Novel scaling relations in systems with absorbing phase transitions

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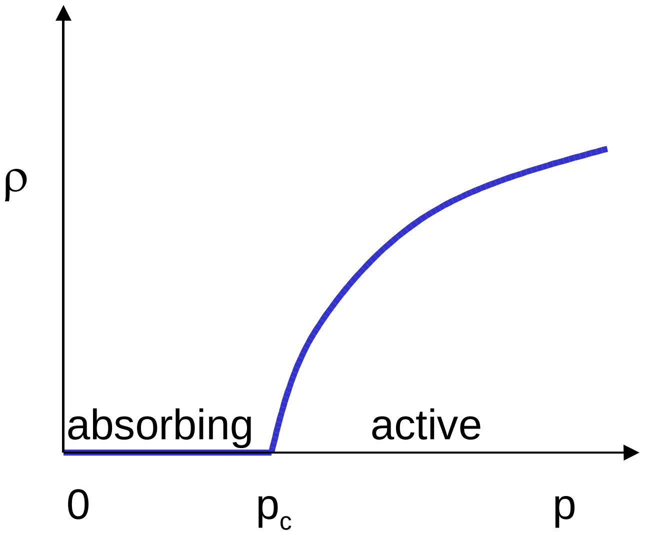
1. Reminder: Standard scaling relations in APT systems
2. Unidirectionally coupled two-species systems
3. Bosonic pair contact process with diffusion: exact results
4. Conclusions

talk based on:
S.K.+G.M.S., Physica A (in press); S.K.+G.M.S. (in prep)
M.P.+G.M.S., J.Phys. A **37** (2004), 4709-4722
M.P. cond-mat/0406xxx

1. Standard scaling relations in APT systems

Standard models: exclusion interaction + lattice reaction-diffusion kinetics

Contact process: (CP)	A → AA (infection)	rate p
	A → 0 (death)	rate 1
	[A0 ←→ 0A (diffusion)	rate d]
Pair contact process: (PCP)	AA → AAA	rate p
	AA → A	rate 1
	[A0 ←→ 0A	rate d]
Branching-annihilating walk: (BAW(2))	A → AAA	rate p
	AA → 0	rate 2
	A0 ←→ 0A	rate 1



Prototypical example for nonequilibrium phase transition: nonconservative reaction kinetics with transition into frozen absorbing state

Continuous phase transition:

offcriticality (stationary)

$$\Delta = p - p_c \rightarrow 0; \quad t = \infty$$

density: $\rho \sim \Delta^{\beta}$

survival probability: $P \sim \Delta^{\beta'}$

correlation length: $\xi \sim \Delta^{-\nu_\perp}$

relaxation time: $\tau \sim \Delta^{-\nu_{||}}$

critical point (spreading)

$$\Delta = 0; \quad t \rightarrow \infty$$

spreading density: $\rho_s \sim t^{-\alpha}$

survival probability: $P_s \sim t^{-\delta}$

spreading distance: $R \sim t^{1/z}$

particle number: $N \sim t^{\eta}$

Dynamical scaling: $\alpha = \beta/\nu_{||}$, $\delta = \beta'/\nu_{||}$, $z = \nu_\perp/\nu_{||}$

Finitely many absorbing states: $\beta = \beta'$

Hyperscaling relation: $\boxed{\eta = d/z - \alpha - \delta}$ $(d < d_c)$

Universality classes (symmetries, dimensionality, …) ?
Role of diffusion ?

Well-established universality classes:

3) directed percolation (DP)

CP, CPD, PCP, …

$$\alpha = \delta \approx 0.159464(6)$$
$$1/z \approx 0.632613(4)$$
$$\eta \approx 0.313686(8)$$

2) parity conserving (PC)

BAW(2), …

$$\alpha = \delta \approx 0.285(2)$$
$$1/z \approx 0.570(1)$$
$$\eta \approx 0$$

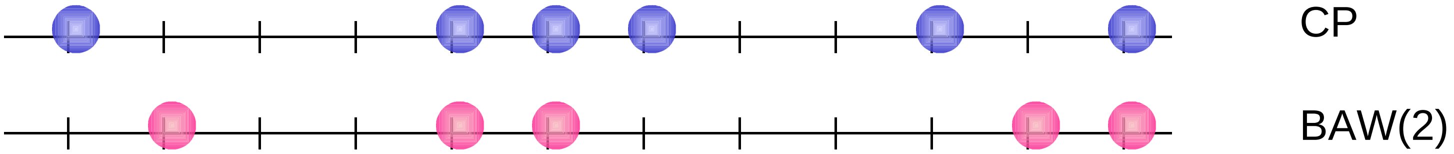
Role of diffusion in CP:
diffusion = A0 → 0A = A0 → AA → 0A = branching→decay ➔ CP ~ CPD

Role of diffusion in PCP:
0A0A0 (frozen) ➔ infinitely many absorbing states (yet DP-universality)
PCPD:
diffusion = 0A0A0 (frozen) → 00AA0 (active) ➔ universality class?

Controversial issue due to ambiguous numerical results:
2) DP
3) own universality class
4) dependence on D

2. Unidirectionally coupled two-species processes

Example: CP → BAW(2)



CP: A → 0, A0,0A → AA; BAW(2): B0 ←→ 0B, 2B→0, B00,00B → BBB

Coupling: A00 → ABB, 00A → BBA

General linear coupling:

master slave	CP	BAW(2)
CP	A → A+B	A → A+B
BAW(2)	A → A+2B	A → A+2B

Common critical point:
- master system is in A-universality class
- slave system: For DP→DP coupling Goldschmidt et al. (PRE 1999) suggest:

$$\eta_B = d/z_B - \delta_B - \delta_A$$

with $z_B = z_{DP},\ \eta_B \neq \eta_{DP},\ \delta_B = \alpha_B \neq \alpha_{DP}$

→ new critical exponents
→ how does hyperscaling relation generalize (and is it correct)?

Source average and slave average

Slave average (Goldschmidt et al):

5) Initialize system with single master particle
6) Average over all realizations that survive up to *slave level* up to time t

Source average (Kwon and G.M.S.):

11) Initialize system with single master particle
12) Average only over realizations that survive at *master level* up to time t

REASON: After source is extinguished, slave evolves autonomously according to its own standard universality class → coupling is lost

→ slave average is average over coupled system with (presumably) new exponents and uncoupled system with different (known) exponents

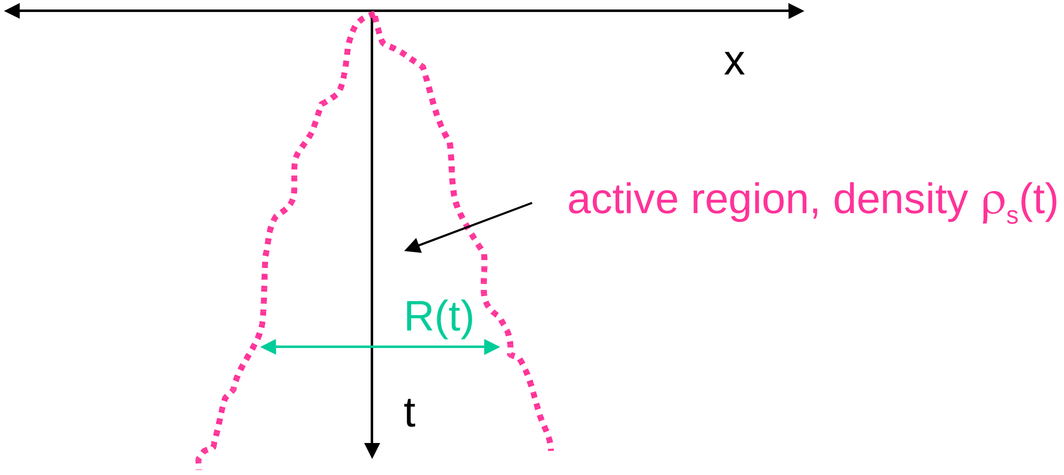
Geometric interpretation of hyperscaling relation

spreading volume → $V(t)$; spreading density → $\rho_s(t)$; survival probability → $P_s(t)$

average particle number (all realizations) →

$$N(t) = V(t) \times \rho_s(t) \times P_s(t)$$

$$V(t) \sim R^d(t)$$



➔ Hyperscaling relation with different exponents in uncoupled and coupled case

➔ valid also in discontinuous transitions into absorbing states $(\alpha \neq \delta)$

Hyperscaling relation in unidirectionally coupled systems: (I) spatial inhomogeneity

$$\boxed{R^{slave}(t) > R^{master}(t)}$$

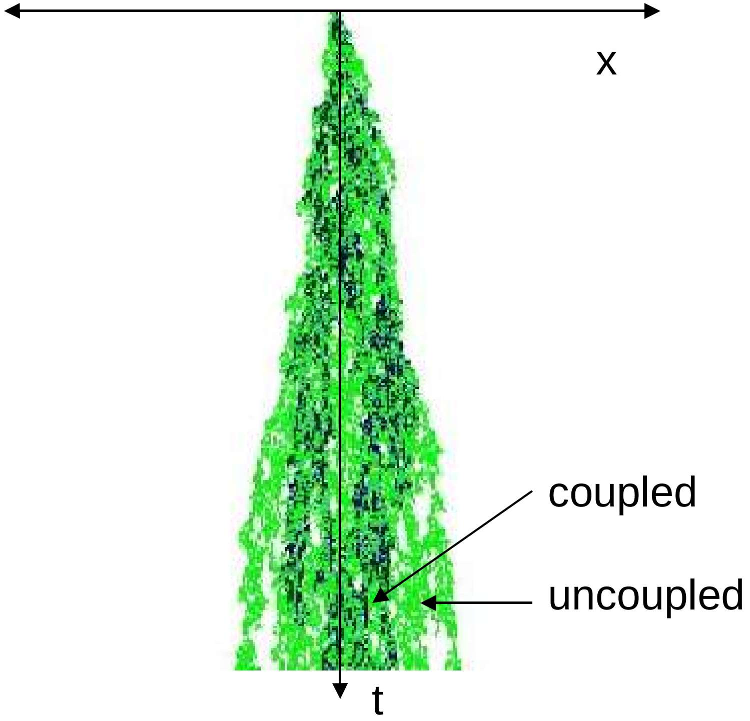
- coupled and uncoupled region
- spatially inhomogeneous particle distribution

$$R(t) = R^C(t) + R^U(t)$$
$$N(t) = N^C(t) + N^U(t)$$

new exponents:

$$z^U,\ \eta^C,\ \eta^U,\ \alpha^C,\ \alpha^U$$
$$z^C = z_A$$

$$1/z_B = \max\{1/z_A, 1/z^U\},$$
$$\eta_B = \max\{\eta^C, \eta^U\},$$
$$\alpha_B = \min\{\alpha^C, \alpha^U\}$$



Hyperscaling relation in unidirectionally coupled systems: (II) source average

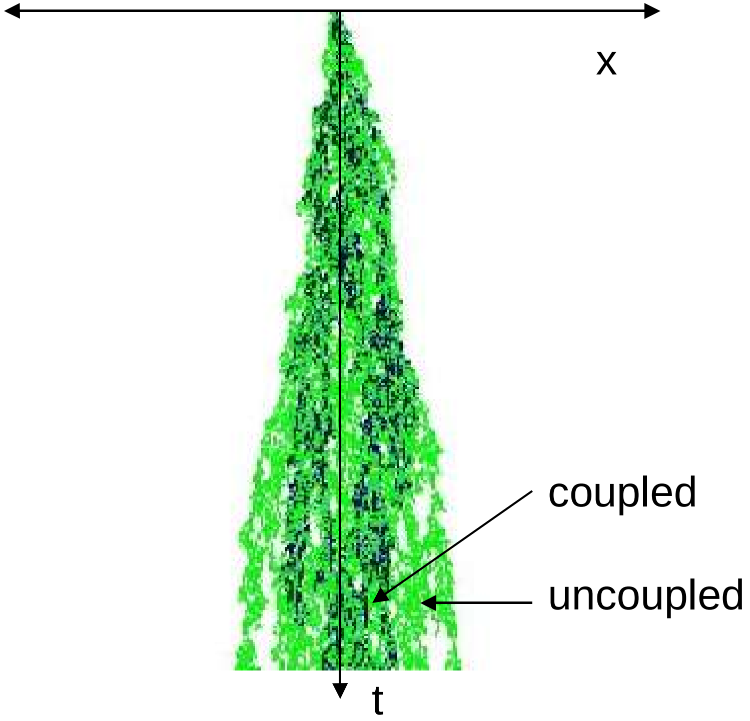
$$\boxed{P_s^{slave}(t) = P_s^{master}(t)}$$

➔ exponents: $\delta_B = \delta_A$

$\alpha_B \neq \delta_B$

- explains origin of δ_A in hyperscaling relation
- replace δ_B → α_B

- valid only for source average



➔ novel hyperscaling relations:

$$\boxed{\left.\begin{aligned} \eta^C &= d/z_A - \alpha^C - \delta_A \\ \eta^U &= d/z^U - \alpha^U - \delta_A \end{aligned}\right\} \quad \eta_B = d/z_B - \alpha_B - \delta_A}$$

Numerical results:

1) DP → DP $\eta^C = d/z_{DP} - \alpha^C - \delta_{DP}$ $\alpha^C \approx 0.08(1),\ \eta^C \approx 0.40(1)$

3) DP → PC $\eta^C = d/z_{DP} - \alpha^C - \delta_{DP}$ $\alpha^C \approx 0.13(1),\ \eta^C \approx 0.34(1)$

5) PC → DP $\eta^U = d/z_{DP} - \alpha^C - \delta_{PC}$ $\alpha^C \approx 0.14(2),\ \eta^U \approx 0.185(5)$

7) PC → PC $\eta^C = d/z_{PC} - \alpha^C - \delta_{PC}$ $\alpha^C \approx 0.21(1),\ \eta^C \approx 0.08(1)$

Comparison: extended hyperscaling ←→ measurement

	η^C_{meas}	η^C_{pred}	η^U_{meas}	η^U_{pred}
DP → DP	0.40(1)	0.39(1)	0.31(1)	0.31(2)
DP → PC	0.34(1)	0.34(1)	0.14(1)	0.16(2)
PC → DP	0.14(2)	0.15(2)	0.185(5)	0.19(2)
PC → PC	0.08(1)	0.08(1)	0.01(1)	0.00(1)

3. Bosonic pair contact process with diffusion: exact results

Importance of diffusion in APT: (1) reentrant phase transitions

(2) universality class of PCPD

Canet et al. (2004), Odor (2004),
see also Kwon and Park (1995)
Odor (2000,2003); Dickman and
de Menezes (2002)

We cannot answer the question of universality class of PCPD, but:

bosonic (unrestricted) PCPD (Howard and Tauber (1997)

$AX \leftrightarrow XA$	rate D	(n.n. hopping without interaction)
$2A \to (2+k)A$	rate μ	(on-site branching)
$2A \to (2-l)A$	rate λ	(on-site annihilation)

D=0: decoupling into single-site processes (d=0), infinitely many absorbing states

D>0: two absorbing states

any D, $\lambda > \lambda_c : \rho^* = 0$ (absorbing state); $\lambda < \lambda_c : \rho^* = 1$ (active state) any D, $\lambda = \lambda_c = \mu\, k/l$ (critical point): equations of motion for n-point functions decouple

exact results for dynamical properties

1) density and particle number: $< N(t) >= N(0)$

➔ $\rho_c = \rho_0$

➔ $\eta = 0$

Introduce effective reactivity $\alpha = \frac{\mu k(k+l)}{2D}$

define

$$\alpha_c = \left[2\int_{-\pi}^{\pi}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{2d - 2\sum_{i=1}^{d}\cos q_i}\right]^{-1} = \begin{cases} 0 & d < 2 \\ C_d & d > 2 \end{cases}$$

2) local fluctuations:

$$\sigma^2(t) \sim \begin{cases} e^{t/\tau} & \alpha > \alpha_c \\ t^y & \alpha = \alpha_c \\ const. & \alpha < \alpha_c \end{cases}$$

$$y = \begin{cases} d/2 - 1 & 2 < d < 4 \\ 1 & d > 4 \end{cases}$$

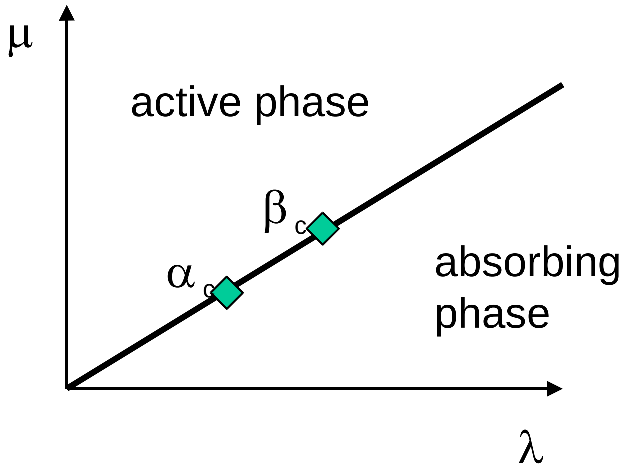
3) correlation function

$$C(r,t) = \begin{cases} \mathrm{e}^{t/\tau} G(r/\xi) & \alpha > \alpha_c \\ r^{4-d} F_d^{(1)}(r^2/t) & \alpha = \alpha_c \\ r^{2-d} F_d^{(2)}(r^2/t) & \alpha < \alpha_c \end{cases}$$

➔ z = 2

4) third moment has exponential divergence for $\alpha > \beta_c > \alpha_c$

- exactly known scaling functions for d>4
- exponentially divergent correlations, yet dynamical scaling $\tau \sim \xi^2$
- phase diagram:



Hyperscaling relation (2<d<4)

geometrical nature: separate hyperscaling for particles, pairs, occupied sites

exact: $z = 2,\ \eta^{part} = 0,\ \eta^{pair} = \begin{cases} -d/2 & \alpha < \alpha_c \\ -(2-d/2) & \alpha = \alpha_c \end{cases}$

➔ $\delta = 0$ (transience of random walk for d>2)

➔ $\eta^{site} = 0$ (vanishing pair density)

➡ $\alpha^{part} = \alpha^{site} = d/2,\ \alpha^{pair} = \begin{cases} d & \alpha < \alpha_c \\ 2 & \alpha = \alpha_c \end{cases}$

simulation data (d=3) $\delta, \eta^{site} < 0.01$

D α	sites	particles	pairs
D=0.5	1.48 (1.5)	1.49 (1.5)	--
D=0.76	1.49 (1.5)	1.49 (1.5)	1.92 (2.0)
D=10	1.50 (1.5)	1.50 (1.5)	3.58 (3.0)

Hyperscaling d=1: no scaling for exponentially divergent pairs

simulation: $\delta \approx 0.471,\ \eta^{sites} \approx -0.450$

$$\alpha^{site}_{pred} = 0.48,\ \alpha^{part}_{pred} = 0.03$$

$$\alpha^{site}_{meas} = 0.57,\ \alpha^{part}_{meas} = 0.52$$

violation of hyperscaling?

- conjecture: $\delta = \alpha^{sites} = -\eta^{sites} = 1/2,\ \alpha^{part} = 0$
- numerical errors due to huge fluctuations
- use hyperscaling in bosonic systems for occupied sites

4. Conclusions

1. unidirectional coupling of critical APT systems generates new critical behaviour where

   > importance of averaging method (source average)
   > heterogeneity of particle distribution

   ↳ - novel extended hyperscaling
   - generalization of known hyperscaling to arbitrary couplings

2. master decays more slowly than slave:
   - conjecture (a) $z_{\text{slave}} = z_{\text{source}}$, (b) other exponents new

   master decays faster than slave:
   - slave unchanged after long enough time, since master dies out (slave average yields standard exponents for slave system)

17. bosonic PCPD:

   - some exact critical exponents and scaling functions
   - transition in fluctuations depending on D
   - extended hyperscaling relations with best numerical results for occupied sites