSION

"Why is $\alpha_n \neq 0$? This innocent question will become more and more of a nuisance until it almost takes over the subject."

— David Masser (2000)

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - ∗ — in our framework, we may admit variables in Ω
  - ∗ — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - ∗ — viewed as the inverse of Constant Zero Problem
  - ∗ — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - ∗ — positive results from TNT!
  - ∗ — approaches to non-robustness in CAD
  - ∗ — theorem proving
- Elementary Problems (Choi, Richardson)
  - ∗ — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - ∗ — rich interplay possible between algebraic and analytic views
  - ∗ — complexity of zero problems and real approximation

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - * — in our framework, we may admit variables in Ω
  - * — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - * — viewed as the inverse of Constant Zero Problem
  - * — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - * — positive results from TNT!
  - * — approaches to non-robustness in CAD
  - * — theorem proving
- Elementary Problems (Choi, Richardson)
  - * — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - * — rich interplay possible between algebraic and analytic views
  - * — complexity of zero problems and real approximation

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - — in our framework, we may admit variables in Ω
  - — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - — viewed as the inverse of Constant Zero Problem
  - — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - — positive results from TNT!
  - — approaches to non-robustness in CAD
  - — theorem proving
- Elementary Problems (Choi, Richardson)
  - — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - — rich interplay possible between algebraic and analytic views
  - — complexity of zero problems and real approximation

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - $*$ — in our framework, we may admit variables in Ω
  - $*$ — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - $*$ — viewed as the inverse of Constant Zero Problem
  - $*$ — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - $*$ — positive results from TNT!
  - $*$ — approaches to non-robustness in CAD
  - $*$ — theorem proving
- Elementary Problems (Choi, Richardson)
  - $*$ — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - $*$ — rich interplay possible between algebraic and analytic views
  - $*$ — complexity of zero problems and real approximation

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - $*$ — in our framework, we may admit variables in Ω
  - $*$ — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - $*$ — viewed as the inverse of Constant Zero Problem
  - $*$ — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - $*$ — positive results from TNT!
  - $*$ — approaches to non-robustness in CAD
  - $*$ — theorem proving
- Elementary Problems (Choi, Richardson)
  - $*$ — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - $*$ — rich interplay possible between algebraic and analytic views
  - $*$ — complexity of zero problems and real approximation

Other Zero Problems (This Workshop)

- Functional Zero Problems (Van Der Hoeven)
  - ∗ — in our framework, we may admit variables in Ω
  - ∗ — cf. Constant Zero Problem, Singularity Theory, Canonicity
- Zero Finding (Myunghi, Sharma, Shub, Verschelde)
  - ∗ — viewed as the inverse of Constant Zero Problem
  - ∗ — condition numbers as counter part to zero bounds
- Applications (Choi, Farouki, Wang)
  - ∗ — positive results from TNT!
  - ∗ — approaches to non-robustness in CAD
  - ∗ — theorem proving
- Elementary Problems (Choi, Richardson)
  - ∗ — frontier of solvable Zero Problems
- Theory of Real Computation (Ziegler)
  - ∗ — rich interplay possible between algebraic and analytic views
  - ∗ — complexity of zero problems and real approximation

END OF TALK

Thanks for Listening!

"A rapacious monster lurks within every computer,
and it dines exclusively on accurate digits."
– B.D. McCullough (2000)

- Software and Papers can be downloaded from
  http://cs.nyu.edu/exact/

  - ⋆ Core Library Software
  - ⋆ "Theory of Real Computation according to EGC",
    Issue from Dagstuhl Workshop on Real Computation, 2006.
  - ⋆ "Complete Subdivision Algorithms 1: Intersecting Bezier Curves",
    SoCG'06.
  - ⋆ "Complete Numerical Isolation of Real Zeros in General Triangular Systems",

(with Jin-San Cheng and Xiao-shan Gao, ISSAC'07

END OF TALK

Schönhage's Pointer Model

-

- Δ-graph G: a labelled digraph
  - Pointer machine transforms G by re-directing edges

Type	Name	Instruction	Meaning
(i)	Node Assignment	$w \leftarrow w'$	$[w]_{G'} = [w']_G$
(ii)	Node Creation	$w \leftarrow$ **new**	$[w]_{G'}$ is new
(iii)	Node Comparison	**if** $w \equiv w'$ **goto** L	$G' = G$
(iv)	Halt and Output	**HALT**(w)	Output $G\|w$
(v)	Value Comparison	**if** $(w \circ w')$ **goto** L where $\circ \in \{=, <, \leq\}$	Compare $Val_G(w) \circ Val_G(w')$
(vi)	Value Assignment	$w := f(w_1, \ldots, w_m)$ where $f \in \Omega$ and $w, w_i \in \Delta^*$	$Val_{G'}(w) = f(Val_G(w_1), \ldots, Val_G(w_n))$

Schönhage's Pointer Model

-

- Δ-graph G: a labelled digraph
  - $*$ Pointer machine transforms G by re-directing edges

Type	Name	Instruction	Meaning
(i)	Node Assignment	$w \leftarrow w'$	$[w]_{G'} = [w']_G$
(ii)	Node Creation	$w \leftarrow$ **new**	$[w]_{G'}$ is new
(iii)	Node Comparison	**if** $w \equiv w'$ **goto** L	$G' = G$
(iv)	Halt and Output	**HALT**(w)	Output $G\|w$
(v)	Value Comparison	**if** $(w \circ w')$ **goto** L where $\circ \in \{=, <, \leq\}$	Compare $Val_G(w) \circ Val_G(w')$
(vi)	Value Assignment	$w := f(w_1, \ldots, w_m)$ where $f \in \Omega$ and $w, w_i \in \Delta^*$	$Val_{G'}(w) = f(Val_G(w_1), \ldots, Val_G(w_n))$