Zeroes, Nullstellensatz, and Effective Elimination

Computational Algebra Workshop

Zeroes 2007

Korean Institute for Advanced Study
Seoul, Korea

19–21 July, 2007

W. Dale Brownawell

A Bit of Transcendence

A. Irrationality of e

- $\mathrm{P_N} := \sum_{\mathrm{n} \leq \mathrm{N}} (-1)^{\mathrm{n}} \frac{\mathrm{N!}}{\mathrm{n!}}$
- $\mathrm{Q_N} := \mathrm{N!}$

$$|\mathrm{P_N} - \mathrm{Q_N}/\mathrm{e}| = \mathrm{N!}\left(\frac{1}{(\mathrm{N}+1)!} - \frac{1}{(\mathrm{N}+2)!} + \frac{1}{(\mathrm{N}+3)!} + \cdots\right)$$

A Bit of Transcendence

A. Irrationality of e

- $\mathrm{P_N} := \sum_{\mathrm{n} \le \mathrm{N}} (-1)^{\mathrm{n}} \frac{\mathrm{N}!}{\mathrm{n}!}$
- $\mathrm{Q_N} := \mathrm{N}!$

$$|\mathrm{P_N} - \mathrm{Q_N}/\mathrm{e}| = \mathrm{N}!\left(\frac{1}{(\mathrm{N}+1)!} - \frac{1}{(\mathrm{N}+2)!} + \frac{1}{(\mathrm{N}+3)!} + \cdots\right)$$

Three Main Remarks: (say with N odd)

- The quantity $\mathrm{P_N} - \mathrm{Q_N}/\mathrm{e}$ is non-zero (even positive)!
- If e were rational, then for all $\mathrm{N} > \mathrm{N_0}$, $\mathrm{Q_N}/\mathrm{e}$ and therefore $\mathrm{P_N} - \mathrm{Q_N}/\mathrm{e}$ would be an integer.
- But $0 < \mathrm{P_N} - \mathrm{Q_N}/\mathrm{e} < 1/(\mathrm{N}+1)$.

Here we see the basic steps: We consider algebraic expressions in the number we are interested in. The expressions are shown to be non-zero. Then we show that the expressions are too small to allow the number to be rational.

We need to recall the following basic fact from classical elimination theory:

Classical Resultant Expansion

$$\mathrm{Res}(P, Q) \quad = \quad b_n^m \prod_{Q(\alpha)=0} P(\alpha) =$$

$$\det \begin{vmatrix} a_0 & a_1 & \dots & a_m & 0 & 0 & & \dots & 0 \\ 0 & a_0 & \dots & a_{m-1} & & a_m & 0 & \dots & 0 \\ & & \dots & \dots & & & & & \\ & & \dots & \dots & a_0 & a_1 & & & a_m \\ b_0 & b_1 & \dots & & b_{n-1} & b_n & 0 & \dots & 0 \\ 0 & b_0 & \dots & b_{n-1} & b_n & \dots & 0 & & \\ & & \dots & \dots & & & & & \\ & & \dots & \dots & & & & & b_n \end{vmatrix} =$$

$$\det \begin{vmatrix} a_0 & a_1 & \dots & a_m & 0 & 0 & & \dots & x^{n-1}P(x) \\ 0 & a_0 & \dots & a_{m-1} & & a_m & 0 & \dots & x^{n-2}P(x) \\ & & \dots & \dots & & & & & \vdots \\ & & \dots & \dots & a_0 & a_1 & & & P(x) \\ b_0 & b_1 & \dots & & b_{n-1} & b_n & 0 & \dots & x^{m-1}Q(x) \\ 0 & b_0 & \dots & b_{n-1} & b_n & \dots & 0 & & \\ & & \dots & \dots & & & & & \vdots \\ & & \dots & \dots & & & & & Q(x) \end{vmatrix}$$

Moral(s) of story

Moral(s) of story

- Relatively prime polynomials cannot be simultaneously small.
- Small polynomials at algebraic points vanish there: $\mathrm{P}(\alpha) = 0, \gcd(\mathrm{P}, \mathrm{Q}) = 1 \Rightarrow$
$$|\mathrm{Q}(\alpha)| \geq (\mathrm{m}+\mathrm{n})^{-(\mathrm{m}+\mathrm{n})}\mathrm{H}(\mathrm{Q})^{-\deg \mathrm{P}}\mathrm{H}(\mathrm{P})^{-\deg \mathrm{Q}}$$

Moral(s) of story

- Relatively prime polynomials cannot be simultaneously small.
- Small polynomials at algebraic points vanish there: $\mathrm{P}(\alpha) = 0, \gcd(\mathrm{P}, \mathrm{Q}) = 1 \Rightarrow$
$$|\mathrm{Q}(\alpha)| \geq (\mathrm{m}+\mathrm{n})^{-(\mathrm{m}+\mathrm{n})} \mathrm{H}(\mathrm{Q})^{-\deg \mathrm{P}} \mathrm{H}(\mathrm{P})^{-\deg \mathrm{Q}}$$
- Liouville's example (1844): $\lambda := \sum_{\mathrm{n}} 10^{-\mathrm{n}!} \notin \overline{\mathbb{Q}}$.

Moral(s) of story

- Relatively prime polynomials cannot be simultaneously small.
- Small polynomials at algebraic points vanish there: $\mathrm{P}(\alpha) = 0, \mathrm{gcd}(\mathrm{P}, \mathrm{Q}) = 1 \Rightarrow$
$$|\mathrm{Q}(\alpha)| \geq (\mathrm{m}+\mathrm{n})^{-(\mathrm{m}+\mathrm{n})}\mathrm{H}(\mathrm{Q})^{-\deg \mathrm{P}}\mathrm{H}(\mathrm{P})^{-\deg \mathrm{Q}}$$
- Liouville's example (1844): $\lambda := \sum_{\mathrm{n}} 10^{-\mathrm{n}!} \notin \overline{\mathbb{Q}}$.
- For $\mathrm{Q}_{\mathrm{N}}(\mathrm{x}) := (10)^{\mathrm{N}!}\mathrm{x} - (10)^{\mathrm{N}!}\sum_{\mathrm{n}=0}^{\mathrm{N}} 10^{-\mathrm{n}!}$, $\deg \mathrm{Q}_{\mathrm{N}} = 1$, $\mathrm{H}(\mathrm{Q}_{\mathrm{N}}) \sim (10)^{\mathrm{N}!}, |\mathrm{Q}_{\mathrm{N}}(\lambda)| \sim (10)^{-(\mathrm{N}+1)!} \sim \mathrm{H}(\mathrm{Q}_{\mathrm{N}})^{-(\mathrm{N}+1)}$

Moral(s) of story

- Relatively prime polynomials cannot be simultaneously small.
- Small polynomials at algebraic points vanish there: $P(\alpha) = 0, \gcd(P, Q) = 1 \Rightarrow$
$$|Q(\alpha)| \geq (m+n)^{-(m+n)} H(Q)^{-\deg P} H(P)^{-\deg Q}$$
- Liouville's example (1844): $\lambda := \sum_n 10^{-n!} \notin \overline{\mathbb{Q}}$.
- For $Q_N(x) := (10)^{N!}x - (10)^{N!} \sum_{n=0}^{N} 10^{-n!}$, $\deg Q_N = 1$, $H(Q_N) \sim (10)^{N!}, |Q_N(\lambda)| \sim (10)^{-(N+1)!} \sim H(Q_N)^{-(N+1)}$
- Criterion used in every proof of transcendence or independence.

Fredholm Series

Theorem (Mahler (1929))

The Fredholm series $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ assumes transcendental values at every non-zero $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$.

Sketch of Proof (for $\alpha = 1/2$):

a) Thue-Siegel: For $N \in \mathbb{N}$, choose $a_0(z), \ldots, a_N(z) \in \mathbb{Z}[z]$ not all zero s.t. $\deg a_i \leq N$, $|\text{coeffs } a_i| \leq N^2$ and

$$F_N(z) = \sum_0^N a_i(z) f(z)^i$$

vanishes at $z = 0$ to an order $N' \geq N^2/2$.

b) Crude Zero Estimate: The functional equation

$$f(z^2) = f(z) - z$$

shows that $f(z)$ is not an algebraic function and therefore $N' < \infty$.

Fredholm Continued

c) Upper Bound: (Analysis)

$$|F_N(\alpha^{2^k})| \sim |C_{N'}||\alpha|^{2^k N'} \leq |C_{N'}||\alpha|^{2^{k-1}N^2}$$

d) Lower Bound: (Algebra and Assumption that $f(\alpha)$ is Algebraic)
Again from the Functional Equation,
$f(z^{2^k}) = f(z) - z - z^2 - \cdots - z^{2^{k-1}}$

$$2^{kN} F_N(\alpha^{2^k}) = P_N(f(\alpha)),$$

where $P_N \in \mathbb{Z}[x]$,

$$\deg_x P_N \leq N, \qquad |\text{coeffs } P_N| \leq e^{c_1 2^k N}$$

So by the resultant inequalities

$$|F_N(\alpha^{2^k})| \geq e^{-c_2 2^k N}.$$

Transcendence Proofs

1. Produce Auxiliary Function in functions whose values give the numbers we want.

2. Show that certain interesting values are non-zero.

3. Estimate the absolute values of the values from above (analysis).

4. Show that the non-zero values cannot be small if the numbers are algebraic.

Role of Zero Estimates in Transcendence Proofs

1. Produce Auxiliary Functions in functions whose values ~~give~~ involve the numbers we want.

2. Show that certain interesting ideal of polynomials in the values are non-zero (perhaps in a certain region).

3. Estimate the absolute values of the function values from above (analysis).

4. Show that the non-zero values cannot be ~~small~~ nearly zero if the numbers are too algebraically simple/dependent.

Smallness vs. Zeroes

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

- Order of a zero (of a vector of functions) at a point gives a notion of smallness

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

- Order of a zero (of a vector of functions) at a point gives a notion of smallness
- Absolute value gives a notion of smallness

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

- Order of a zero (of a vector of functions) at a point gives a notion of smallness
- Absolute value gives a notion of smallness

From the arithmetic point of view of effective elimination theory where coefficients do not count (Nesterenko-Philippon-Brownawell), there is NO difference!

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

- Order of a zero (of a vector of functions) at a point gives a notion of smallness
- Absolute value gives a notion of smallness

From the arithmetic point of view of effective elimination theory where coefficients do not count (Nesterenko-Philippon-Brownawell), there is NO difference!

From the point of view of transcendence, BOTH are (so far) necessary.

Smallness vs. Zeroes

For a polynomial, or an ideal of polynomials:

- Order of a zero (of a vector of functions) at a point gives a notion of smallness
- Absolute value gives a notion of smallness

From the arithmetic point of view of effective elimination theory where coefficients do not count (Nesterenko-Philippon-Brownawell), there is NO difference!

From the point of view of transcendence, BOTH are (so far) necessary. The duplication “feels” redundant, but we don’t know how to unify the considerations.

Zeroes are Maniable

From the geometric point of view of multiplicity (Bezout's Theorem à la Brownawell-Masser-Wüstholz-Philippon), there is a difference.

But wait! There is a common refinement, which can treat the zeros in either of two different ways. (G. Rémond thesis) Both essentially components of Faltings heights, cf. Arakelov theory, whatever that is.

The Two Points of View

The Two Points of View

Arithmetic:

Polynomials have

- Degree and
- Coefficients from a ring with a valuation:
    - deg for polynomials

The Two Points of View

Arithmetic:

Polynomials have

- Degree and
- Coefficients from a ring with a valuation:
  - deg for polynomials
  - absolute value for integers, height for algebraic numbers.

The Two Points of View

Arithmetic:

Polynomials have

- Degree and
- Coefficients from a ring with a valuation:
  - deg for polynomials
  - absolute value for integers, height for algebraic numbers.

The Two Points of View

Arithmetic:

Polynomials have

- Degree and
- Coefficients from a ring with a valuation:
  - deg for polynomials
  - absolute value for integers, height for algebraic numbers.

Geometric: Polynomials have

- Multidegrees,

i.e. degrees with respect to various (usually two) sets of variables. Here one degree keeps track of the degrees of "coefficients".

The Two Points of View

Arithmetic:
Polynomials have

- Degree and
- Coefficients from a ring with a valuation:
  - deg for polynomials
  - absolute value for integers, height for algebraic numbers.

Geometric: Polynomials have

- Multidegrees,

i.e. degrees with respect to various (usually two) sets of variables. Here one degree keeps track of the degrees of "coefficients".

Before beginning with the first approach, we give parallel considerations:

Simple case: Zeroes of $F(z) = P(z, e^z) \in \mathbb{C}[z, e^z]$

More naive than Tijdeman: We count only zeroes of order at least two, assume P irreducible, $\deg_z P = L$, $\deg_{e^z} = M$: In both approaches, differentiate to obtain $G(z) = P_x(z, e^z) + P_y(z, e^z)e^z$.

Arithmetic Approach

Geometric Approach

$H(z) := \mathrm{Res}_y(F, G) \in \mathbb{C}[z]$

$\deg H \leq 2LM$,
$\mathrm{ord}_{z_i} H \geq \min \mathrm{ord}_{z_i} F, \mathrm{ord}_{z_i} G$
$\geq \mathrm{ord}_{z_i} F - 1$

$\sum_{z_i} \mathrm{ord}\, F - 1 \leq 2LM.$

Simple case: Zeroes of $F(z) = P(z, e^z) \in \mathbb{C}[z, e^z]$

More naive than Tijdeman: We count only zeroes of order at least two, assume P irreducible, $\deg_z P = L$, $\deg_{e^z} = M$: In both approaches, differentiate to obtain $G(z) = P_x(z, e^z) + P_y(z, e^z)e^z$.

Arithmetic Approach

$H(z) := \mathrm{Res}_y(F, G) \in \mathbb{C}[z]$

$\deg H \leq 2LM$,

$\mathrm{ord}_{z_i} H \geq \min \mathrm{ord}_{z_i} F, \mathrm{ord}_{z_i} G$

$\geq \mathrm{ord}_{z_i} F - 1$

$\sum_{z_i} \mathrm{ord}\, F - 1 \leq 2LM.$

Geometric Approach

Homogenize wrt x, y; Elements of $I = (F^h, G^h)$ vanish to order $\geq \mathrm{ord}_{z_i} F - 1$

I "is" irrelevant wrt y, has degree at most 2LM wrt x. So if we accept arbitrary powers of e^z, we can get polys in x of degree $\leq 2LM$ in I.

$\sum_{z_i} \mathrm{ord}\, F - 1 \leq 2LM.$

Chow Forms for Effective Elimination Theory

- Chow Form of a Prime Ideal, of a Cycle
- Chow Form of a Principal Ideal
- Resultant of a Chow Form and a Homogeneous Polynomial
- Degree of a Chow Form, of a Resultant (Bezout's Theorem)
- Value of a Chow Form at a Projective Point
    - For a Principal Ideal
    - For a Resultant
- Notion of complexity in constant ring (degree or size) which carries over to Chow Forms of resultants
- Lower bound on constants in terms of complexity (Liouville: upper bound on number of zeros or lower bound on algebraic numbers)

Multidegrees for Zero Estimates

- Multidegrees for multihomogeneous ideals
- Bezout-Krull-van der Waerden Theorem (Intersection degrees)

Multidegrees for Zero Estimates

- Multidegrees for multihomogeneous ideals
- Bezout-Krull-van der Waerden Theorem (Intersection degrees) Hilbert-Samuel multiplicity

Multidegrees for Zero Estimates

- Multidegrees for multihomogeneous ideals
- Bezout-Krull-van der Waerden Theorem (Intersection degrees) Hilbert-Samuel multiplicity

Q: Why bother with the more complicated looking Chow form set-up?

Multidegrees for Zero Estimates

- Multidegrees for multihomogeneous ideals
- Bezout-Krull-van der Waerden Theorem (Intersection degrees) Hilbert-Samuel multiplicity

Q: Why bother with the more complicated looking Chow form set-up?

A: Most transcendence applications require the first approach in the “second elimination” anyway.
Exception: Siegel-Shidlovsky. But historically even there the first approach partially effectivized the zero estimate known as “Shidlovsky’s Lemma”.

Chow Forms

Preliminary Remark: Planes through a Point in $\mathbb{P}^n$

Let x be represented by $[x_0, \ldots, x_n]$. All linear relations $x^{tr} \cdot u = \sum_i u_i x_i = 0$ on the coordinates x_i are generated linearly by the ones with

$$u \; = (0, \ldots, x_k, \ldots, -x_j, \ldots, 0) =: \sigma_{jk} \cdot x,$$

where

$$\sigma_{jk} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

with entries 0 except in the (j, k) and (k, j) places. between the various pairs of coordinates. A generic skew-symmetric matrix has the form $S = \sum_{j<k} s_{jk}\sigma_{jk}$ with indeterminates s_{jk}, the coefficients of a generic hyperplane through x can be given by $u := Sx$.

Chow Form of a Variety/Prime Ideal

Let R be UFD with field of quotients k. Let V be an irreducible variety in $\mathbb{P}^n$ of dimension d corresponding to a homogeneous prime ideal $\mathfrak{P}$ in $k[x_0, \ldots, x_n]$. For $j = 0, \ldots, d$, we have hyperplanes

$$H_j \colon u_{j0}x_0 + \cdots + u_{jn}x_n = 0.$$

Then

$$H_0 \cap \cdots \cap H_{d-1} \cap V =: \{\alpha_1, \ldots, \alpha_g \subset \mathbb{P}^n\}$$

are generic zeros of $\mathfrak{P}$. Normalizing say so that some coordinate of the α_i equals 1, there is $a \in R[u_0, \ldots, u_{d-1}]$ s.t.

$$F_{\mathfrak{P}}(u_0, \ldots, u_d) := a \prod_j \alpha_j \cdot u_d$$

is irreducible. Then the Chow Form

$$F_{\mathfrak{P}}(u_0, \ldots, u_d) = 0$$

exactly when $H_0 \cap \cdots \cap H_{d-1} \cap H_d \cap V$ is nonempty. This extends to cycles.

Chow Ideal: Motivation for Value of Chow Form

Writing

$$\mathrm{F}_{\mathfrak{P}}(\mathrm{xS}_0, \ldots, \mathrm{xS}_\mathrm{d}) = \sum \mathrm{p}_\mu(\mathrm{x})\mu$$

where μ are monomials in entries of the generic skew symmetric S_i gives generators of the Chow ideal of $\mathfrak{P}$, "almost generators" of $\mathfrak{P}$. In fact,

$$\langle \mathrm{p}_\mu(\mathrm{x}) \rangle = \mathfrak{P} \bigcap \mathcal{E},$$

where $\mathcal{E}$ is embedded in $\mathfrak{P}$. This motivates the definition of the projective "absolute value" of $\mathfrak{P}$ at a point:

$$|\mathrm{F}_{\mathfrak{P}}|_\omega := \max |\mathrm{p}_\mu(\omega)| / \|\omega\|^{(\mathrm{d}+1)\deg \mathfrak{P}}.$$

Resultant of a Chow Form and a Homogeneous Polynomial (Form)

Theorem (Nesterenko, ... Arithmetic Chow Version of Bézout for Hypersurface Sections)

If $Q \in R[x]$ is homogeneous, $Q \notin \mathfrak{P}$, then

$$\mathrm{Res}(F_{\mathfrak{P}}, Q) := a^{\deg Q} \prod Q(\alpha_i) = \prod F_j^{e_j}$$

where the F_j are the Chow forms of the isolated prime ideals $\mathfrak{P}_j$ of $(\mathfrak{P}, Q)$ and e_j is the length (multiplicity) of the $\mathfrak{P}_j$-primary component.

When $R = \mathbb{C}[x]$, and z_i a point of $\mathbb{C} \times \mathbb{C}^n$,

$$\begin{aligned} \deg_x \mathrm{Res}(F_{\mathfrak{P}}, Q) &\leq \deg F_{\mathfrak{P}} \cdot \deg Q, \\ \mathrm{ord}_{z_i} \mathrm{Res}(F_{\mathfrak{P}}, Q) &\geq \min\{\mathrm{ord}\, F_{\mathfrak{P}}, \mathrm{ord}\, Q\} \end{aligned}$$

Dimension Zero Case & Estimates over $\mathbb{Z}$

Case $\dim \mathfrak{P} = 0$

"Base" Case When $\dim \mathfrak{P} = 0$, $\mathrm{Res}(F_{\mathfrak{P}}, Q) \in R$ is simply non-zero, with the same estimate on degree when $R = \mathbb{C}[z]$.

Case $R = \mathbb{Z}$

Over $\mathbb{Z}$, (with the analogous remark when $\dim \mathfrak{P} = 0$),

$$\begin{aligned}\log \mathrm{ht}(\mathrm{Res}(F_{\mathfrak{P}}, Q)) \leq C_n(&\deg_{u_0} F_{\mathfrak{P}} \log \mathrm{ht}\, Q + \\ &\deg Q \log \mathrm{ht}\, F_{\mathfrak{P}} + \deg Q \deg F_{\mathfrak{P}}).\end{aligned}$$

$$\log |\mathrm{Res}(F_{\mathfrak{P}}, Q)|_{\omega} \leq \mathrm{RHS} + \max \log\{|F_{\mathfrak{P}}|_{\omega}, |Q|_{\omega}\}.$$

Meta-Outline for Zero Estimates

Setting: We are given a polynomial Q in z and functions $f_1, \ldots, f_n$ and a set of points Σ.

Induction: Generate "new" polynomials using properties of the f_i,

- Differential Equation or
- Addition Formula,

without losing much of order of zeros.

End.Game: Eliminate variables $x_1, \ldots, x_n$ to obtain a polynomial in z of known (bounded) degree and almost as many zeros as $Q(z, f_1, \ldots, f_n)$. Compare.

Meta-Outline for Zero Estimates

Setting We are given a polynomial Q in z and functions $f_1, \ldots, f_n$ and a set of points Σ.

Induction Generate "new" polynomials using properties of the f_i,

- Differential Equation or
- Addition Formula,

without losing much of order of zeros.

End.Game Eliminate variables $x_1, \ldots, x_n$ to obtain a polynomial in z of known (bounded) degree and almost as many zeros as $Q(z, f_1, \ldots, f_n)$. Compare.

What is a "new" polynomial?

“New” Polynomials

“New” Polynomials

Properties

- Order of zero relatively large.

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.
- Allow elimination of variables, i.e. decrease dimension

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.
- Allow elimination of variables, i.e. decrease dimension of some isolated components

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.
- Allow elimination of variables, i.e. decrease dimension of some isolated components having high order of vanishing w.r.t. degree(s)

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.
- Allow elimination of variables, i.e. decrease dimension of some isolated components having high order of vanishing w.r.t. degree(s)

How are they generated?

Two starting points

- Throw out “weak” primary components & use the original polynomial
- Concentrate on the “strong” prime components & use some good generator of the prime ideal

Properties Used

- Differential Equations (Differential Equations)
- Translation Formulas (Algebraic Groups)

“New” Polynomials

Properties

- Order of zero relatively large.
- Degree not too large.
- Allow elimination of variables, i.e. decrease dimension of some isolated components having high order of vanishing w.r.t. degree(s)

How are they generated?

Two starting points

- Throw out “weak” primary components & use the original polynomial
- Concentrate on the “strong” prime components & use some good generator of the prime ideal

Properties Used

- Differential Equations (Differential Equations)
- Translation Formulas (Algebraic Groups)

Setting of Nesterenko's Zero Estimate

Setting: A differential ring $\mathrm{S} = \mathbb{C}[\mathrm{z}, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}] = \mathrm{R}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ with operator D mimicking the effect of derivation on the ring generated by a solution of a system of linear differential equations over $\mathbb{C}[\mathrm{z}]$,

$$\mathrm{Y}' = \mathrm{A}(\mathrm{z})\mathrm{Y} :$$

- $\mathrm{Dz} := \mathrm{T}(\mathrm{z})$
- $\mathrm{Dx}_\mathrm{i} := \sum \mathrm{T}(\mathrm{z})\mathrm{a}_{\mathrm{ij}}(\mathrm{z})\mathrm{x}_\mathrm{j}$,

where $\mathrm{T}(\mathrm{z}) \in \mathrm{R}$ is a LCD of the $\mathrm{a}_{\mathrm{ij}}(\mathrm{z})$. (Differentiation alone introduces denominators from the a_{ij}.)
In this setting, Chow forms F for polynomial ideals in S and cycles have both

- $\deg \mathrm{F} := \deg_{\mathrm{u}_0} \mathrm{F}$ and
- $\deg_\mathrm{z} \mathrm{F} := \deg_\mathrm{z} \mathrm{F}$.

Fundamental Fact: Derivation Lowers Order.

Fix a solution Y of this system and define the order (of vanishing) of any polynomial $P(z, x_1, \ldots, x_n)$ at a point, say $z = 0$ to be

$$O(P) := \operatorname{ord} P := \operatorname{ord}_0(P \circ Y).$$

The basic point is that the operator D lowers $\operatorname{ord} P$ whenever $\operatorname{ord} P$ is positive.

For a prime ideal $\mathfrak{P}$ with Chow Form $F := F_{\mathfrak{P}}$, set

$$O(F) := \operatorname{ord} F := \min_{P \in \mathfrak{P}} \operatorname{ord} P.$$

Lemma

If $0 < O(\mathfrak{P}) < \infty$ and $T(0) \neq 0$, then there is a polynomial $P \notin \mathfrak{P}$ with

- $\operatorname{ord} P = (\operatorname{ord} \mathfrak{P}) - 1$ and
- $\deg_z P \leq \deg_z \mathfrak{P} + \deg T$, $\deg_x P \leq \deg \mathfrak{P}$.

D-Operator Eventually Escapes All (Interesting) Primary Components

Lemma (Nesterenko, Br-Masser)

If $\mathfrak{Q}$ is an isolated component of $\mathfrak{I}$,and $T(0) \neq 0$, then for some $k \leq$ exponent$(\mathfrak{Q})$, $D^k(\mathfrak{Q}) \not\subset \mathfrak{P}$.

The exponent of a $\mathfrak{P}$-primary ideal $\mathfrak{Q}$ is the least $e \in \mathbb{N}$ such that $\mathfrak{P}^e \subseteq \mathfrak{Q}$; it is at most the multiplicity of $\mathfrak{Q}$.

Proof of Lemma: Take $F \notin \mathfrak{P}$ but in all other primary components of $\mathfrak{I}$ and take $Q \in \mathfrak{Q}$ with k least such that $D^k Q \notin \mathfrak{P}$. Then by the Product Rule

$$D^k(FQ) \equiv FD^kQ \not\equiv 0 \mod \mathfrak{P}. \quad \square$$

Setting for Zero Estimates for Solutions of DEs

Notation.

- Let $\mathfrak{P}_d$ be the (homogenization wrt x of the) ideal of algebraic relations holding for $y_1, \ldots, y_n$ over $R = \mathbb{C}[z]$. Set $h_d = \deg_z F_{\mathfrak{P}}$, $D_d := \deg_{u_0} F_{\mathfrak{P}_d} := \deg \mathfrak{P}_d$, $t := \deg \max T(z) a_{ij}$.
- Let $Q_d := Q$ be a given polynomial (think with a high order E_0 of vanishing at $z = 0$). Set $\deg_z Q = h_Q$, $\deg_x Q = D_Q$.

- Goal: We want a function, say B, such that, if $Q_d \notin \mathfrak{P}$, then $O(Q_d) \leq B(Q_d)$.

Setting for Zero Estimates for Solutions of DEs

Notation.

- Let $\mathfrak{P}_d$ be the (homogenization wrt x of the) ideal of algebraic relations holding for $y_1, \ldots, y_n$ over $R = \mathbb{C}[z]$. Set $h_d = \deg_z F_{\mathfrak{P}}$, $D_d := \deg_{u_0} F_{\mathfrak{P}_d} := \deg \mathfrak{P}_d$, $t := \deg \max T(z) a_{ij}$.
- Let $Q_d := Q$ be a given polynomial (think with a high order E_0 of vanishing at $z = 0$). Set $\deg_z Q = h_Q$, $\deg_x Q = D_Q$.

- Goal: We want a function, say B, such that, if $Q_d \notin \mathfrak{P}$, then $O(Q_d) \leq B(Q_d)$.

- Strategy: Differentiate our way out of ideals.

Outline of Nesterenko's Proof when $T(0) \neq 0$

Recursion. Set $i = d$. While $i \geq 0$, let F_{i-1} be a Chow form obtained by omitting from $\mathrm{Res}(F_i, Q_i)$ the factors based on prime ideals $\mathfrak{P}$ with $O(\mathfrak{P}) = 0$. Then

- $O(F_{i-1}) \geq \min O(F_i), O(Q_i) \quad$ and
- $\deg_z F_{i-1} \leq (d-i)h_Q D_d D^{d-i-1} + h_d D_Q^{d-i} + (d-i)tD_d^2 D^{2(d-i)}$,
- $\deg F_{i-1} \leq \deg F_i \cdot \deg Q_i \leq D_d D_Q^{d-i}$.

Let Q_{i-1} be a polynomial produced by taking sufficiently generic integral combinations of the polynomials produced by differentiating out of the primes of F_{i-1} with

$$\text{derivatives of order} \leq \deg F_{i-1} \leq D_d D_Q^{d-i}$$

so that $\deg_x Q_{i-1} \leq D_Q$

- $\deg_z Q_{i-1} \leq h_Q + tD_d(D + \cdots + D^{d-i+1})$
- $O(Q_{i-1}) \geq 0(F_i) - D_d D_Q^{d-i}$.

Upshot

Theorem (Nesterenko $\pm\epsilon$)

If $P(z, x) \in \mathbb{C}[z, x_1, \ldots, x_n]$ with $O(P) < \infty$, then

$$O(P) \leq dh_Q D_d D^{d-1} + h_d D_Q^d + dtD_d^2 D^{2d} + dD_d D^d.$$

What happens when $T(0) = 0$? Then the upper bound increases essentially by a constant factor depending on the system.

Lemma (Nesterenko)

There is a constant $B \geq 0$ s.t. if $\mathfrak{P}$ is a prime ideal of S with $D(\mathfrak{P}) \subset \mathfrak{P}$, then $O(\mathfrak{P})$ is either ∞ or $\leq B$.

Reason: If there is a prime ideal invariant under $T(z)D$ and properly containing the ideal of relations on Y, then there is a minimal one $\mathfrak{P}_0$.

Remark on Method

Cancelling "weak" primary components allowed us to generate new polynomials based on the original one. Nesterenko's first method used derivatives of the the Chow ideal. So at each step the degrees w.r.t. x of the Q_i would essentially square. So the bound would involve $h_Q D^{2^n}$. Actually Nesterenko had:

$$h_q D^{(n+1)^{n+1}},$$

which he later improved to the optimal(!) $h_q D^d$.
The key to this improvement is to show that, in fact, the degrees of an ideal behave almost as if the ideal were a complete intersection!

Degrees of Polynomial Elements and Hilbert Functions

Theorem (Nesterenko)

Let the prime ideal $\mathfrak{P} \subset \mathbb{C}[z, x_0, \ldots, x_n]$ be homogeneous in the x. Then there is $P \in \mathfrak{P}$ which is homogeneous in the x such that

- $\deg_x P \leq C_1(\deg \mathfrak{P})^{1/(1+\operatorname{codim} \mathfrak{P})}$ and
- $\deg_z P \leq C_2(\deg_z \mathfrak{P})/(\deg \mathfrak{P})^{\operatorname{codim} \mathfrak{P}/(1+\operatorname{codim} \mathfrak{P})}$.

This has the effect of letting the "new polynomials" always have about the same degrees as the original one. Thus the degree in z after all the resultants have been formed is, up to a constant, $h_d D^d + h_Q D_d D^{d-1} + D_d D^d$.
However other results still utilize the approach outlined of omitting "weak primes".

Liouville-Łojasiewicz Inequality

Theorem (Br-…-Hickel-Br)

Let $P_1, \ldots, P_m \in \mathbb{Z}[x]$ be (ordinary) non-zero polynomials of degrees at most D with $\log|\text{coeffs}| \leq h$. Then there are integers e_0, e_1, e_2 depending only on the P_i such that, if Z_0 denotes the isolated finite zeros of the P_i, then

$$\max \frac{|P_i(\omega)|}{1+\|\omega\|^{\deg P_i}} \geq Ce^{-C(h+D)D^n} \frac{\mathrm{dist}(\omega, Z)^{e_1}}{1+\|\omega\|^{e_2}} \min\{1, \mathrm{dist}(\omega, Z_0)^{e_0}\},$$

where $e_0 + e_1 + e_2 \leq D^n$, $e_1 \leq e_2$, and the C's are explicit constants depending only on n, d.

The exponents e_0, e_1 refer to multiplicities of finite components of zeroes of the P_i. So if there are none, $e_0 = e_1 = 0$.
Whenever $P_i \in k[x]$ where k is an algebraically closed field with valuations satisfying a product formula, a similar result holds.
Without the product formula, the C are no longer explicit.

Philippon's Criterion for Algebraic Independence

Theorem (Philippon)

Let $\omega \in \mathbb{C}^n$ and suppose that $\mathfrak{P} \in \mathbb{Z}[x]$ is a prime ideal of dimension d and $\deg \mathfrak{P} + \log h(\mathfrak{P}) \leq \sigma_d$ such that $\omega \in Z(\mathfrak{P})$. For $a > 1$ and $N \geq N_0$, let $\{D_N\}, \{S_N\}$ denote monotonically increasing, unbounded sequences of positive integers such that $D_{N+1} \leq D_N$, $S_{N+1} \leq aS_N$. Assume that $C > 0$ is sufficiently large and that for each $N \geq N_0$ there is an ideal J_N generated by homogeneous polynomials $P_k \in \mathbb{Z}[x]$ such that

- J_N has only finitely many zeros within the ball $B_{\rho_N}(\omega)$ of radius $\rho_N := \exp(-CD_N^d S_n \sigma_d)$ centered about ω,
- $\deg P_k \leq D_N, \quad \deg P_k + \log|\text{coeffs } P_k| \leq S_N$, and
- $\log|P_k(\omega)| \leq C^d \log \rho_N$.

Then for all $N \geq N_1$, the point ω is a zero of J_N.

Philippon's Alternative (Beginning)

One basic property used previously was that, if both F and Q are small at ω, then so is $\mathrm{Res}(F, Q)$. Philippon noted another sufficient reason for $\mathrm{Res}(F, Q)$ to be small. It uses the notion of projective distance:

$$\mathrm{dist}(\omega, \theta) := \frac{\max |\omega_i \theta_j - \omega_j \theta_i|}{(\max |\omega_i|)(\max |\theta_i|)}$$

Lemma (Philippon)

If F is the Chow form of a homogeneous prime ideal of $\mathbb{C}[x]$ intersecting $\mathbb{Z}$ only in 0, and if, for each of its zeros β,

$$\|Q\|_\omega \le \mathrm{dist}(\omega, \theta)^\mu,$$

where $0 \le \mu \le 1$, then

$$\| \mathrm{Res}(F, Q) \| \le |F|_\omega^\mu H(F)^{\deg Q} H(Q)^{\deg F} e^{8n(\deg F \deg Q)}.$$

Philippon's Partial Converse

Lemma

If F is the Chow form of a homogeneous prime ideal of $\mathbb{C}[x]$ of dimension $d \geq 0$, then for every $\omega \in \mathbb{P}^n(\mathbb{C})$, there is a zero β of $\mathfrak{P}$ such that

$$\mathrm{dist}(\omega, \beta)^{\deg F} \leq \|F\|_\omega e^{3n^2 \deg F}.$$

This allows Philippon to use either Nesterenko's inequality or

$$\|\operatorname{Res}(F, Q)\| \leq |F|_\omega^\mu H(F)^{\deg Q} H(Q)^{\deg F} e^{8n(\deg F \deg Q)}.$$

of the previous slide depending on the location of common zeros in establishing his criterion.

Linear Independence in Commutative Algebraic Groups (Gelfond-Schneider-Baker Theory)

Theorem (Gelfond-Schneider)

If $e^a \in \overline{\mathbb{Q}}, a \neq 0, b \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then $(e^a)^b \notin \overline{\mathbb{Q}}$.

Gelfond's Proof.

Consider $z \mapsto (e^z, e^{bz}) \subset \mathbb{G}_m(\mathbb{C})^2$. Form an auxiliary function $F_N(z) = \sum_{i,j \leq N} c_{i,j} e^{(i+jb)z}$ having zeroes

- at points $\Gamma([\sqrt{N \log N}]) = \{0, a, \ldots, [\sqrt{N \log N}] \cdot a = \ell_0 a\}$
- of order $N_0 \geq [\sqrt{N^3 / \log N}]$ This leads to non-zero derivative of order $\approx N^{3/2}/(\log N)^{1/2}$ with
- absolute value $\log |F_N^{N_1}(l_1)| \approx -N^2 \log N$, $0 \leq \ell_1 \leq \ell_0$ but
- size $(ca.N^{3/2}(\log N)^{1/2})$ to obtain a contradiction.

Thus this proof differentiates its way out of the ideal of polynomials vanishing at the points $(\mathrm{e}^{\mathrm{a}}, \mathrm{e}^{\mathrm{a}}\mathrm{b}) \in \mathbb{G}_{\mathrm{m}}(\mathbb{C})^2$.

Thus this proof differentiates its way out of the ideal of polynomials vanishing at the points $(e^a, e^a b) \in \mathbb{G}_m(\mathbb{C})^2$.

Schneider's Proof.

Consider $z \mapsto (z, e^{az}) \subset \mathbb{G}_a(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$. Form an auxiliary Function $F_N(z) = \sum c_i(z)e^{iaz}$

- $\deg c_i \leq N^3/(\log N)^2$, $i \leq N$
- at points of $\Gamma(N) = \{j + \ell b \colon j, \ell \leq L_0 = N^2/\log N\}$
- This leads to a non-zero value at $j + \ell b$ with $j, \ell \leq C_0 L_0$
  - with absolute value $|F_N(j + \ell b)| \leq -N^4/(\log N)$ but
  - size ca. $N^3/\log N$.

□

Thus this proof shifts out of the ideal of functions vanishing on the image of $\Gamma(N)$, i.e. points of the form $(\gamma, e^{a\gamma})$, $\gamma \in \Gamma(N)$.

Degrees of Homogeneous Ideals

Hilbert Functions, Hilbert Polynomials

Hilbert function of a homogeneous ideal I (unmixed) in $\mathrm{R} = \mathrm{k}[\mathrm{x}]$:

$$\mathrm{H}(\mathrm{t}, \mathrm{I}) := \dim \dim_{\mathrm{k}} \mathrm{H}_{\mathrm{t}}.$$

where the H_{t} are the homogeneous components of the graded ring $\mathrm{R}/\mathrm{I} = \mathrm{H}_0 + \mathrm{H}_1 + \dots$.

Theorem (Hilbert)

Given I, there is a polynomial $\mathrm{P}(\mathrm{I}, \mathrm{t}) \in \mathbb{Z}[\mathrm{t}]$ such that

- for every $\mathrm{t} \geq \mathrm{t}_0(\mathrm{I})$, $\mathrm{H}(\mathrm{I}, \mathrm{t}) = \mathrm{P}(\mathrm{I}, \mathrm{t})$,
- $\deg \mathrm{P} =$ (homogeneous) $\dim \mathrm{I}$, say g, and
- $\mathrm{P}(\mathrm{I}, \mathrm{t}) = (\deg \mathrm{I})/\mathrm{g}!\mathrm{t}^{\mathrm{g}} +$ lower order terms.

Multiplicities and Degrees in Unmixed Homogeneous Ideals

Fact 1 If $\mathfrak{Q}$ is $\mathfrak{P}$-primary of length $\ell_{\mathfrak{Q}}$, then $\deg \mathfrak{Q} = \ell_{\mathfrak{Q}} \deg \mathfrak{P}$.

Fact 2 If $\mathrm{I} = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_\mathrm{r}$ is a primary decomposition of the unmixed homogeneous ideal I, then $\deg \mathrm{I} = \sum \mathfrak{Q}_\mathrm{i} = \sum \ell_{\mathfrak{Q}_\mathrm{i}} \deg \mathfrak{P}_\mathrm{i}$.

Fact 3 (Bezout's Theorem Krull) If I is unmixed as above and Q is homogeneous, $\mathrm{Q} \notin \bigcup_\mathrm{i} \mathfrak{P}_\mathrm{i}$, then $\deg(\mathrm{I}, \mathrm{Q}) = (\deg \mathfrak{P}) \cdot (\deg \mathrm{Q})$.

Vanishing Along an Analytic Subgroup of a Commutative Algebraic Group

Let $\exp_{\mathrm{G}} \colon \mathbb{C}^{\mathrm{n}} \to \mathbb{G}(\mathbb{C})$ be the exponential map of a commutative algebraic group. Let $\Psi \colon \mathbb{C}^{\mathrm{k}} \to \mathbb{G}(\mathbb{C})$ be analytic with $\Psi(\mathrm{z}_0) = \mathrm{g}_0$, and let P be a polynomial. Then we say that P vanishes at g_0 to order at least T along Ψ if $\mathrm{P} \circ \Psi$ vanishes at z_0 to order at least T.

If $\mathrm{e} \in \Sigma \subset \mathbb{G}$, then we define $\Sigma(\mathrm{n}) := \{\sigma_1 + \cdots + \sigma_{\mathrm{n}} \colon \sigma_{\mathrm{i}} \in \Sigma\}$. In the statement of the theorem of Masser-Wüstholz-Philippon, let $\mathbb{G} \subset \mathbb{P}^{\mathrm{n}}(\mathbb{C})$ be a commutative connected algebraic group of dimension d, $\mathrm{e} \in \Sigma \subset \mathbb{G}(\mathbb{C})$, finite, and let Ψ be an analytic subgroup whose image contains Σ.

Zero Estimates on Commutative Algebraic Groups

Theorem (Masser-Wüstholz-Philippon)

Let P be a homogeneous polynomial of degree D not vanishing identically on $\mathbb{G}$, but vanishing at each $\Sigma(\mathrm{d})$ to order at least $\mathrm{dT}+1$, $\mathrm{T} \geq 0$. Then for some proper connected algebraic subgroup H

$$\binom{\mathrm{T}+\mathrm{s}}{\mathrm{s}} \mathrm{Card}\left(\frac{\Sigma+\mathrm{H}}{\mathrm{H}}\right)(\deg \mathrm{H})\mathrm{D}^{\dim \mathrm{H}} \leq (\deg \mathbb{G})(\mathrm{cD})^{\dim \mathbb{G}}.$$

Moreover H is contained in a translate of $\mathbb{G} \cap \{\mathrm{P}=0\}$, and it is defined in $\mathbb{G}$ as a component of the zeros of additional polynomials of degrees $\leq \mathrm{cD}$, where c depends only on $\mathbb{G}$.

Some Ideas in the Proof

The addition formulas for addition by elements $\sigma \in \Sigma$ give an action τ_σ on $\mathrm{R} = \mathbb{C}[\mathrm{x}]$. For a homogeneous ideal $\mathrm{I} \subset \mathrm{R}$, define $\tau(\mathrm{I}) := (\mathrm{I}, \ldots, \tau_\sigma \mathrm{I}, \ldots)$, where $\sigma \in \Sigma$. Let $\mathrm{I}_0 := (\mathrm{I}(\mathbb{G}), \mathrm{P})$ and define inductively

$$\mathrm{I}_0 \subset \mathrm{I}_1 \subset \mathrm{I}_2 \subset \cdots \subset \mathrm{I}_\mathrm{d},$$

where $\mathrm{I}_{\mathrm{j}+1} = \tau(\mathrm{I}_\mathrm{j})$. We have a chain of $\mathrm{d} + 1$ ideals properly containing the ideal I which has dimension d. So some

$$\dim \mathrm{I}_\mathrm{j} = \dim \mathrm{I}_{\mathrm{j}+1},$$

i.e. translation by Σ does not escape some isolated prime. We find that translation permutes certain isolated primes.
The subgroup fixing the related algebraic set is essentially our H.

Derivatives and Applications

This did not take differentiation into account. In some sense – Anderson-Baker-Coates – the action of derivatives and translations commute. So differentiate as well T times to produce $\mathrm{I}_{\mathrm{j}+1}$ from I_{j}.

Derivatives and Applications

This did not take differentiation into account. In some sense – Anderson-Baker-Coates – the action of derivatives and translations commute. So differentiate as well T times to produce $\mathrm{I}_{\mathrm{j}+1}$ from I_{j}.

Applications

In a particular case, show that the conclusion does not hold. Then neither can the polynomial vanish so much. So we obtain a non-zero value. To apply the independence criteria, we need no nearby zeroes! Masser-Wüstholz showed how.

Derivatives and Applications

This did not take differentiation into account. In some sense – Anderson-Baker-Coates – the action of derivatives and translations commute. So differentiate as well T times to produce $\mathrm{I}_{\mathrm{j}+1}$ from I_{j}.

Applications

In a particular case, show that the conclusion does not hold. Then neither can the polynomial vanish so much. So we obtain a non-zero value. To apply the independence criteria, we need no nearby zeroes! Masser-Wüstholz showed how.

Products

We often need to work with products of groups. This means treating different variables differently ...

Multihomogeneous Degrees àla van der Waerden

Let I be an unmixed ideal which is bihomogeneous, i.e. with generators simultaneously homogeneous in two different sets of variables, x, y. Set

$$\mathrm{H}(\mathrm{I}; \mathrm{s}, \mathrm{t}) := \dim_{\mathrm{k}} \mathrm{H}_{\mathrm{s},\mathrm{t}},$$

where $\mathrm{k}[\mathrm{x}, \mathrm{y}]/\mathrm{I} := \sum \mathrm{H}_{\mathrm{i},\mathrm{j}}$ is the bi-graded graded ring.

Fact 1 There is a polynomial $\mathrm{P}(\mathrm{s}, \mathrm{t}) \in \mathbb{Z}[\mathrm{s}, \mathrm{t}]$ such that when $\mathrm{s} + \mathrm{t} \geq \mathrm{n}_0$, $\mathrm{H}(\mathrm{I}; \mathrm{s}, \mathrm{t}) = \mathrm{P}(\mathrm{s}, \mathrm{t})$,

Fact 2 $\deg \mathrm{P} = (\mathrm{proj}) \dim \mathrm{I} (= \mathrm{g})$, say,

Fact 3 $\mathrm{P}(\mathrm{s}, \mathrm{t}) = \sum_{\mathrm{i}+\mathrm{j}=\mathrm{g}} (\mathrm{g}_{\mathrm{i},\mathrm{j}}/(\mathrm{i}!\mathrm{j}!))\mathrm{s}^{\mathrm{i}}\mathrm{t}^{\mathrm{j}} +$ lower terms.

Properties

Then, as before,

Fact 4 If $\mathfrak{Q}$ is $\mathfrak{P}$-primary of length $\ell_{\mathfrak{Q}}$, then each $\mathrm{g}_{\mathrm{ij}}(\mathfrak{Q}) = \ell_{\mathfrak{Q}} \mathrm{g}_{\mathrm{i,j}}(\mathfrak{P})$.

Fact 5 If $\mathrm{I} = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_{\mathrm{r}}$ is a primary decomposition of the unmixed bi-homogeneous ideal I, then each $\mathrm{g}_{\mathrm{i,j}}(\mathrm{I}) = \sum_{\mathrm{l}} \mathrm{g}_{\mathrm{i,j}}(\mathfrak{Q}_{\mathrm{l}}) = \sum_{\mathrm{l}} \ell_{\mathfrak{Q}_{\mathrm{l}}} \mathrm{g}_{\mathrm{i,j}}(\mathfrak{P}_{\mathrm{l}})$.

Fact 6 (Bezout's Theorem) If I is unmixed bihomogeneous as above and Q is bi-homogeneous, $\mathrm{Q} \notin \bigcup_{\mathrm{l}} \mathfrak{P}_{\mathrm{l}}$, then $\mathrm{g}_{\mathrm{i,j}}(\mathrm{I}, \mathrm{Q}) = \mathrm{g}_{\mathrm{i+1,j}}(\mathrm{I})(\deg_{\mathrm{x}} \mathrm{Q}) + \mathrm{g}_{\mathrm{i,j+1}}(\mathrm{I})(\deg_{\mathrm{y}} \mathrm{Q})$, written as $(\deg \mathrm{I}) * (\deg \mathrm{Q})$.

Theorem (Masser-Wüstholz-Philippon)

Let $T \in \mathbb{N}$, P be a multi-homogeneous polynomial in the blocks of variables $x_1, \ldots, x_p$, each of size $N_1, \ldots, N_p$, of multidegree $D_1, \ldots, D_p$. Assume that P vanishes along the analytic subgroup A to order $\geq nT + 1$ at every point of $\Sigma(n)$. Then there is a connnected algebraic subgroup G' of G such that:

- G' is an isolated component of an ideal generated by $I(G)$ and polynomials of multidegrees at most $(c_1 D_1, \ldots, c_p D_p)$,
- G' is contained in $G \cap \{P = 0\}$, and

$$\binom{T + \operatorname{codim}_A(A \cap G')}{\operatorname{codim}_A(A \cap G')} \operatorname{Card}\left(\frac{\Sigma + G'}{G'}\right) \times$$

$$\deg(G') * (D_1, \ldots, D_p)^{*\dim G'} \leq \deg(G) * (c_1 D_1, \ldots, c_p D_p)^{*\dim G}.$$