Applications of Transcendental Zero Bounds to Geometric Computation

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Geometric Computation: An Example

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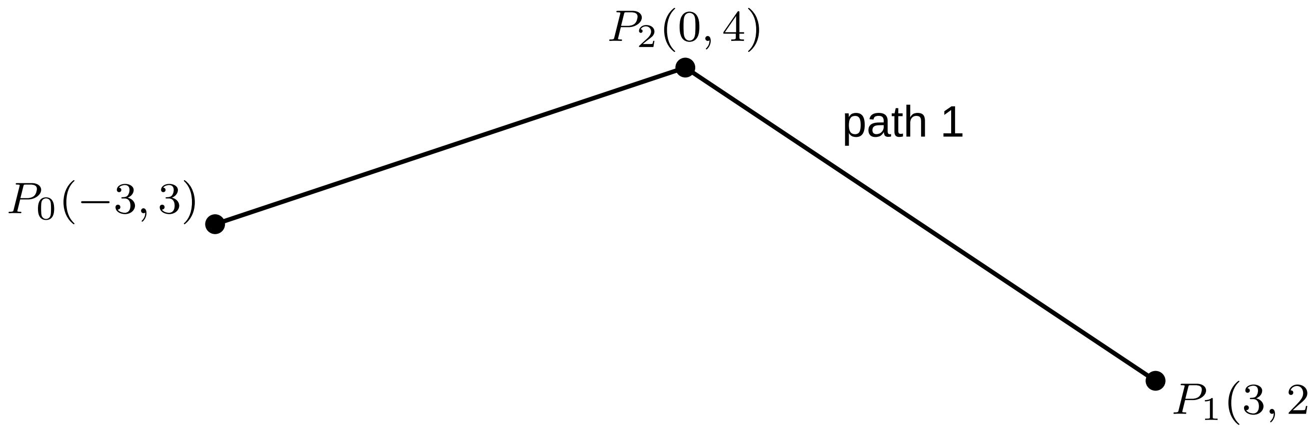

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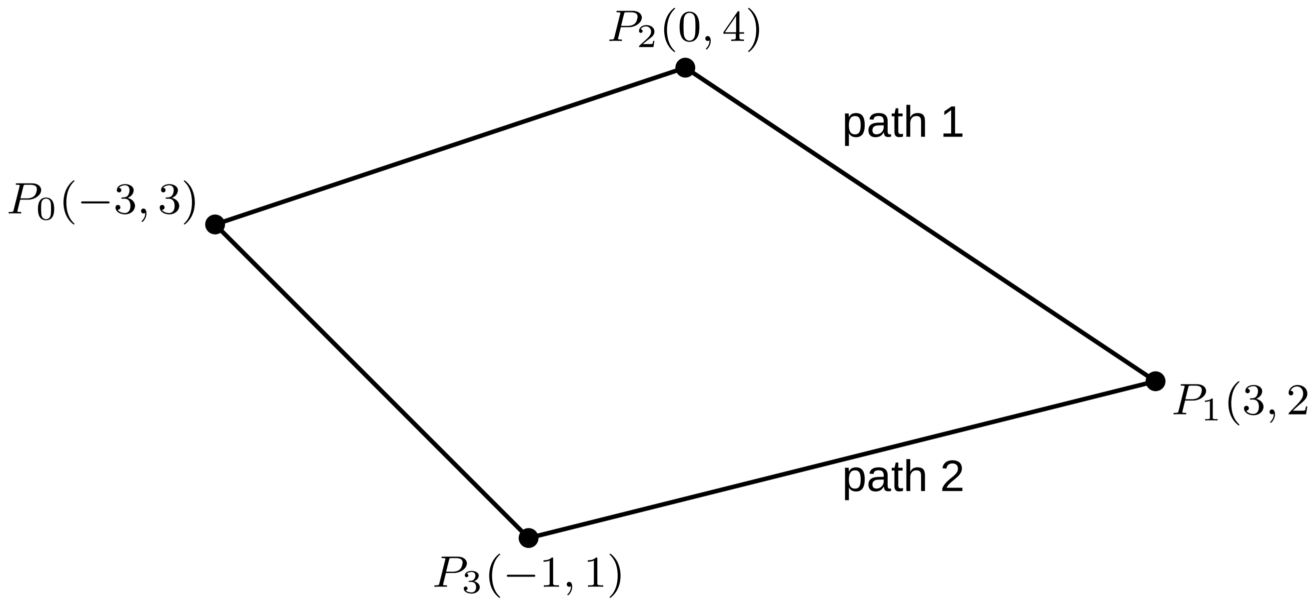

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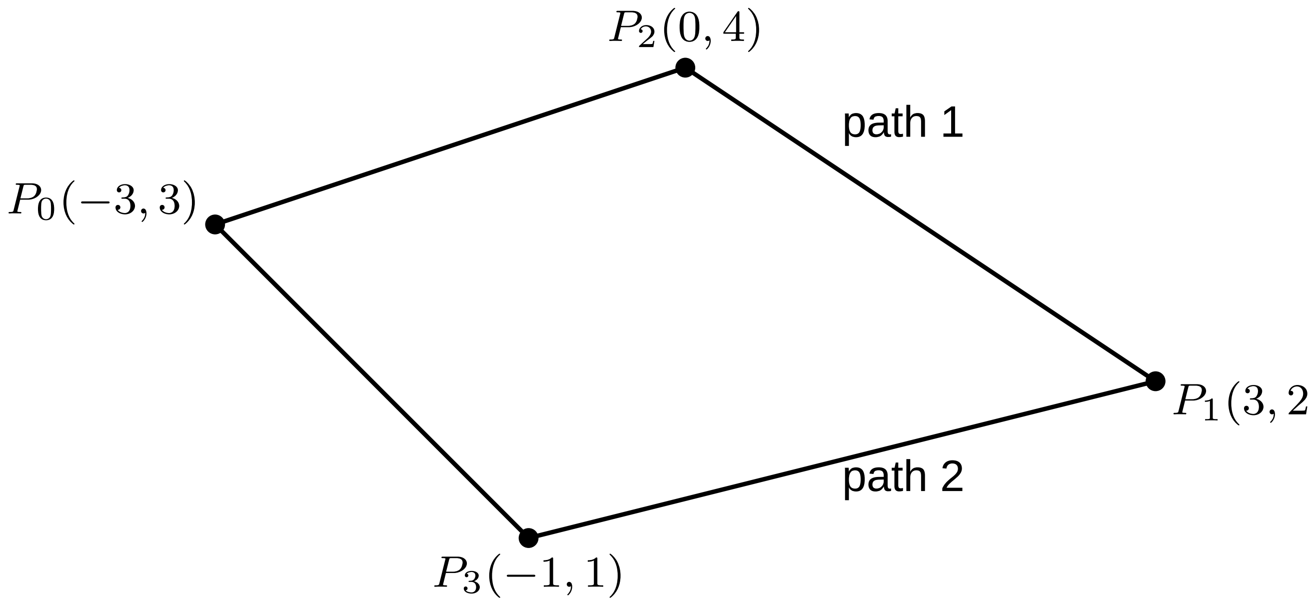

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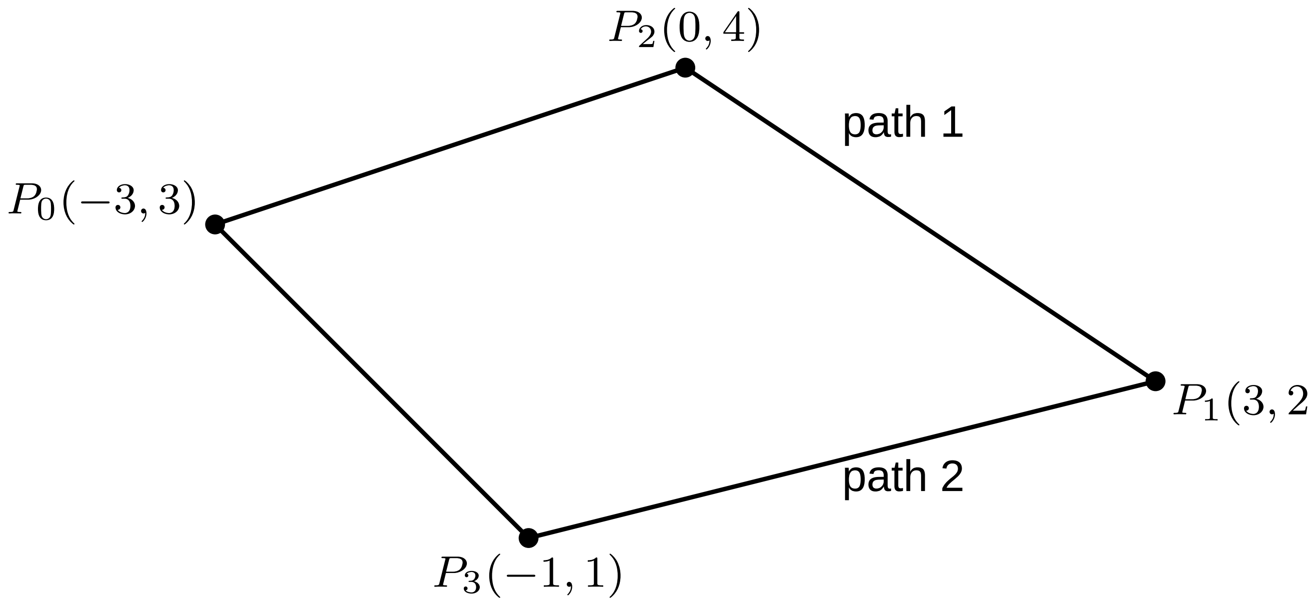

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$$\begin{aligned}
\overline{\text{path 1}} &= \overline{P_0P_2} + \overline{P_2P_1} = \sqrt{(-3-0)^2 + (3-4)^2} + \sqrt{(0-3)^2 + (4-2)^2} \\
&= \sqrt{10} + \sqrt{13} \\
\overline{\text{path 2}} &= \overline{P_0P_3} + \overline{P_3P_1} = \sqrt{(-3-(-1))^2 + (3-1)^2} + \sqrt{(-1-3)^2 + (1-2)^2} \\
&= \sqrt{8} + \sqrt{17}
\end{aligned}$$

Determine the sign of: $\overline{\text{path 1}} - \overline{\text{path 2}} = \boxed{\sqrt{10} + \sqrt{13} - \sqrt{8} - \sqrt{17}}$

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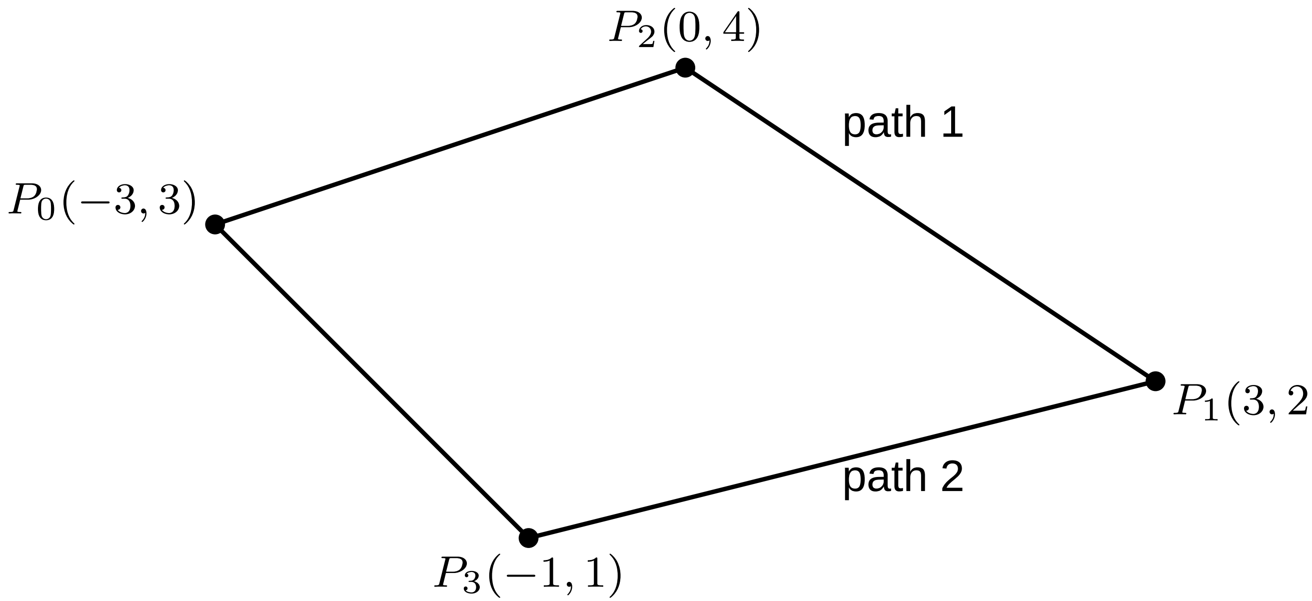

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Determine the sign of: $\overline{\text{path 1}} - \overline{\text{path 2}} = \boxed{\sqrt{10} + \sqrt{13} - \sqrt{8} - \sqrt{17}} = -0.183704\ldots$

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$\sqrt{2} + \sqrt{5 - 2\sqrt{6}}$?? $\sqrt{3}$

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[Scheinerman, Amer. Math. Monthly, 2000]

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$\longrightarrow$ Zero problem becomes important!

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    - always gives an exact answer (for algebraic expressions).
    - hard for general *transcendental expressions*.
    - uniformly slow.
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  $\longrightarrow$ We adopt numerical approach!
- Trivial with Real RAM model – not realistic
- We need decidability with TM!

Bounding Roots of a Polynomial Away From Zero

Example: $x = \sqrt{2} + \sqrt{5 - 2\sqrt{6}} - \sqrt{3}$.

$\longrightarrow x^4(x^4 - 40x^2 + 16) = 0$

Suppose $x \neq 0$. Then $x^4 - 40x^2 + 16 = 0$.

$\longrightarrow \left(\frac{1}{x}\right)^4 - \frac{5}{2}\left(\frac{1}{x}\right)^2 + \frac{1}{16} = 0$.

$\longrightarrow$ Cauchy's bound: $\left|\frac{1}{x}\right| < 1 + \max\left\{|1|, \left|-\frac{5}{2}\right|, \left|\frac{1}{16}\right|\right\} = \frac{7}{2}$.

$\longrightarrow |x| > \frac{2}{7}$.

But $x = 0.0\ldots$, a contradiction. $\longrightarrow x = 0$.

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♣ Classical bounds: *Inefficient and Ineffective!*

$\rightarrow$ *Constructive Root Bounds*

Constructive Root Bound

- Some modern bounds: Degree-Measure [Mignotte (1982)], Degree-Height & Degree-Length [Yap-Dubé (1994)], BFMS [Burnikel et al (1989)], Eigenvalue [Scheinerman (2000)], Conjugate [Li-Yap (2001)], BFMSS [Burnikel et al (2001)], k-ary [Pion-Yap (2002)]
- For each step of operations $\{\pm, \times, /, \sqrt[k]{\ }\}$, can determine the resulting sufficient precision bit for the zero test in terms of those of the arguments.
- $\alpha = \sqrt{x} + \sqrt{y} - \sqrt{x + y + 2\sqrt{xy}}$, $\quad x = a/b$, $y = c/d$, $\quad a, b, c, d$: L-bit integers
  $\rightarrow$ The number of bits sufficient to determine the zero problem for α: 96L + 30 (BFMSS), 28L + 60 (Li-Yap), . . .
- Key to Exact Geometric Computation (EGC).
- *No general bound for transcendental expressions!*

Algebraic Numbers and Expressions

Def: $\alpha \in \mathbb{C}$ is *algebraic*, if $p(\alpha) = 0$ for some nontrivial $p \in \mathbb{Z}[x]$.

- Natural numbers, rational numbers, $\sqrt{2}$, i, ...
- Closed under $\pm$, $\times$, $\div$, $\texttt{RootOf}()$
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Algebraic Expressions:

- Start with $\mathbb{N}$, and successively apply operations $\{\pm, \times, \div, \mathtt{RootOf}()\}$. (e.g., $\sqrt{2} + \sqrt{5 - 2\sqrt{6}} - \sqrt{3}$)
- Values are algebraic numbers if well-defined. (closedness)
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Def: $\alpha \in \mathbb{C}$ is *transcendental*, if α is not algebraic.

- e, π, ...
- most of the numbers are transcendental (uncountable)
- *transcendental expressions*: $\exp(1 - \cos 5)$ (cf., $\exp(\log 3)$)

Algebraic Problems

- Inputs are *algebraic*. (often $\mathbb{Z}$ or $\mathbb{Q}$)
- Involves zero problems for algebraic expressions only.
  Ex. relative configuration of line & circle:
  Given a line $l : ax + by + c = 0$ and a circle $C : (x - d)^2 + (y - e)^2 = r^2$
  with rational inputs a, b, c, d, e, r, determine the relation between them.
  $\longrightarrow$ Determine the sign of the discriminant D, which is *algebraic*.
- Most of the known problems in discrete algorithm.
- *Decidable in TM-sense.*

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- Input: algebraic
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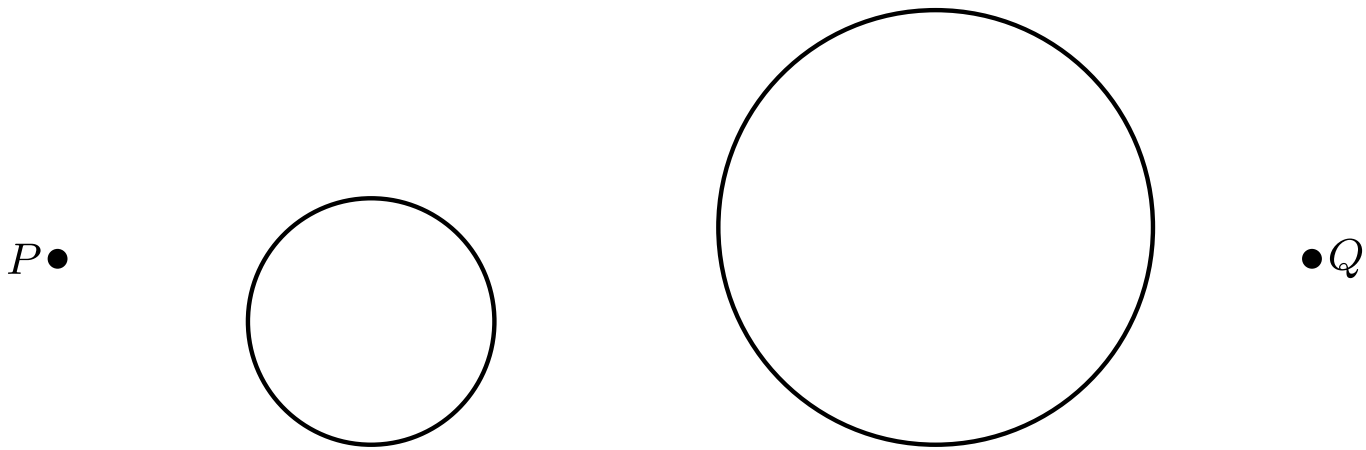

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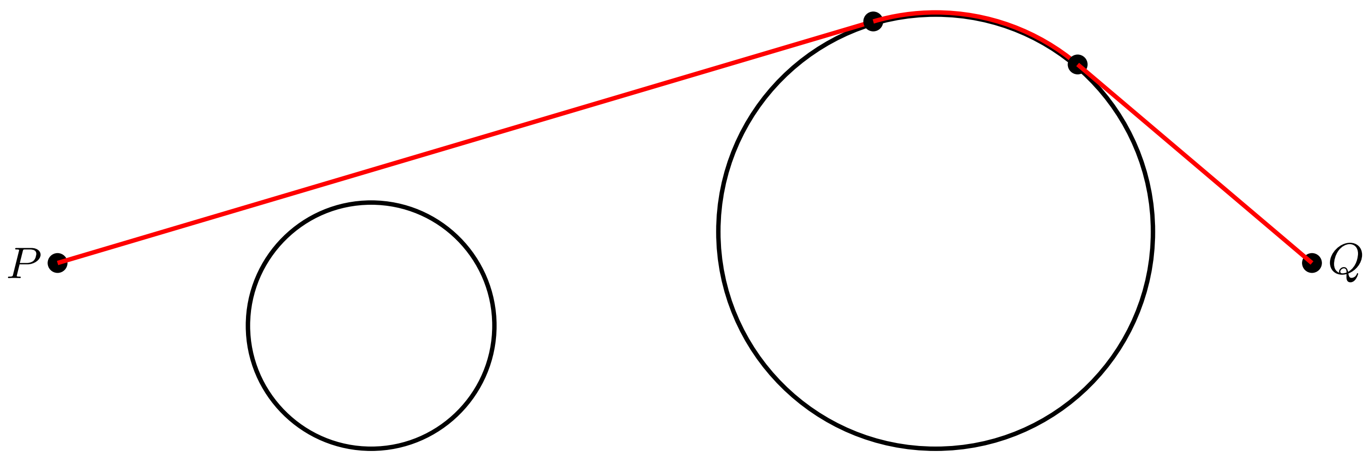

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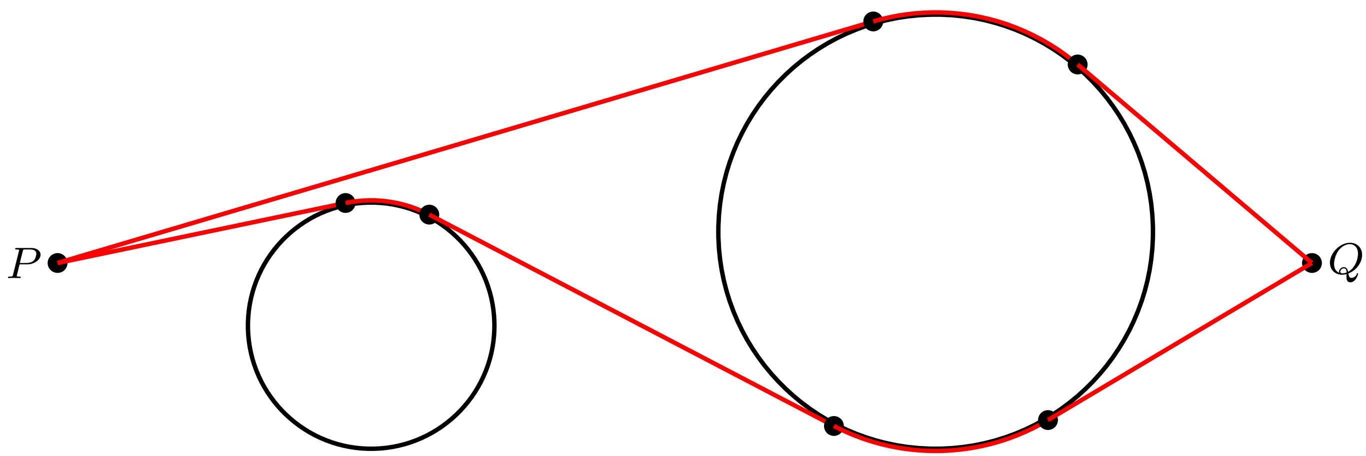

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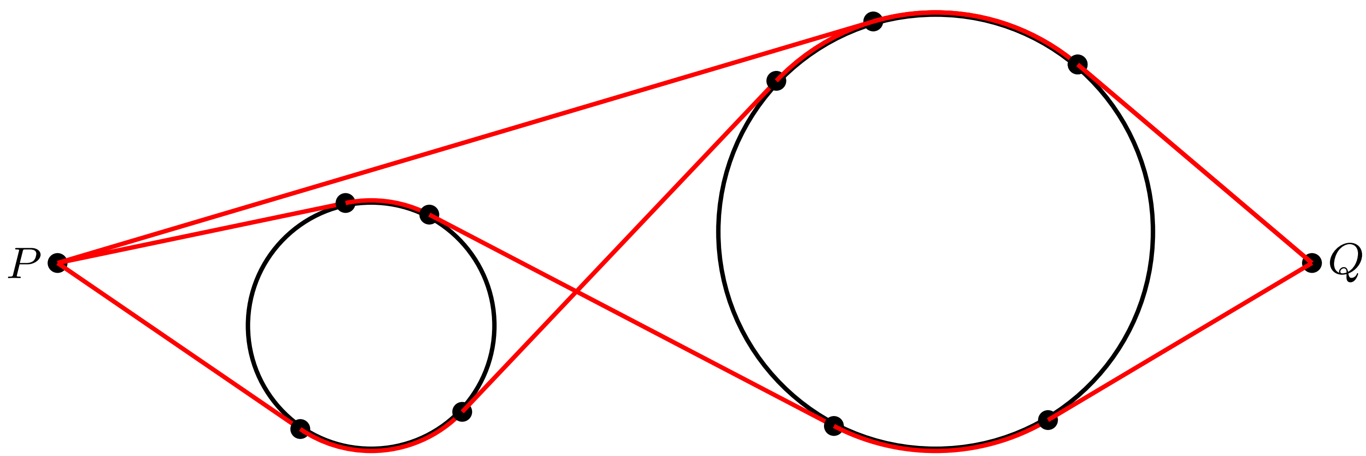

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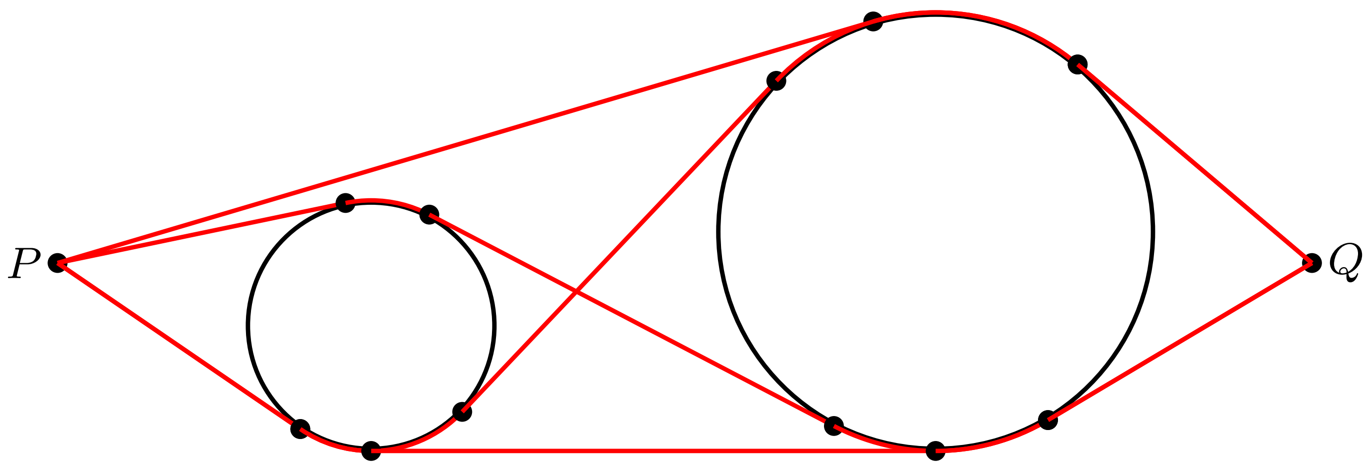

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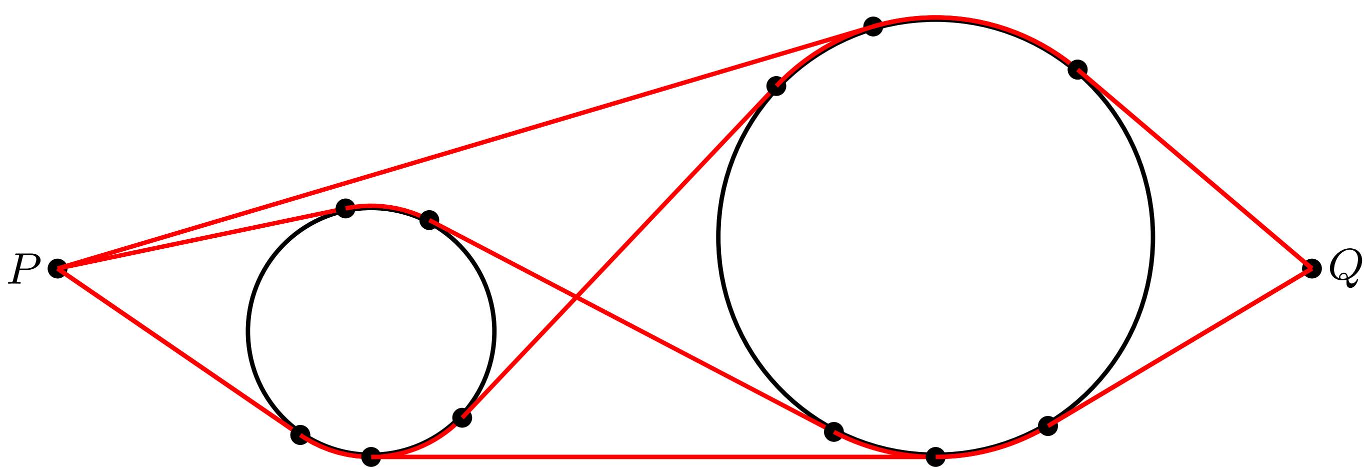
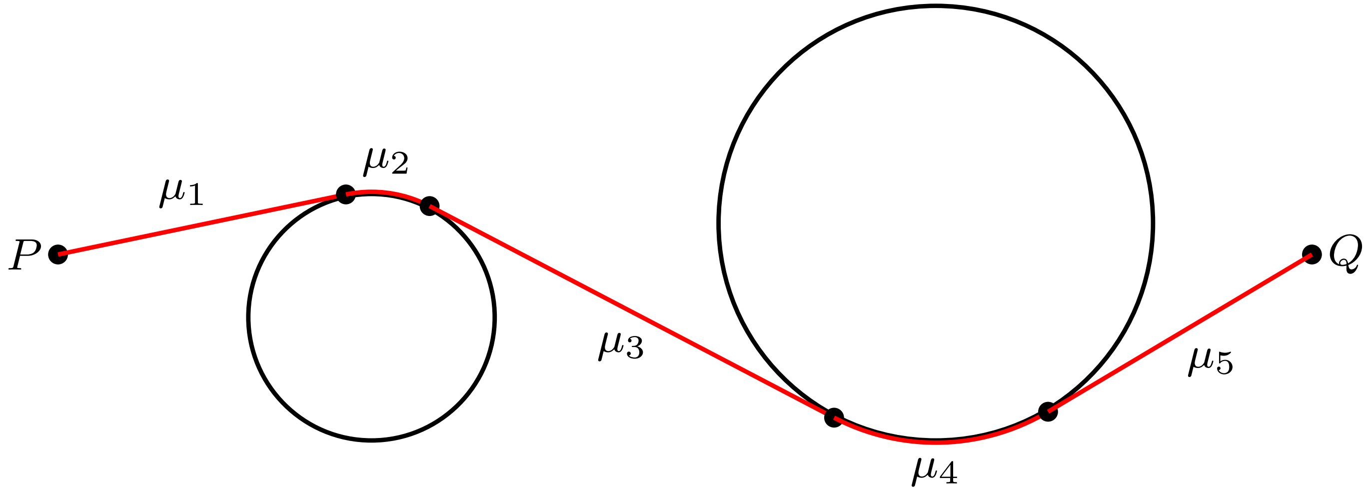


- Assume: each coord. of P, Q, centers of C_i, radii of C_i are all algebraic.
- Seemingly a typical problem in computational geometry – *feasible paths*.
- *The first* nontrivial example of a transcendental problem which turned out to be TM decidable. [Chang et al, Int. J. Comp. Geom. Appl. 2006]

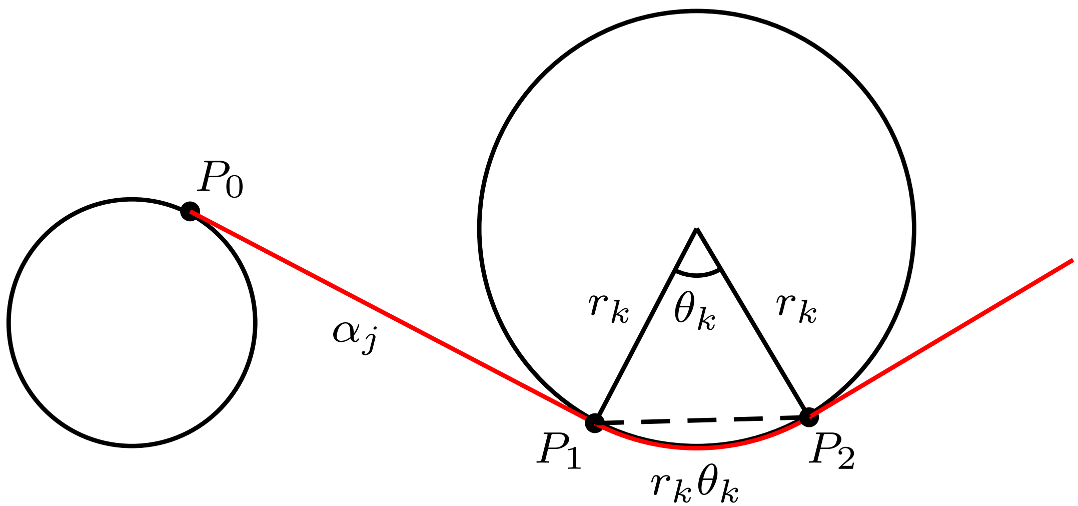
Overall Approach



- Find Feasible Paths: $\mu = \mu_1; \mu_2; \cdots; \mu_k$
  - Alternating between line segments and circular arcs
  - Boundary points are *algebraic*.
  - Sum up the lengths of $\begin{cases} \text{line segments:} & \sqrt{(\cdot - \cdot)^2 + (\cdot - \cdot)^2} \\ \text{circular arcs:} & r \cdot \theta \end{cases}$
- Apply Dijkstra's Algorithm:
  - Compute a combinatorial (weighted) graph $G = (V, E)$, where vertices V: discs & edges E: joining two discs.
  - $O(n^2 \log n)$, where n: # discs

Length of Feasible Path

$$\rightarrow \quad d(\mu) = \sum_i d(\mu_i) = \sum_j \alpha_j + \sum_k r_k \theta_k$$



- $\sum \alpha_j$: length of line segments $\Rightarrow \alpha_j = \overline{P_0 P_1}$ algebraic
- $\sum r_k\theta_k$: length of circular arcs
- $\cos\theta_k = \frac{r_k^2 + r_k^2 - \overline{P_1 P_2}^2}{2 r_i \cdot r_i}$: algebraic $\Rightarrow \theta_k$: transcendental (Lindemann's Lemma)

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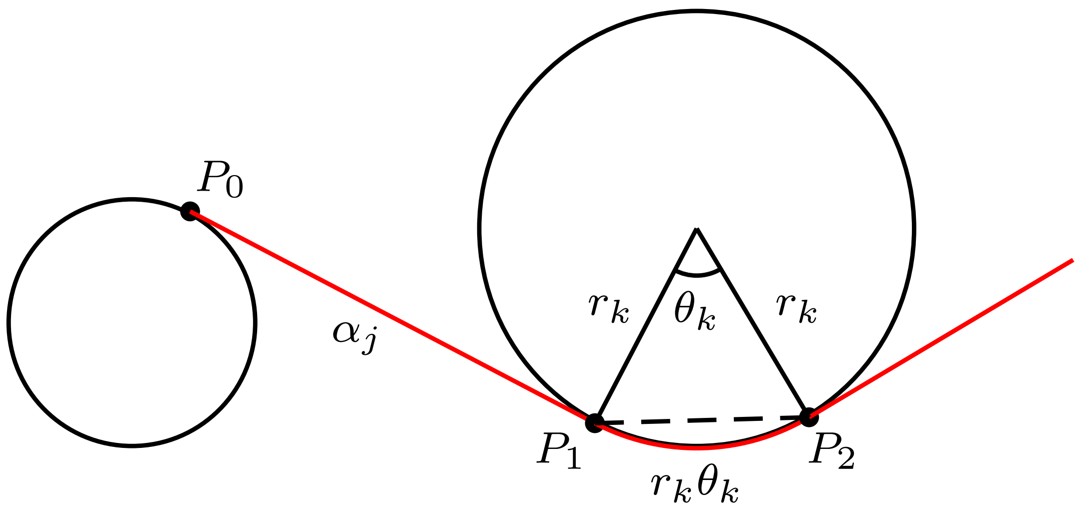

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Comparison of Two Feasible Paths:

$$d(\mu_1) - d(\mu_2) \rightarrow \alpha + r_1\theta_1 + \cdots + r_n\theta_n \quad \alpha, r_i\text{: algebraic}, \theta_i\text{: transcendental}$$

Decidability

We have to solve the zero problem for:

$$\begin{aligned}
\overline{\Lambda} &= \alpha + r_1\theta_1 + \cdots + r_n\theta_n \\
&= \alpha - ir_1 \log e^{i\theta_1} - \cdots - ir_n \log e^{i\theta_n} \\
&= \boxed{\alpha + (-ir_1)\log\left(\cos\theta_1 \pm i\sqrt{1-\cos^2\theta_1}\right) + \cdots + (-ir_1)\log\left(\cos\theta_1 \pm i\sqrt{1-\cos^2\theta_1}\right)}
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Baker's Theorem Let $\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ be nonzero algebraic numbers, with their degrees $\leq d$ and heights $\leq H$. let

$$\Lambda = \alpha_0 + \alpha_1 \log \beta_1 + \cdots \alpha_n \log \beta_n \quad \text{(linear forms in logarithms).}$$

If $\Lambda \neq 0$, then $\exists$ constant $C = C(n, d, H)$ s.t. $|\Lambda| > 2^{-C}$.

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- So the problem is *transcendental* but *decidable*!
- How many bits are needed to solve the zero problem?

Some Definitions

$\alpha \in \mathbb{C}$: algebraic & $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$: its *minimal polynomial*

- *Conjugates*: roots of p
- *Degree*: $\deg(\alpha) := \deg(p) = n$
- *Height*: $H(\alpha) := \max_{0 \le i \le n} |a_i|$
- *Absolute logarithmic height*: $h(\alpha) := \frac{1}{\deg(\alpha)} \log M(\alpha)$
- *Mahler measure*: $M(\alpha) := |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}$, where $\alpha_1, \cdots, \alpha_n$ are all the conjugates of α.

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Example: $\alpha = p/q$, ($p, q \in \mathbb{Z}$ are relatively prime.)

$\longrightarrow$

$$
\begin{aligned}
\text{minimal poly.} &= qx - p, & \deg(\alpha) &= 1, \\
\text{conjugates} &= p/q, & H(\alpha) &= \max\{|p|, |q|\}, \\
M(\alpha) = |q| \max\{1, |p/q|\} &= \max\{|p|, |q|\}, & h(\alpha) &= \max\{\log|p|, \log|q|\}
\end{aligned}
$$

Effective Bound from Transcendental Number Theory

Theorem. (Waldschmidt) For $n \geq 2$, let $\gamma_0, \gamma_1, \cdots, \gamma_n$ be algebraic numbers, and let $\beta_1, \cdots, \beta_n$ be nonzero algebraic numbers. If

$$\Lambda := \gamma_0 + \gamma_1 \log \beta_1 + \cdots + \gamma_n \log \beta_n \neq 0,$$

then

$$|\Lambda| > \exp\{-2^{8n+51} n^{2n} D^{n+2} V_1 \cdots V_n (W + \log(EDV_n^+))(\log(EDV_{n-1}^+))(\log E)^{-n-1}\},$$

where

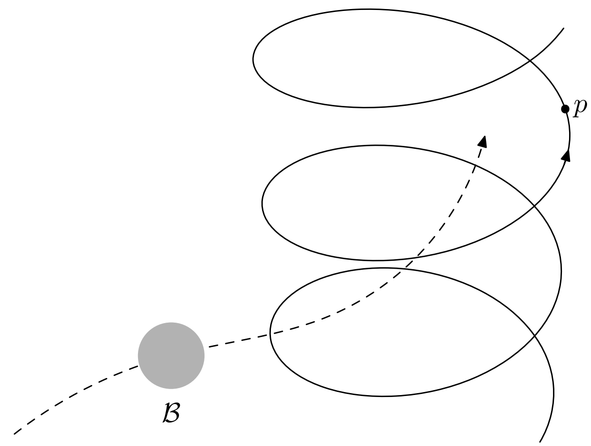
$$\begin{aligned} D &\geq [\mathbb{Q}(\gamma_0, \gamma_1, \cdots, \gamma_n, \beta_1, \cdots, \beta_n) : \mathbb{Q}], & W &\geq \max_{0 \leq j \leq n} \{h(\gamma_j)\}, \\ V_j &\geq \max\{h(\beta_j), |\log \beta_j|/D, 1/D\}, & V_1 &\leq \cdots \leq V_n, \\ V_{n-1}^+ &= \max\{V_{n-1}, 1\}, & V_n^+ &= \max\{V_n, 1\}. \\ 1 < E &\leq \min\{e^{DV_1}, \min_{1 \leq j \leq n} \{4DV_j/|\log \beta_j|\}\}. \end{aligned}$$

Bit Complexity

- Assume the input is L-*bit rational numbers* (P/Q, where P, Q are L-bit integers. ($|P|, |Q| < 2^L$)), and N is the number of discs.
- Detailed estimation gives: $|\overline{\Lambda}| > \exp\left[-2^{O(N^2+N\log L)}\right]$.
- The number of bits we need to expand to compare the lengths of two feasible paths is $2^{O(N^2+N\log L)}$.

Collision Detection Involving Helical Motion

Given a helical motion $h(t) = (\cos t, \sin t, s \cdot t)$ of a point p and an algebraic motion $c(t) = (c_1(t), c_2(t), c_3(t))$ of a ball $\mathcal{B}$ with radius r, determine exactly whether they will collide.



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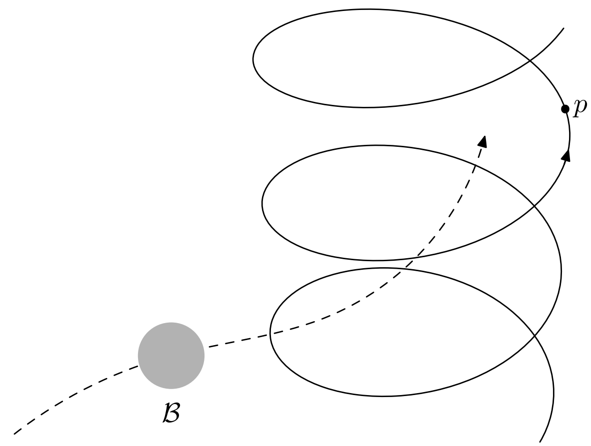

- Assume algebraic input: s, r, c_i algebraic
  - $c_i(t)$ algebraic, if $\exists P(x,y) \in \mathbb{Z}[x,y]$ $s.t.$ $P(c_i(t), t) \equiv 0$
- Natural question (e.g. in CAD)
- If both motions are algebraic $\longrightarrow$ becomes an algebraic problem.

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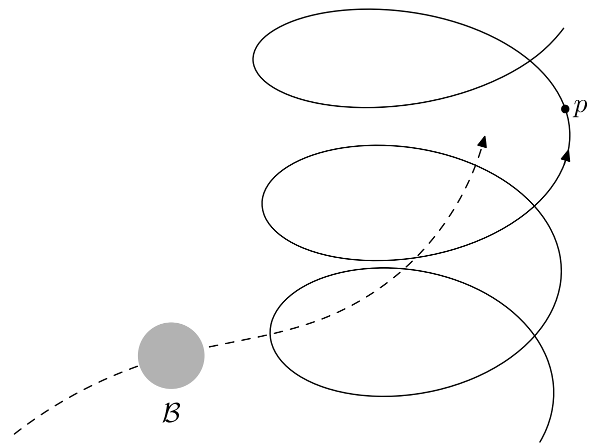

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- If both motions are algebraic $\longrightarrow$ becomes an algebraic problem.
- *Turns out to be another (the second) nontrivial transcendental problem which is decidable with TM.* [Choi et al, Real Numbers and Computers 7, 2006]

How?

$$?\exists t,\ ||h(t) - c(t)|| \leq r$$

Natural assumption: no collision initially

$$\begin{aligned}\Leftrightarrow ?\exists t,\ r^2 &= ||h(t) - c(t)||^2 \\ &= -2c_1(t)\cos t - 2c_2(t)\sin t + \left\{c_1(t)^2 + c_2(t)^2 + (st - c_3(t))^2 + 1\right\}\end{aligned}$$

$$\Leftrightarrow ?\exists t,\ a(t)\cos t + b(t)\sin t + d(t) = 0,$$

where $a(t) = -2c_1(t)$, $b(t) = -2c_2(t)$, $d(t) = c_1(t)^2 + c_2(t)^2 + (st - c_3(t))^2 + 1 - r^2$.

$$\Leftrightarrow \begin{cases} ?\exists t,\ a(t) = b(t) = d(t) = 0 \quad \longrightarrow \text{ algebraic problem} \\ \textit{or } ?\exists t,\ \alpha(t)\cos t + \beta(t)\cos t = \delta(t), \\ \quad \text{where } \alpha(t) = \frac{a(t)}{\sqrt{a(t)^2+b(t)^2}}, \beta(t) = \frac{b(t)}{\sqrt{a(t)^2+b(t)^2}}, \delta(t) = -\frac{d(t)}{a(t)^2+b(t)^2} \end{cases}$$

$$\leadsto ?\exists t,\ \cos\left(t \pm \arccos(\alpha(t))\right) = \delta(t)$$

$$\Leftrightarrow ?\exists t,\ t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) = 0 \mod 2\pi$$

$$\boxed{\Leftrightarrow ?\exists t,\ t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) + 2k\pi = 0, \quad (k\text{: between zeros of } \delta(t) \pm 1)}$$

Linear Form in Logarithms Again

$$F(t) := t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) + 2k\pi$$

$\longrightarrow$ Determine (exactly) the signs of all extremal points of F.
An extremal point t_* satisfy:

$$\begin{array}{rcl} F'(t_*) & = & 1 \pm \dfrac{\alpha'(t_*)}{\sqrt{1-\alpha(t_*)^2}} \pm \dfrac{\delta'(t_*)}{\sqrt{1-\delta(t_*)^2}} = 0 \\ \text{or} \quad \alpha(t_*) \pm 1 & = & 0 \\ \text{or} \quad \delta(t_*) \pm 1 & = & 0 \end{array}$$

$\longrightarrow$ t_* is algebraic.
$\longrightarrow$ Determine the sign of:

$$\begin{array}{rcl} F(t_*) & = & t_* \pm \arccos(\alpha(t_*)) \pm \arccos(\delta(t_*)) + 2k \arccos(-1) \\ & = & \boxed{t_* \pm i \log\left\{\alpha(t_*) \pm i\sqrt{1-\alpha(t_*)}\right\} \pm i \log\left\{\delta(t_*) \pm i\sqrt{1-\delta(t_*)}\right\} \pm 2ki \log(-1)} \end{array}$$

$\longrightarrow$ Linear forms in logarithms! $\longrightarrow$ Decidable by Baker's Theorem

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- Input Assumption:
  - $c_1(t), c_2(t), c_3(t) \in \mathbb{Q}[t]$, $s, t \in \mathbb{Q}$.
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- By Waldscmidt's theorem, we get:
  - $|F(t_*)| > \exp\left[-O\left(L^3 \log L \cdot N^2 8 (\log N)^{13}\right)\right]$, if $F(t_*) \neq 0$.
  - We need $O\left(L^3 \log L \cdot N^{28} (\log N)^{13}\right)$ bits to solve the zero problem for one $F(t_*)$. $\longrightarrow$ polynomial time!

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Thank you!