Complex Networks: Structure and Dynamics

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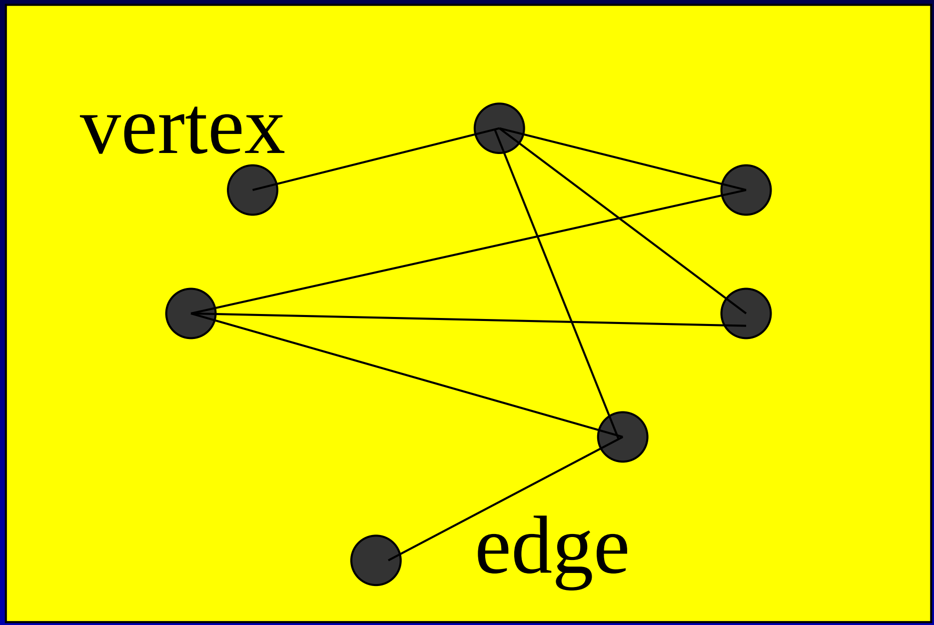
What is Network?

- System composed of vertices (nodes) and edges (links).



Examples of Networks

- Social networks: actors in movies, sexual contacts, citations in papers, scientific collaborations, karate club, etc.
- Biological networks: protein network, neuronal network, etc.
- Information networks: World-Wide-Web, Internet, etc.
- Flight connections, power grids, etc

Network is everywhere!

Complex Networks

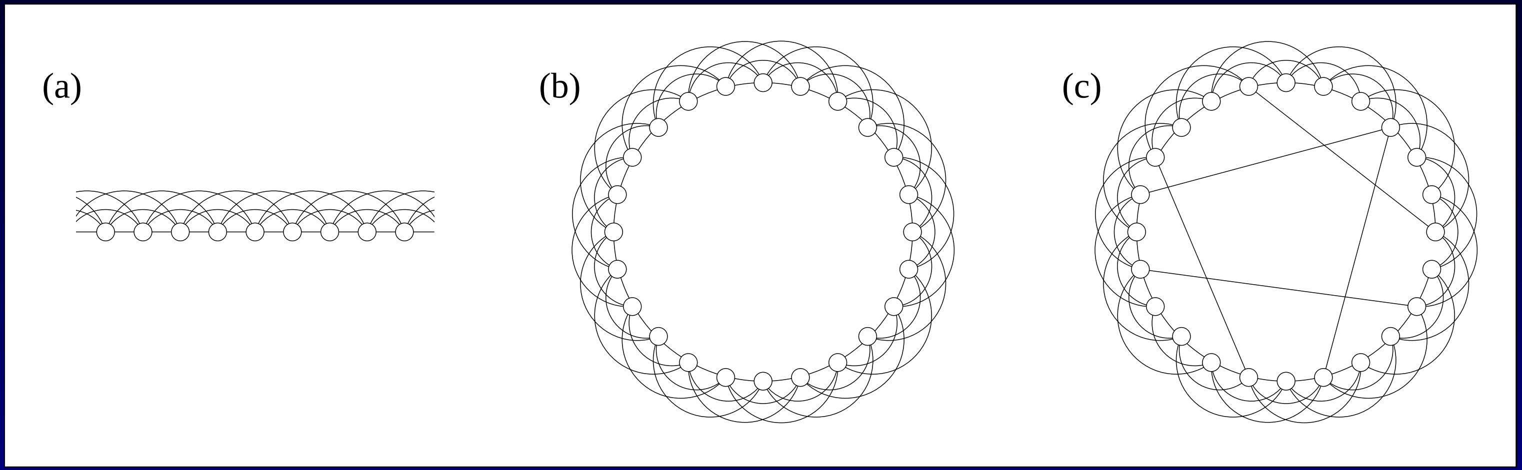
- New emerging paradigm to understand various systems in terms of their underlying topological network structures.
  - Why Internet is so vulnerable to attack by hackers and why random vaccination of AIDS virus is doomed to fail?
    $\Leftarrow$ Identical explanation.
- Two approaches
  - Study **of** complex network structure.
  - Study of a system **on** complex network.

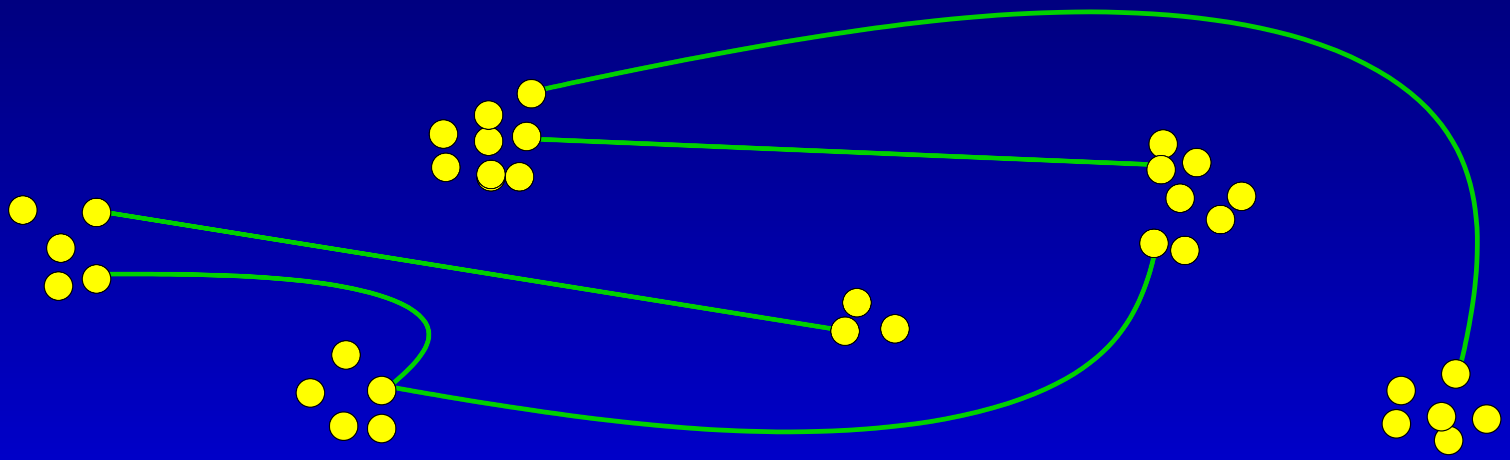
Complex Networks

In real networks, dynamics of network structure and dynamics of the system defined on network are intermingled.
Form follows function (by architect Louis Sullivan),
Function follows form.

- Zeroth order approximation: Consider vertices as empty objects and study only structural properties of network.
- First order approximation: Consider dynamic behaviors of the system on network but the network structure is assumed to be frozen.
- Still long way to go . . .

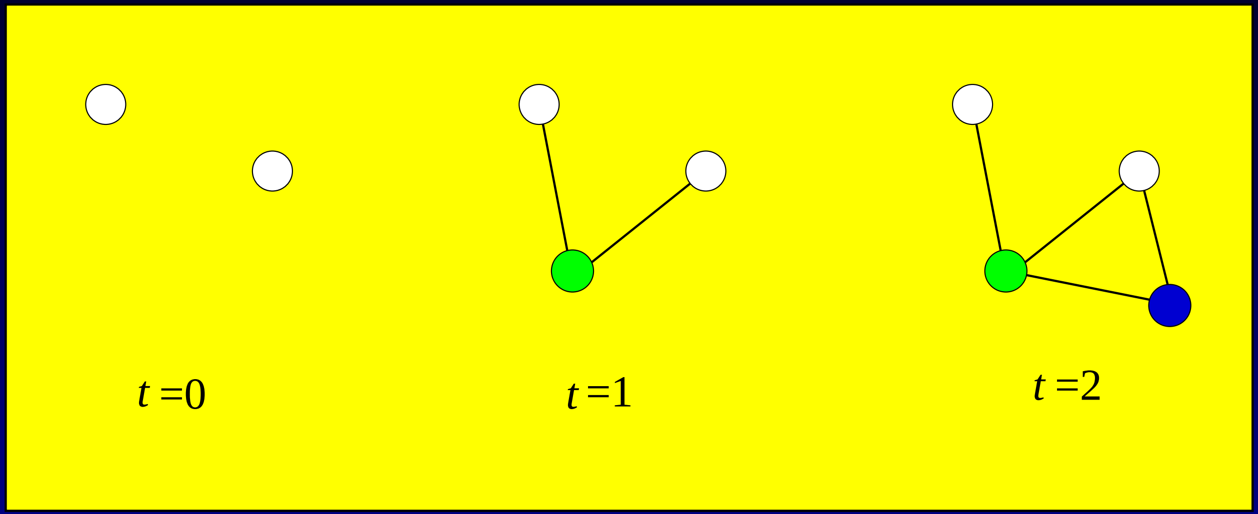
Watts-Strogatz (WS) network





Local connection + long range shortcuts.
Connection range r and rewiring probability p.

Barabasi-Albert (BA) network



- Growth: Add one vertex at every time step.
- Preferential attachment: The richer become richer. New HTML document hyperlinks well known website.

Important quantities

- Characteristic path length l(how many connection steps involved between two vertices).
  - Large-world network: $l \sim N$ (N = network size).
  - Small-world network: $l \sim \ln N$.
- Clustering coefficient γ (two of my friends are likely to be friends to each other).
- Degree distribution $P(k)$ (distribution of number of friends)
  - Scale-free networks: $P(k) \sim k^{-\alpha}$.
  - Exponential networks: $P(k)$ decays exponentially at large k.

Dynamics on Networks

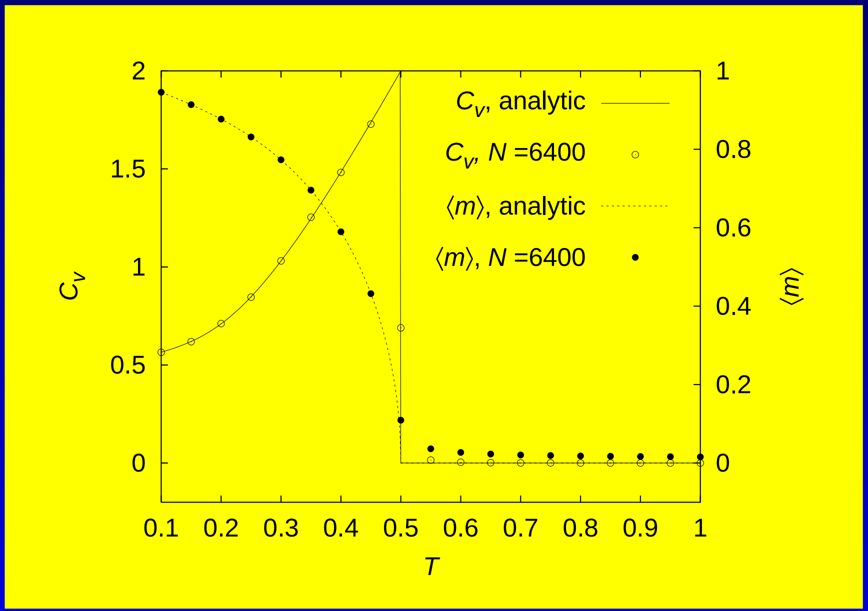
Study dynamic systems on networks: How function follows form?

- XY model on network
- Prisoner's dilemma game
- Quantum and classical diffusion
- Fiber bundle model of fracture
- Hopfield neural network model

XY model on WS network

Motivation

- Mermin-Wagner theorem: $T_c = 0$ for the XY model in local 1D network.
- Globally coupled XY model: $T_c \neq 0$.



- Hubbard-Stratonovich transform
- $\beta = 1/2$ in $m \sim (T_c - T)^\beta$

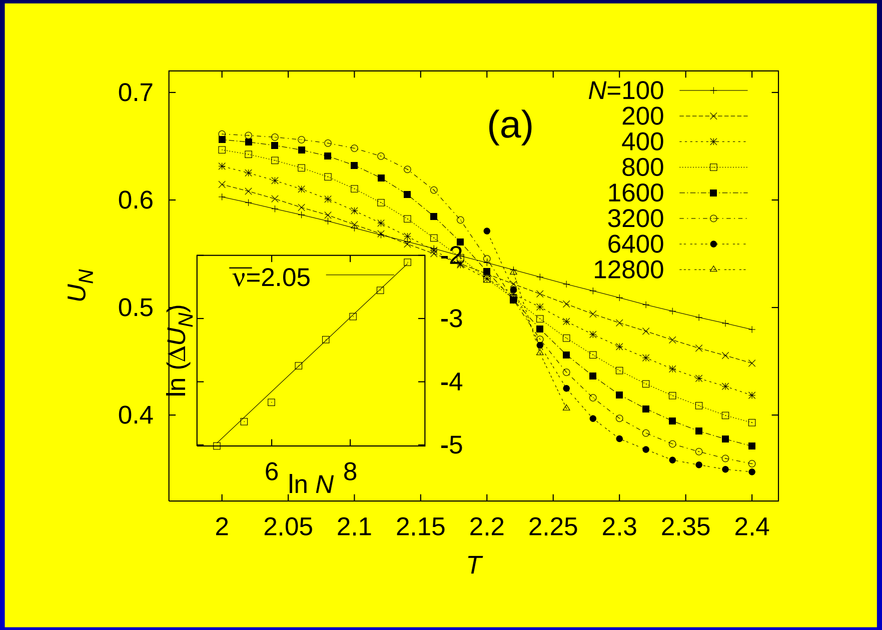
- Phase transition of XY model on WS network?

XY model on WS network

Monte Carlo simulation and Finite-size scalings

- Binder's fourth-order cumulant

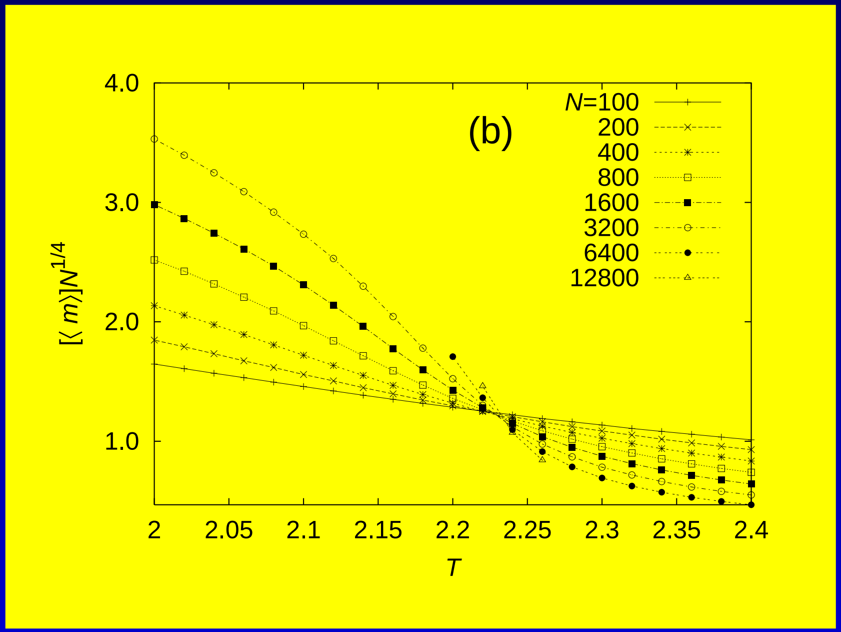
$$U_N(T) \equiv 1 - \frac{[\langle m^4 \rangle]}{3[\langle m^2 \rangle]^2}, \quad \text{with } m = \frac{1}{N}\sum_j e^{i\phi_j}.$$



For $r = 3$ and $p = 0.2$, $T_c \approx 2.235$ and $\bar{\nu} \approx 2.0$ found.
Crossing is not unique for small sizes.

XY model on WS network

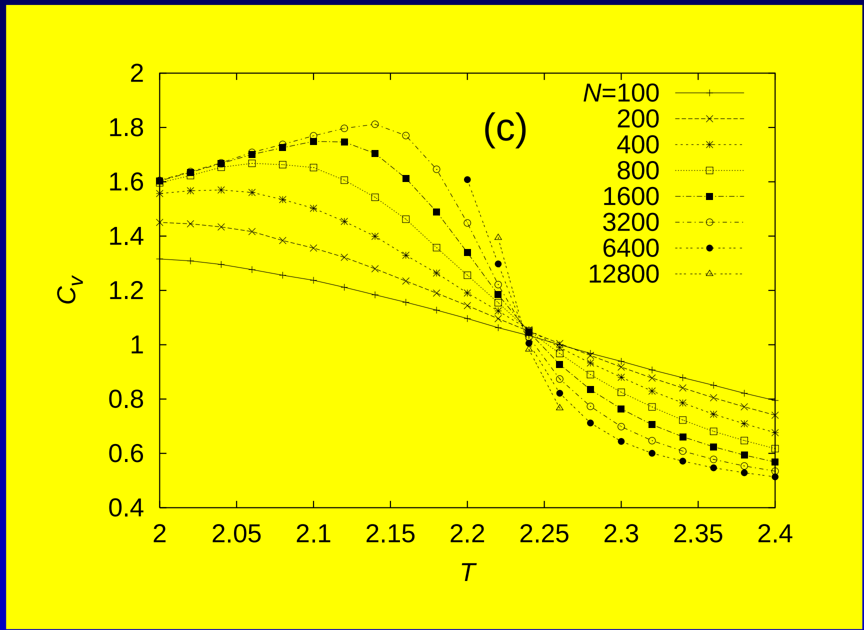
- Magnetization $[\langle m \rangle] \sim (T_c - T)^\beta$ with $[\langle m \rangle] N^{\beta/\bar{\nu}} = g\Big((T - T_c)N^{1/\bar{\nu}}\Big)$. The number of correlated vertices diverges as $\xi_N \sim |T - T_c|^{-\bar{\nu}}$.



For $r = 3$ and $p = 0.2$, $T_c \approx 2.235$ and $\beta/\bar{\nu} \approx 1/4$ are found.

XY model on WS network

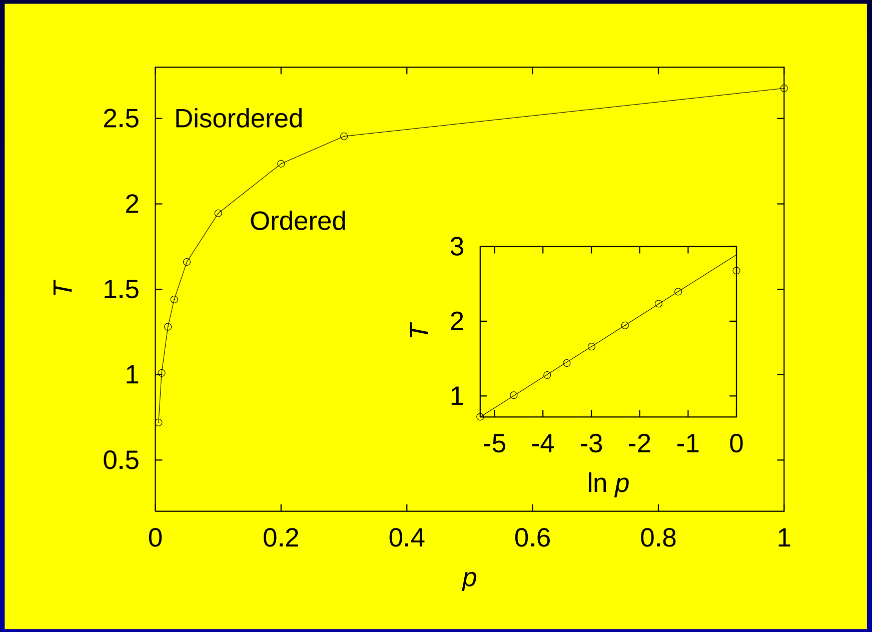
- Mean-field transition has $\bar{\nu} = 2 \Rightarrow \alpha = 0$ from the hyperscaling relation $\bar{\nu} = 2 - \alpha$.
- Specific heat $C_v = h[(T - T_c)N^{1/\bar{\nu}}]$.



For $r = 3$ and $p = 0.2$, $T_c \approx 2.235$ and $\alpha \approx 0$ are found.

XY model on WS network

Phase diagram ($p =$(# of shortcuts)/(total # of connections))



- For $p > 0.005$, finite T_c is found $\Rightarrow p_c < 0.005$, suggesting $p_c = 0$.
- Broad range of phase boundary is well described by $T_c \approx 0.41 \ln p + 2.89$: Open question.

XY model on WS network

Summary and Discussion

- Tiny fraction of shortcuts dramatically changes the phase transition in the system: no long-range order at $p = 0 \Rightarrow$ ferromagnetic long-range order exists for $p > 0$.
- The phase transition in the small-world network with $p > 0$ is of the mean-field nature characterized by $\beta = 1/2$, $\bar{\nu} = 2$, and $\alpha = 0$.
- The large-world to small-world transition at $p = 0$: the small-world network structure and the phase transition closely related.

Prisoner's Dilemma

The police ask two prisoners **A** and **B**, "who committed the crime?"

- If both say "I did not", both get light punishment because the police cannot pinpoint.
- If **A** says "I saw **B** did it", while **B** says "I did not and I do not know **A**", **A** is acquitted while B gets heavy punishment.
- If **A** says "**B** did it" and **B** says "**A** did it", both get punishment.

Dilemma: **A** defects since he worries **B**'s defection, and vice versa $\Rightarrow$ both become defectors although they can get light punishment if they cooperate together.

Prisoner’s Dilemma

There exist various examples in sociology, economics, and politics:

- You borrowed money from your friend. Should you return it?
- Two companies with contract. If one breaks the contract, it gains more in a short term.
- Russian put missile base in Cuba. What is the proper action by USA?
- Congressmen suddenly become cooperative when they decide the budget for their election districts.

Prisoner’s Dilemma

Why the cooperation is abundant in this selfish world?

- Future prospect: Defection gains only in a short term (Iterated PD game).
- Local interaction: Local neighbors tend to cooperate within the group while the same group of people can be hostile to strangers (Spatial PD game).

Emergence of cooperation without iron-fisted central authority (preferred by anarchists).

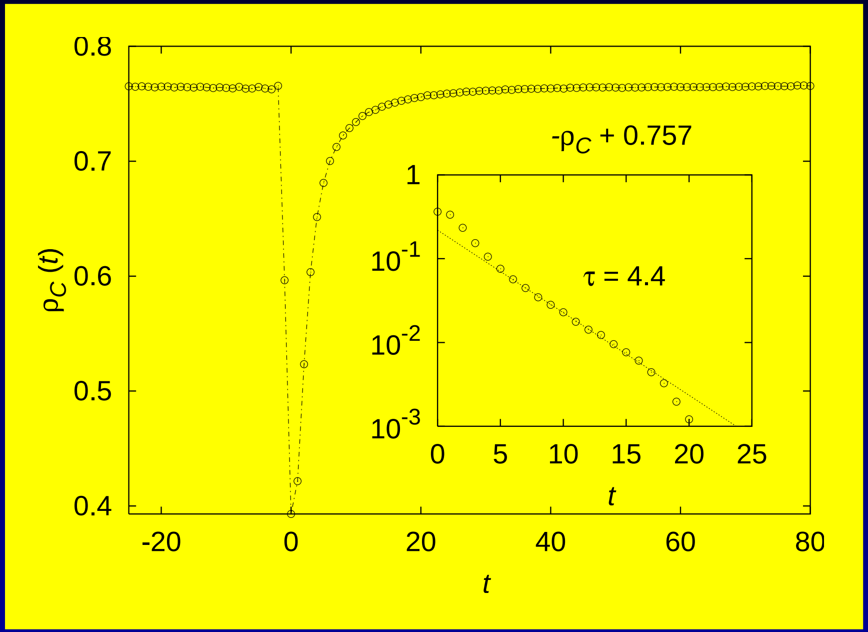
Prisoner’s Dilemma

Motivation of this work

- As far as human society is concerned, the PD game should be played on the complex network (Social game on social network).
- We use the WS network and study effects of dominant vertex in the network (E.g., important politicians, pop stars, mass media) connected to other vertices by directed edges.

Time evolution of PD game on WS network

Prisoner's Dilemma

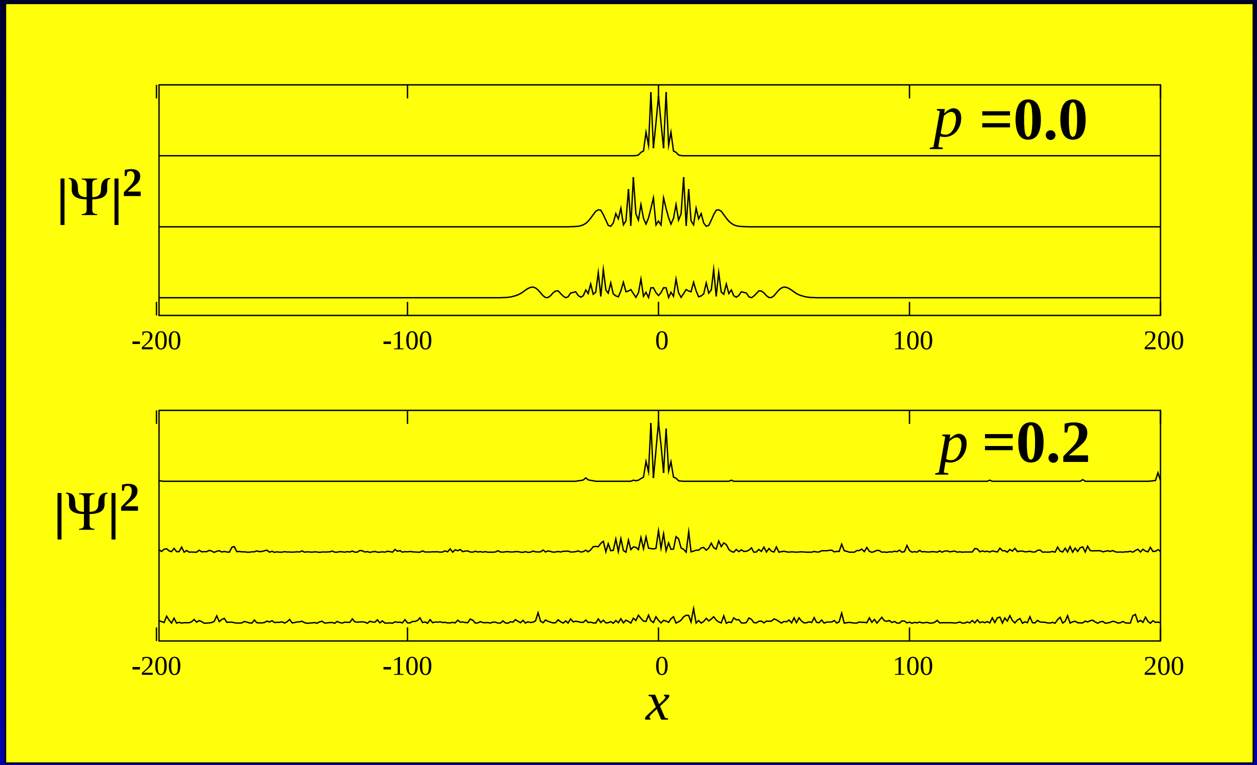


- There exists punctuated equilibrium behavior.
- Sudden breakdowns and slow recovery of cooperation: Once betrayed, the emergence of cooperation takes time.
- Influential people should behave themselves well.

Diffusion: Quantum

- Put tight-binding particle system on WS network.
  - Particle can hop to only directly connected vertices: $\langle u|H|v\rangle \neq 0$ only if u and v are connected by an edge.
  - Solve the time-dependent Schrödinger equation: $i\hbar(\partial/\partial t)|\Psi\rangle = H|\Psi\rangle$.
  - Initial condition: Localized wave packet at one vertex n: $\langle v|\Psi(t=0)\rangle = \delta_{v,n}$.
- Simplest case: $p = 0$ and $r = 1$ (regular 1D) $\Rightarrow |\langle v|\Psi(t)\rangle|^2 = |J_v(2t)|^2$ for $n = 0$.

Diffusion: Quantum



- Fast diffusion for $p \neq 0$.
- Diffusion starts at the other end of shortcut.

Diffusion: Quantum

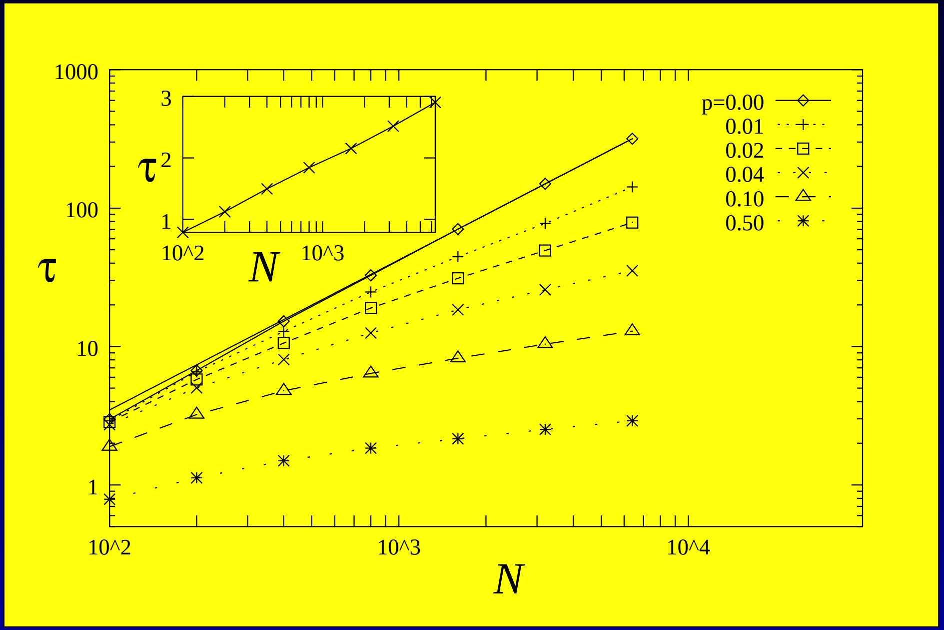
Measure the participation ratio:

$$P_Q(t) \equiv \frac{\sum_n |\Psi_n(t)|^2}{\sum_n |\Psi_n(t)|^4}.$$

- At $t = 0$, $\Psi_n = \delta_{n,0} \Rightarrow P_Q(0) = 1$.
- As $t \to \infty$, $\Psi_n \sim 1/\sqrt{N} \Rightarrow P_Q \sim N$.
- Define the diffusion time τ:

$$P_Q(t = \tau) = 0.25N.$$

Diffusion: Quantum



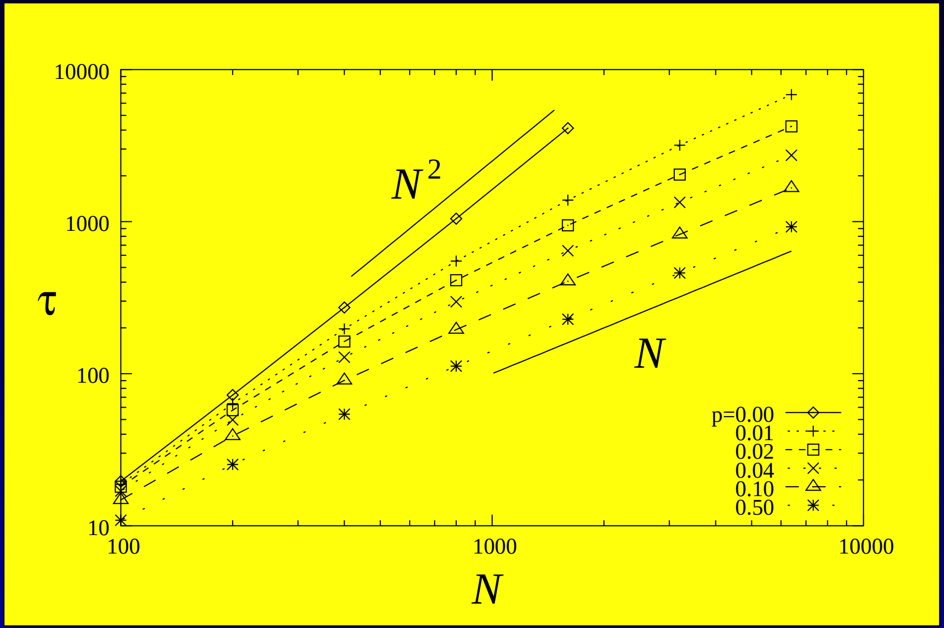
- At $p = 0$, $P_Q(t) \sim t$ for large $t \Rightarrow \tau \sim N$.
- At $p \neq 0$, $\tau \sim \log N$ is found.
- Fast-world transition at $p = 0$

Diffusion: Classical

- Put a particle at a vertex.
- The particle can hop to a directly connected vertex at random.
- In analogy to the participation ratio P_Q in quantum diffusion, define the classical participation ratio P_C as the number of visited vertices during time t.
  - $P_C(t = 0) = 1$ and $P_C(t \to \infty) = N$.
  - Define the classical diffusion time τ from

$$P_C(t = \tau) = 0.1N.$$

Diffusion: Classical



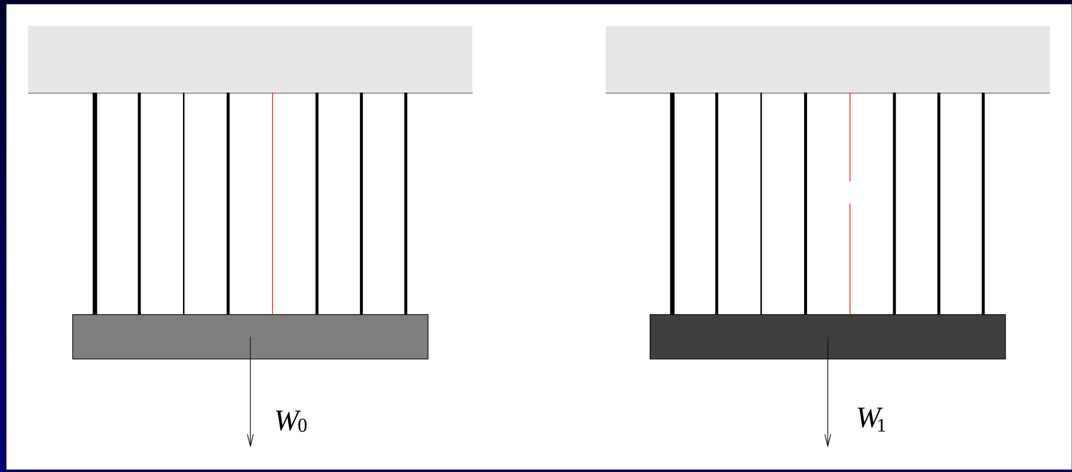
- At $p = 0$, $\tau \sim N^2$ (the well-known random-walk result in 1D).
- At $p \neq 0$, $\tau \sim N$ is found (the same as the result for the d-dimensional random walk for $d \geq 3$).

Diffusion: Summary

- Quantum and classical diffusion on the Watts-Strogatz small-world network.
- Structural transition from the large world to the small world is accompanied by the dynamical transition from the slow world to the fast world.
- In both quantum and classical, $\tau \sim N^a$ at $p = 0$ changes to $\tau \sim N^{a-1}$ at $p \neq 0$.

	$p = 0$	$p \neq 0$
quantum	$\tau \sim N$	$\tau \sim \log N$
classical	$\tau \sim N^2$	$\tau \sim N$
characteristic path length	$l \sim N$	$l \sim \log N$

Fiber Bundle Model (FBM)

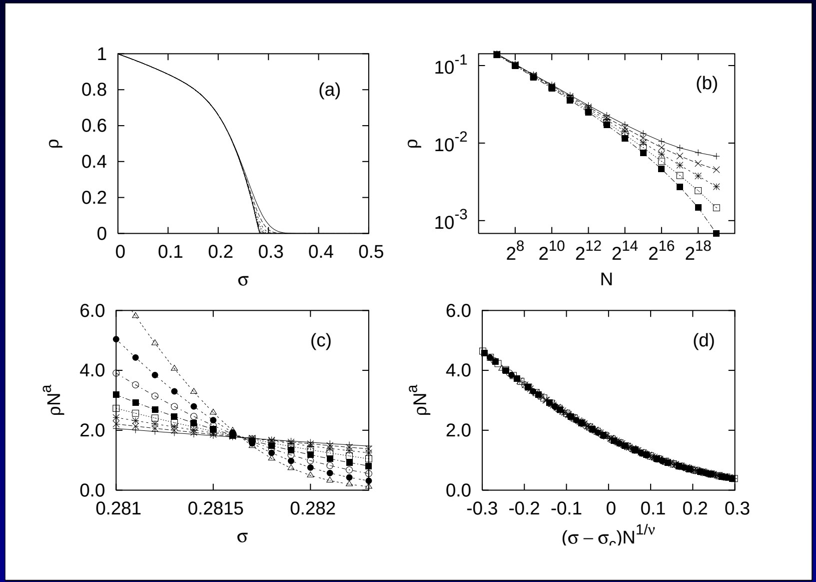


- Fibers attached to a heavy load.
- As the weight W increases ($W_0 < W_1$), the weakest fiber breaks.
- The load which was carried by the broken fiber should now be added to other fibers' burden.

FBM on Network

- On each vertex, assign the capacity c_i at random.
- Apply total load $N\sigma$ (each vertex has $\sigma_i = \sigma$).
- If $\sigma_i > c_i$, break the fiber i and the load σ_i is equally transfered to i-th neighbors ($\sigma_j \to \sigma_j + \sigma_i/K_i$ with the degree K_i). Repeat until no more fiber breaks.
- Increase the load ($\sigma \to \sigma + d\sigma$): $\sigma_i \to \sigma_i + d\sigma(N/N_{\text{unbroken}})$ with the number of unbroken fibers N_{unbroken}.
- Repeat the procedures and measure the density of unbroken fibers $\rho \equiv N_{\text{unbroken}}/N$ ($\rho = 1$ for $\sigma = 0$ and $\rho = 0$ as $\sigma \to \infty$).

FBM on 1D Regular Network



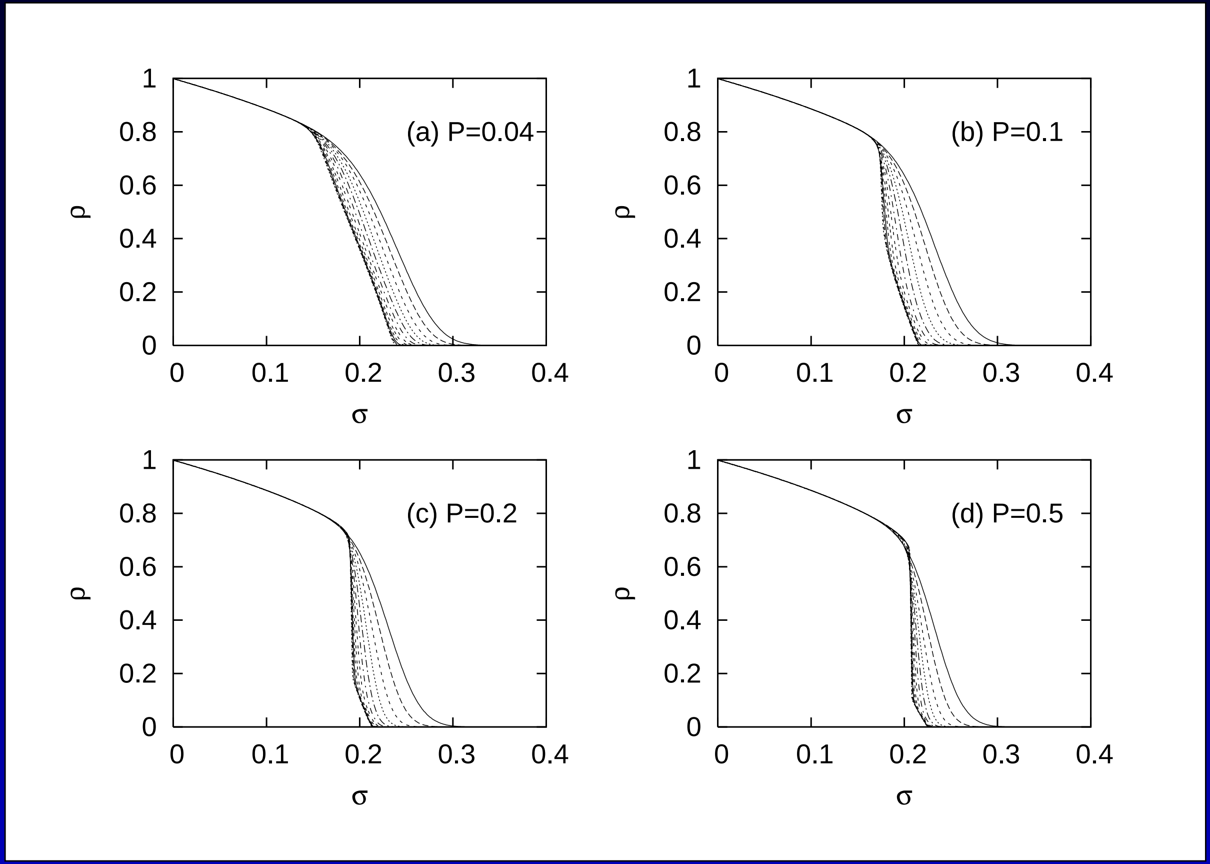
- Coupling upto next-nearest neighbors
- Uniform capacity distribution in [0,1].
- Finite-size scaling: $\rho = N^{-\beta/\nu} f\left((\sigma - \sigma_c)N^{1/\nu}\right)$
- Continuous phase transition at $\sigma_c = 0.28165(2)$ with critical exponents $\nu = 2.0$, $\beta = 1.0$.

FBM on WS network

Motivation

- Previous work (Moreno *et. al.*) on the Barabasi-Albert (BA) network.
    - Discontinuous transition.
    - Network holds tight until it meets final disaster.
    - Once broken, nothing left.
- Effect of the long-range shortcuts in WS?
    - More shortcuts allow the local damage spread more easily.
    - Strength of weak ties: shortcuts may help the whole network to endure the stress from outside.

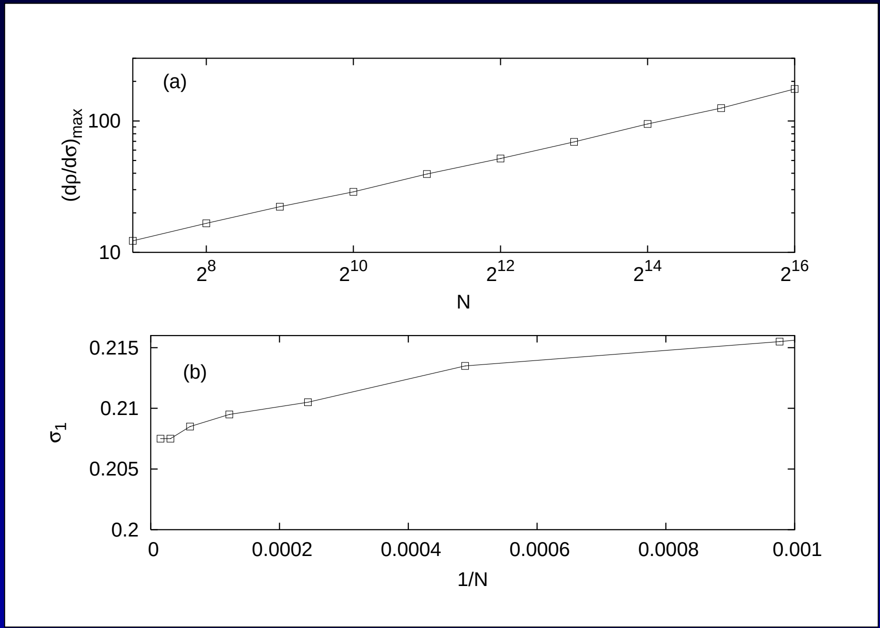
FBM on WS network



- Transitions in two steps at higher P: Sudden discontinuous breakdown at σ_1, and then gradual decay to zero at σ_2.

FBM on WS network

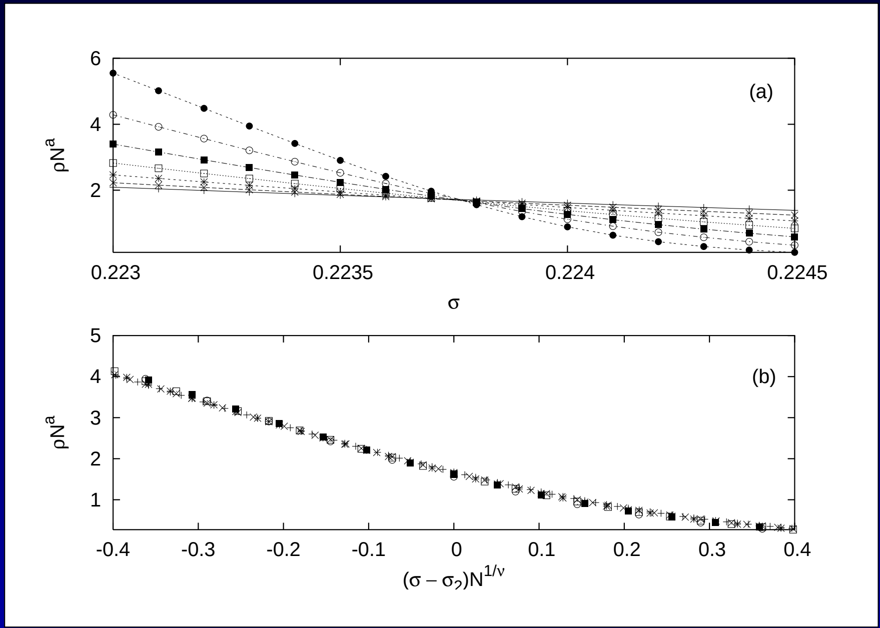
What happens at σ_1 ?



- The derivative $d\rho/d\sigma$ at σ_1 increases with N: Discontinuous transition in thermodynamic limit.
- Extrapolation of $\sigma_1(N)$ to the thermodynamic limit gives the estimation of σ_1 for $N \longrightarrow \infty$.

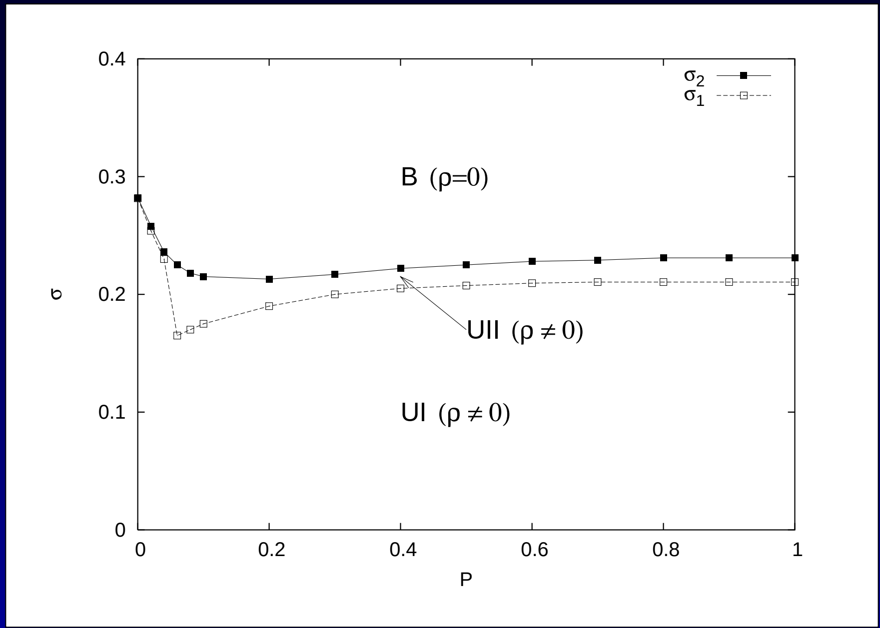
FBM on WS network

What happens at σ_2 ?



- Continuous phase transition confirmed from the finite-size scaling.
- For $P = 0.5$, $\sigma_2 = 0.22376(2)$, $\beta = 1.0$, $\nu = 2.0$. Same universality class as the 1D regular network.

FBM: Phase Diagram



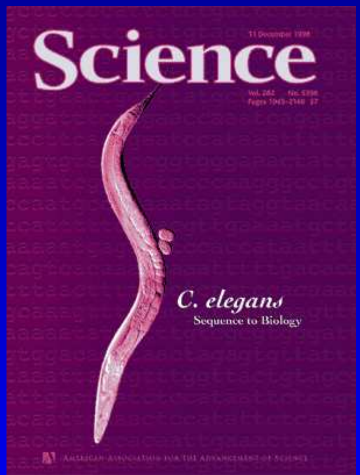
- Two unbroken phase ($\rho \neq 0$) and one broken phase ($\rho = 0$).
- UI to UII (at σ_1) : discontinuous transition.
- UII to B (at σ_2): continuous transition with $\beta = 1.0$ and $\nu = 2.0$.

FBM: Summary

- Fiber bundle model on one-dimensional local regular array exhibits a well-defined continuous phase transition.
- Watts-Strogatz network: Discontinuous (at σ_1) and continuous transitions (at σ_2, same universality class as pure 1D).
- Effects of shortcuts:
  - Local damage spreads across the whole network like an avalanche (discontinuous phase transition at σ_1).
  - Even after the global disaster at σ_1, there still remain a finite fraction of fibers still working (different from BA).

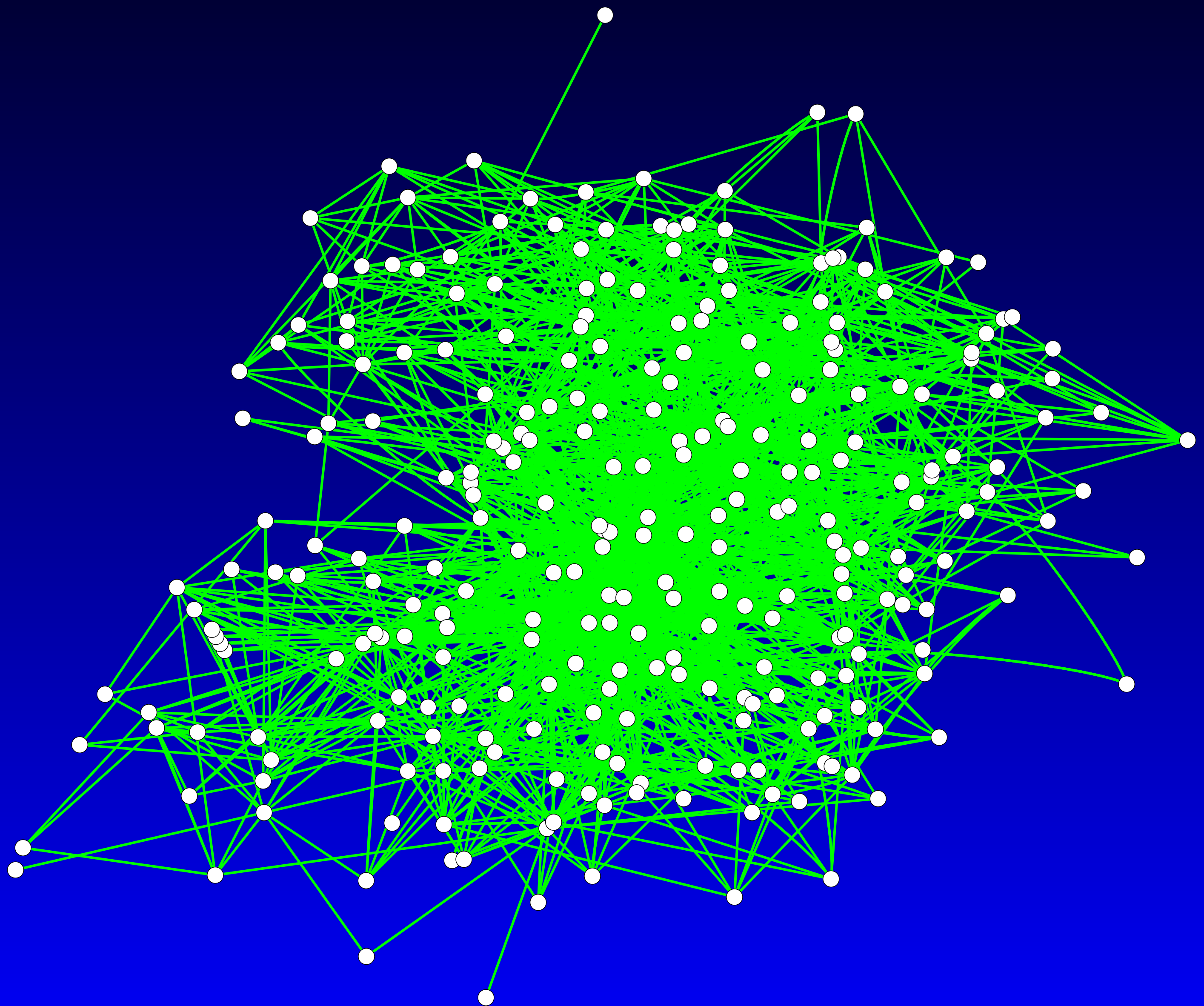
Hopfield on Net: Motivation

- The term "network" is in the scope of statistical physics:
  - Neural network
  - Complex network
- In the subject of complex network, the neuronal network of the worm *C. elegans* has been discussed.

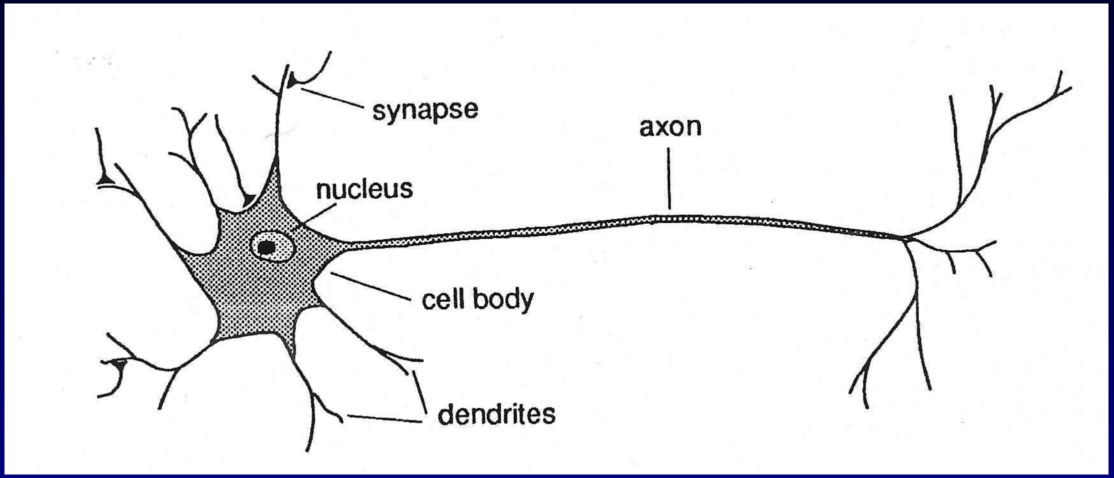


Motivation:
Study the neural network model in statistical physics on biological neuronal network of *C. elegans*.

Hopfield on Net: *C. elegans*

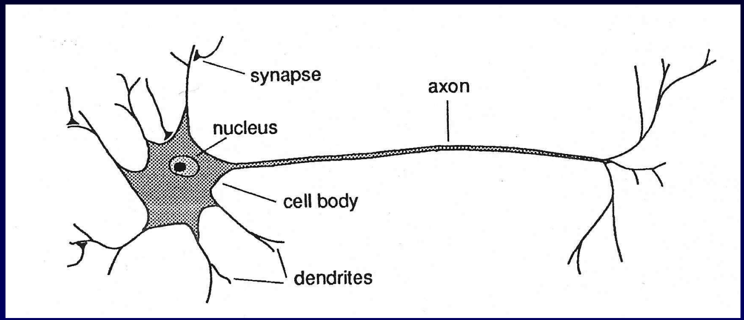


Hopfield on Net: Neurons



- Neurons are coupled to other neurons via synapses.
- Signal from a neuron is transmitted through the axon, and then received by dendrites or cell body of other neuron: Directed couplings.
- A neuron can be in two states: "firing" and "not-firing" (high and low action potential).

Hopfield on Net: Model



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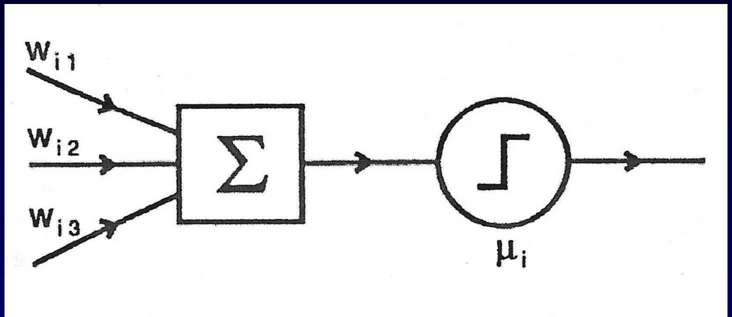


$$n_i(t+1) = \Theta\left(\sum_j w_{ij} n_j(t) - \mu_i\right)$$

- Sum up input signals (from other neurons).
- Depending on the summed value, determine whether or not to fire ($n_i = 1$ or 0).
- Generated signal is transmitted to other neurons.
- Building blocks of the Hopfield model.
- If the two figures look the same, you are a natural-born physicist (or have eyesight problem).

Hopfield on Net: Recognition

- Basic task of learning and recollection:
  - Store a set of p patterns ξ_i^μ $(\mu = 1, 2, \cdots, p)$.
  - When an input pattern σ_i is given, the network's output pattern is the closest to one of the stored pattern: Pattern recognition.
- You are given the word "Einstien", and you recall promptly "relativity" and "mc^2".
  - Can you do that with the type of memory in your computer ($\approx$ telephone book)?
  - Associative content-addressable memory: Different (but similar) input patterns are associated with one stored pattern and thus produce an identical output.

Hopfield on Net: Learning

Change of variable: $n_i(= 1 \text{ or } 0) \Rightarrow \sigma_i(= \pm 1)$:

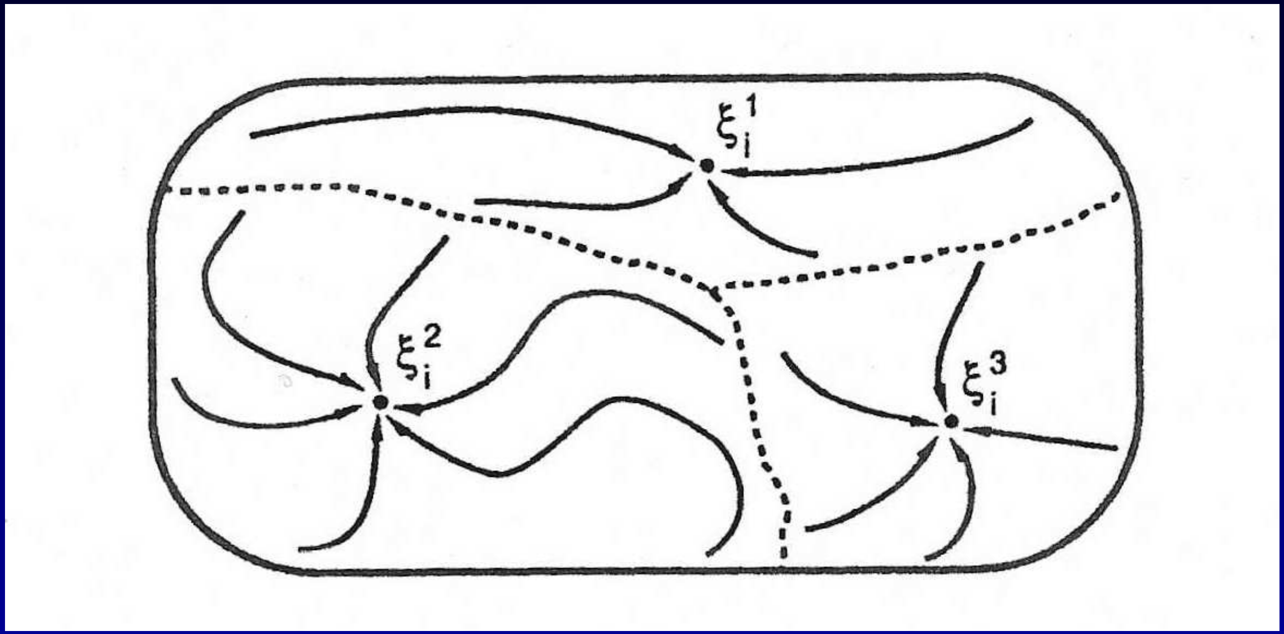
$$\sigma_i(t+1) = \text{sgn}[\sum_j W_{ij}\sigma_j(t)]$$

How to choose W_{ij} for the task of pattern recognition?

- For $p = 1$ (one stored pattern)
  - Stability of $\sigma_i = \xi_i$ when $\text{sgn}[\sum_j W_{ij}\xi_j] = \xi_i$, which is fulfilled with the choice $W_{ij} = \xi_i\xi_j$ since $\xi_j^2 = 1$.
- For p patterns, use the Hebbian learning rule: $W_{ij} = \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$

Bit reversal problem: $\sigma_i = -\xi_i^\mu$ is also a stable fixed point of the dynamics.

Hopfield on Net: Recollection



What is the dynamics doing:

- Search the closest stable fixed point in the phase space composed of 2^N points.
- Impossible if brute force search is used.

Hopfield on Net: Hamiltonian

Examples

- Pattern recognition and Image restoration

Hamiltonian approach: $H = -\sum_{ij} W_{ij}\sigma_i\sigma_j$.

- Looks exactly like the Ising model with long-range interaction (Ising spin glass model).
- For $p = 1$, $\sigma_i = \pm\xi_i$ is the ground state.
- Monte Carlo dynamics at $T = 0$ with single spin reversal ($\sigma_i \to -\sigma_i$): $\Delta E = 4\sum_j W_{ij}\sigma_j\sigma_i$
  - Accept only when $\Delta E < 0$ $\Rightarrow \sigma_i(t+1) = \mathrm{sgn}[\sum_j W_{ij}\sigma_j(t)]$.
- Dynamics of McCulloch-Pitts neuron: MC dynamics at $T = 0$.

Hopfield on Net: Measurement

- $H = -\sum_{ij} \Lambda_{ij} W_{ij} \sigma_i \sigma_j$ with the adjacency matrix Λ: $\Lambda_{ij} = 1(0)$ if i and j are (dis)connected.
- Hebbian learning rule $W_{ij} = \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$.
- Initial configuration contains 20% error bits: $\sigma_i = \xi_i^\nu$ for $0.8N$ neurons and other $0.2N$ neurons have opposite sign.
- Start $T = 0$ MC and wait until stationarity.
- Measure the overlap: $m \equiv (\sum_i \sigma_i \xi_i^\nu)/N$.
  - Full recovery of the stored pattern: $m = 1$.
  - The Hamming distance $d \equiv [\sum(\sigma_i - \xi_i^\nu)^2]/N = 2(1-m)$.

Hopfield on Net: Results

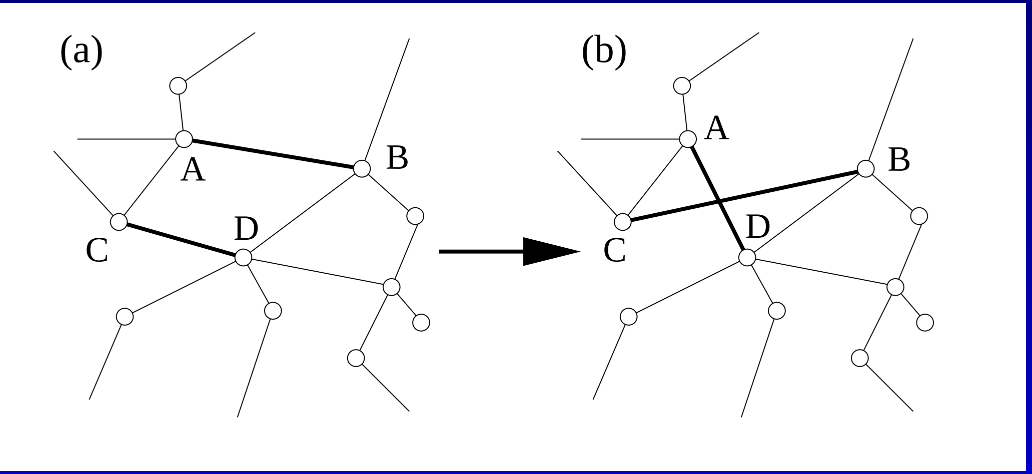
For comparisons, the size and and the average degree of each network are fixed to those for *C. elegans*: $N \approx 300$ and $\langle k \rangle = 14$

Network	$k_{\max}$	γ	m
WS($P = 0.0$)	14	0.69	0.689
WS($P = 0.1$)	17	0.50	0.743
C. elegans	77	0.28	0.798
BA	67	0.11	0.838
WS($P = 1.0$)	22	0.05	0.881

- Both BA and *C. elegans* have broad range of k.
- As the clustering becomes weaker, the neural network does better (larger m).

Hopfield on Net: Results

- Although the degree of each vertex is given, the actual network connection has infinite possibility.
- How to generate such networks with degree of each vertex fixed? ⇒ Use the edge exchange method by Maslov and Sneppen (2002):



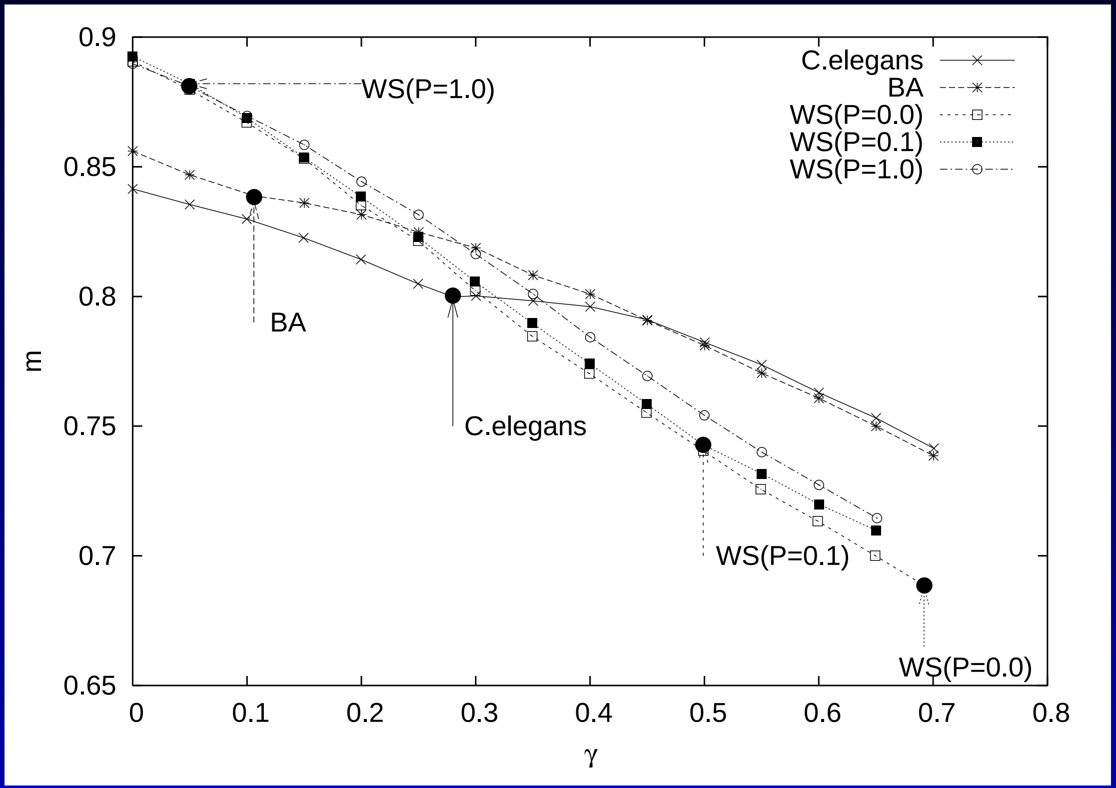
Networks of fixed degrees but with variable clustering coefficient are generated.

Hopfield on Net: Results

Use MC like procedure with $H = \pm \sum_i \gamma_i$:

- Pick two edges at random, and try to exchange.
- If the trial decreases energy, accept the try.
- Repeat the procedure until target value of $\langle \gamma \rangle$ is reached.
- Take the snapshot of the network structure and then put Hopfield neurons on the network to study the performance by measuring m in stationary state.

Hopfield on Net: Results



- Network performance depends much more strongly on the clustering coefficient than on the degree distribution.

Hopfield on Net: Discussions

- Practical viewpoint: Give me your neural network, I will make it much better without changing the degree of each vertex.
- Most previous studies in the research field of "dynamics on network" were focused on the importance of the degree distribution:
  $\Rightarrow$ We have to look other directions too.
- Why less clustering helps better performance?
  $\Rightarrow$ reduction of redundancy.

Hopfield on Net: Discussions

Frankly speaking,

- The original motivation of this work was to find what made the actual structure of the neuronal network of *C.elegans* survive the tremendous evolutionary pressure.
- But found is that this worm is not doing well.
- Why did not *C.elegans* reduce clustering?
  - Room for further evolution?
  - Dark side effect like vulnerability?
  - Geometric information missing: Energy cost to have long-range synaptic couplings?

Concluding Remarks

- Complex networks: New and emerging paradigm to see various systems in terms of their topological structures of networks.
- Dynamic systems on networks: Function follows form.
- How to study “Form follows Function?”