AIRY DISTRIBUTION : FROM HASHING ALGORITHMS TO FLUCTUATING INTERFACE

SATYA N. MAJUMDAR (CNRS - TOULOUSE & ORSAY)

COLLABORATOR : ALAIN COMTET (ORSAY & IHP)

S.M & A. COMTET, cond-mat/0401110

TO APPEAR IN PRL (2004).

PRL, 92, 225501 (2004).

PLAN:

- BRIEF REVIEW ON 'AIRY DISTRIBUTION'

- PROBLEMS IN COMPUTER SCIENCE : HASHING ALGORITHM
  GRAPH THEORY : TREES

- 'AIRY DISTRIBUTION' IN A STATISTICAL PHYSICS PROBLEM

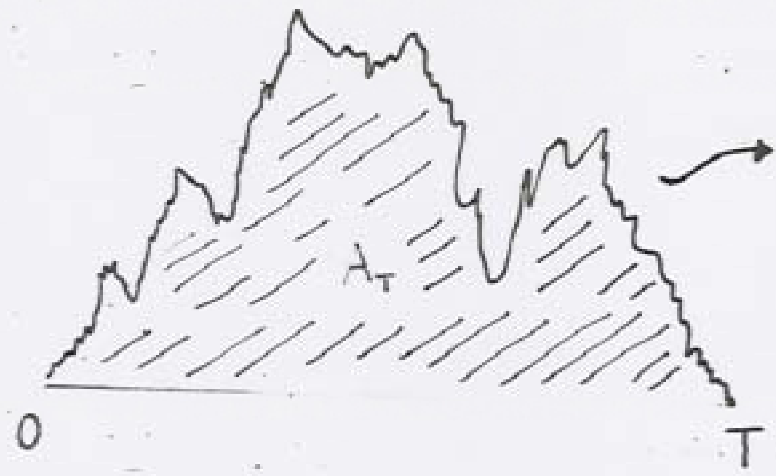
  ↓

  'MAXIMAL' HEIGHT FLUCTUATION OF (1+1)-d INTERFACES

'UNIVERSALITY' AND THE ROLE OF 'BOUNDARY CONDITIONS'

SUMMARY & CONCLUSION

Airy Distribution



BROWNIAN EXCURSION

$A_T \equiv$ AREA UNDER A BROWNIAN EXCURSION $= \int_0^T x(\tau)\,d\tau \sim T^{3/2}$

→ RANDOM VARIABLE

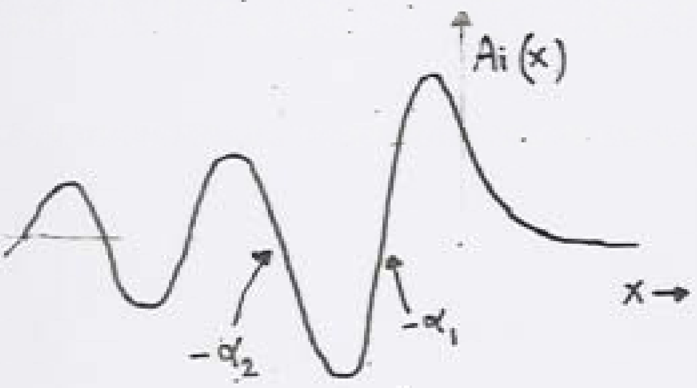
QUESTION: PROBABILITY DISTRIBUTION OF A_T ?

$$P(A, T) = \frac{1}{T^{3/2}} f\left(\frac{A}{T^{3/2}}\right) \; ; \quad f(x) = ?$$

DARLING ('83). LOUCHARD ('84)

$$\boxed{\tilde{f}(s) = \int_0^\infty f(x)\, e^{-sx}\, dx = s \sum_{k=1}^{\infty} \sqrt{2\pi}\; e^{-\alpha_k s^{2/3} 2^{-1/3}}}$$

α_k's → MAGNITUDE OF THE ZEROS OF AIRY FUNCTION $Ai(x)$



$\alpha_1 = 2.3381\ldots$

$\alpha_2 = 4.0879\ldots$

$\alpha_3 = 5.5205\ldots$

$\vdots$

$\alpha_k \sim \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3}$

$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-\frac{b_k}{x^2}}\, b_k^{2/3}\; U\!\left(-\frac{5}{6},\, \frac{4}{3},\, \frac{b_k}{x^2}\right)$$

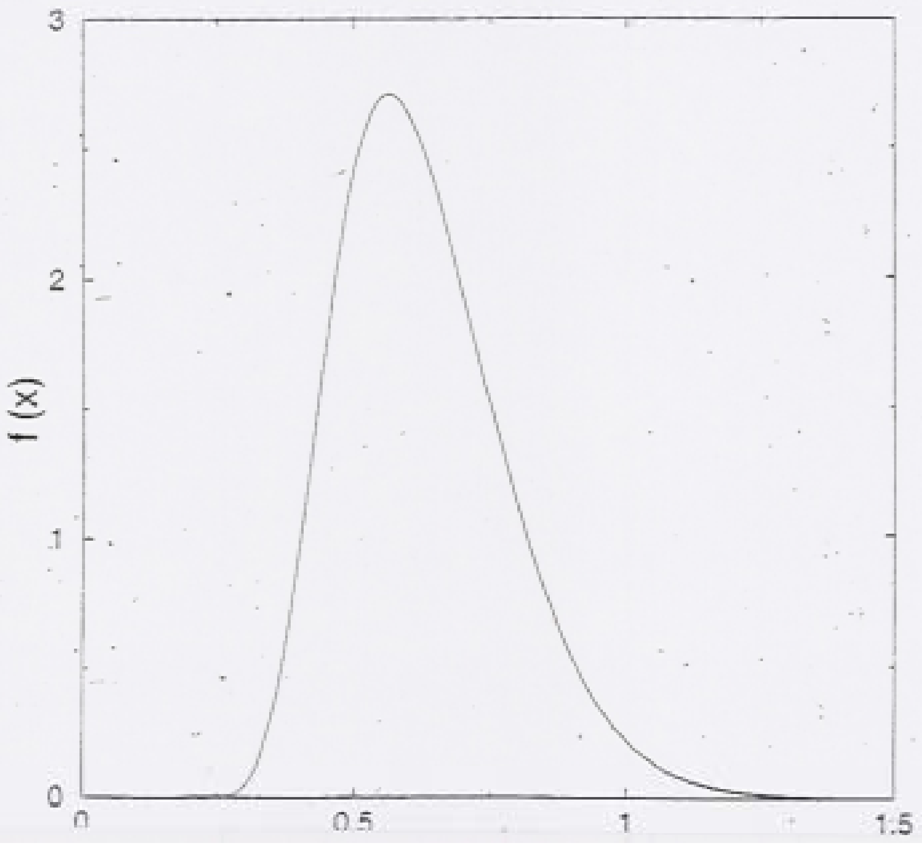
TAKÁCS ('91)

$b_k = \frac{2\alpha_k^3}{27}$; $U(a, b, z) \rightarrow$ CONFLUENT HYPERGEOMETRIC FUNCTION

$$f(x) \xrightarrow{x \to 0} \frac{8}{81}\, \alpha_1^{3/2}\, x^{-5}\, e^{-\frac{2\alpha_1^3}{27x^2}}$$

$$f(x) \xrightarrow{x \to \infty} A\, e^{-6x^2}$$ (CSÖRGO, SHI & YOR, '99)

$\hookrightarrow$ STILL UNKNOWN



MOMENTS:

$$\tilde{f}(s) = s\sqrt{2\pi} \sum_{k=1}^{\infty} e^{-\alpha_k s^{2/3} 2^{-1/3}}$$

MOMENTS ARE NONTRIVIAL TO DERIVE

$$M_n = \int_0^{\infty} f(x)\, x^n \, dx = \frac{4\sqrt{\pi}\; n!}{\Gamma\left(\frac{3n-1}{2}\right) 2^{n/2}} K_n$$

WHERE

$$K_n = \frac{3n-4}{4} K_{n-1} + \sum_{j=1}^{n-1} K_j K_{n-j} \quad , \; n \geq 1 , \qquad K_0 = -\frac{1}{2}$$

TAKÁCS, '91

$$M_0 = 1$$

$$M_1 = \sqrt{\frac{\pi}{8}}$$

$$M_2 = \frac{5}{12}$$

$$M_3 = \frac{15}{32}\sqrt{\frac{\pi}{8}}$$

$\vdots$

$$M_{19} = \frac{62130572562729337 01875}{8398596311897025282048}\sqrt{\frac{\pi}{8}}$$

$$M_{20} = \frac{12120452147224705149184613725}{19459884677136591383371957552}$$

$\vdots$

AIRY DISTRIBUTION → APPEARS IN A NUMBER OF COMPUTER SCIENCE PROBLEMS

EXAMPLE : COST FUNCTION IN 'LINEAR PROBING WITH HASHING'

↓

LPH ALGORITHM (KNUTH, 62)

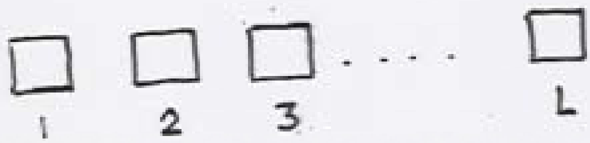
↓

SORTS DATA INTO A LINEAR TABLE

L ITEMS TO BE INSERTED SEQUENTIALLY INTO A LINEAR TABLE WITH L CELLS

↓

$\{x_1, x_2, \cdots x_L\}$



- INITIALLY ALL CELLS ARE EMPTY
- FOR EACH ITEM $x_i \to h(x_i) \in [1, 2, \cdots L]$

  ↳ PREDETERMINED ADDRESS (HASH FUNCTION)

- TRY PUTTING x_i INTO $h(x_i)$

  IF $h(x_i)$ IS EMPTY, THE ITEM x_i GOES THERE

  IF $h(x_i)$ IS OCCUPIED, TRY

  $h(x_i)+1,\ h(x_i)+2, \cdots$ (MODULO L)

  TILL YOU REACH AN EMPTY CELL AND PUT x_i THERE

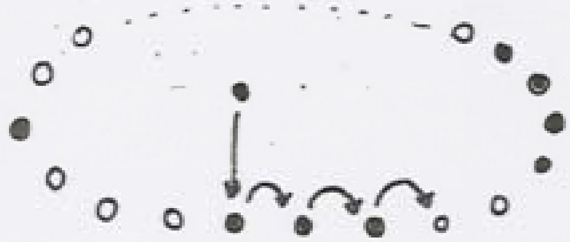
- GO FOR THE NEXT ITEM

IMPORTANT APPLICATION : RANDOM HASH FUNCTION

PHYSICIST'S LANGUAGE : DROP-PUSH MODEL

S.M & D. DEAN, PRL, $\underline{89}$, 115701 (2002)

J. FILIPE & G. J. RODGERS ('95)



- START WITH AN EMPTY LATTICE OF SIZE L WITH PERIODIC B.C.
- CHOOSE A SITE AT RANDOM (WITH PROB. $\frac{1}{L}$) AND ATTEMPT TO PUT A PARTICLE THERE

  → DROP MOVE

  (i) IF THE TARGET SITE IS EMPTY, THE PARTICLE GOES THERE

  (ii) IF OCCUPIED, THE PARTICLE HOPS TO THE RIGHT NEIGHBOR (IF EMPTY IT GOES THERE)

  → NEXT NEAREST NEIGHBOR TO THE RIGHT → ... TILL IT FINDS AN EMPTY SITE AND SETTLES THERE

  → PUSH MOVE

- START (2) WITH THE NEXT PARTICLE
- CONTINUE TILL M PARTICLES ARE INSERTED

CAR PARKING MODEL : WHEN A NEW CAR ARRIVES IN A ONE WAY LANE, IT MOVES FORWARD TILL IT FINDS AN EMPTY SPOT.

→ THE SYSTEM NEVER JAMS (UNLIKE THE USUAL RANDOM SEQ. ADSORPTION MODELS)

WHEN M (no. OF PARTICLES) $=$ L (NO OF SITES)

→ TABLE IS FULLY OCCUPIED AND THE DYNAMICS STOPS

...IRY DISTRIBUTION IN GRAPH THEORY

ROOTED PLANAR TREES WITH $n+1$ VERTICES

Ex : $n=3$

= 3 $\quad$ S = 4 $\quad$ S = 4 $\quad$ S = 5 $\quad$ S = 6

OF SUCH TREES $= C_n = \frac{1}{n+1}\binom{2n}{n}$ → CATALAN NUMBER

ERVABLE : TOTAL INTERNAL PATH LENGTH $= S_n = \sum_{k=1}^{n} d_k$

WHAT IS THE DISTRIBUTION OF S_n : $P(S_n)$?

$\hookrightarrow \frac{1}{n^{3/2}} f\left(\frac{S}{n^{3/2}}\right)$

$f(x) \longrightarrow$ AIRY DISTRIBUTION FUNCTION

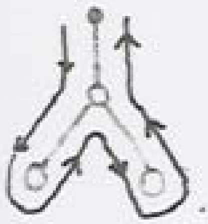
AIRY DISTRIBUTION IN GRAPH THEORY :

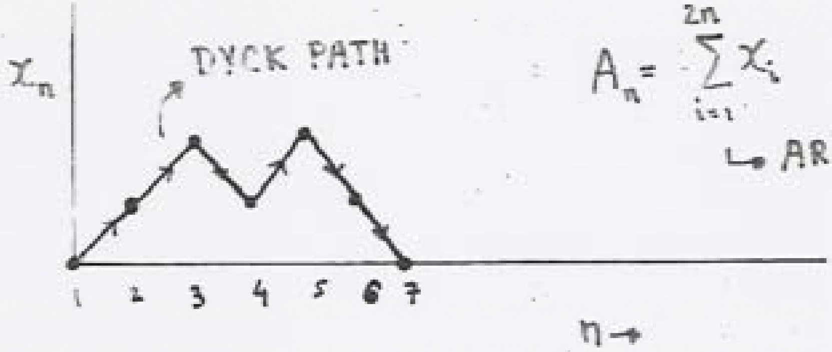
ROOTED PLANAR TREES WITH $(n+1)$ VERTICES

Ex: $n=3$

NO. OF SUCH TREES = $C_n = \frac{1}{n+1}\binom{2n}{n}$ → CATALAN NUMBER

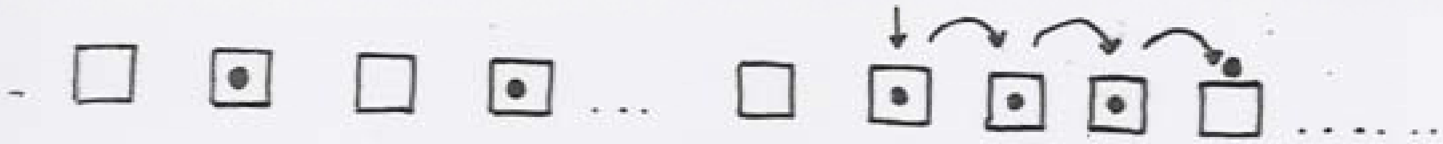
TOTAL INTERNAL PATH LENGTH = $S_n = \sum_{k=1}^{n} d_k$





$A_n = \sum_{i=1}^{2n} x_i$ ↳ AREA

$$S_n = \frac{A_n + n}{2}$$



C_n = COST IN INSERTING THE n-th PARTICLE

= NO. OF UNSUCCESSFUL MOVES BEFORE GETTING ABSORBED

$S_L = \sum_{n=1}^{L} C_n$ = TOTAL COST IN FILLING UP THE TABLE

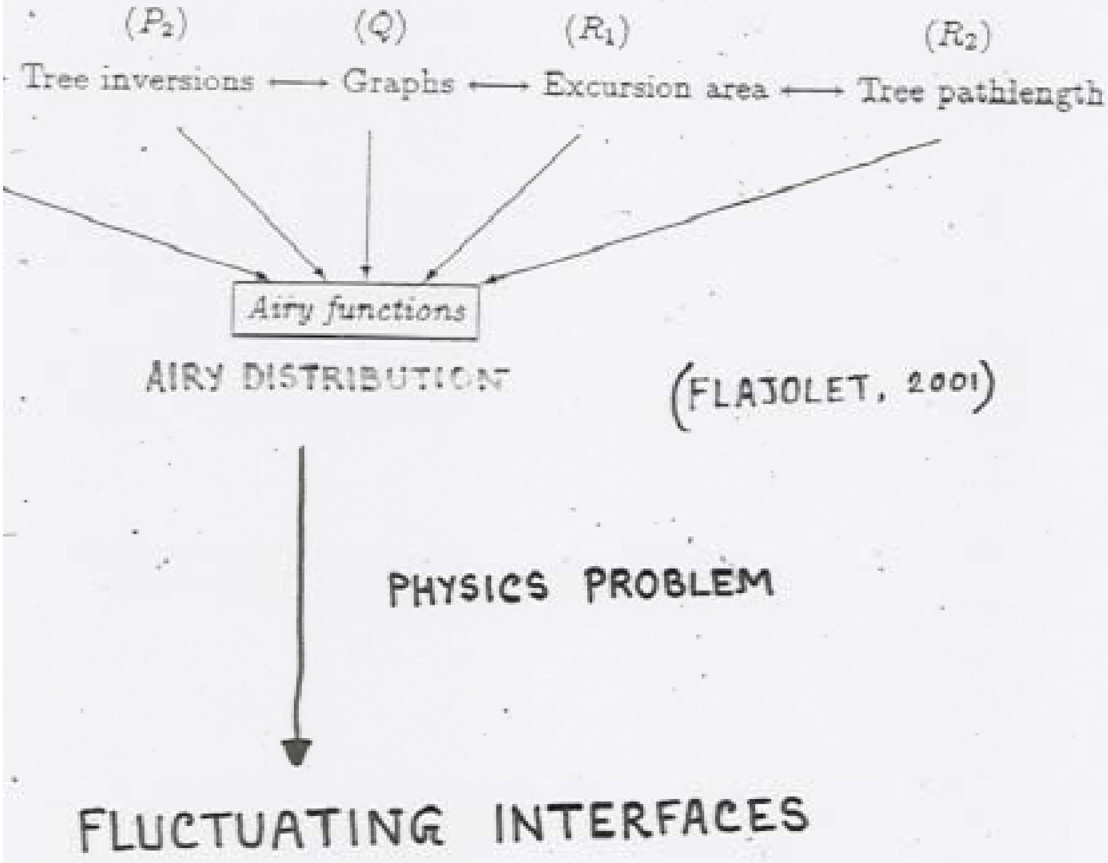
$\longrightarrow$ RANDOM VARIABLE

QUESTION: PROBABILITY DISTRIBUTION OF S_L ?

$$P[S_L] \underset{L\to\infty}{\sim} \frac{1}{L^{3/2}}\, f\left[\frac{S_L}{L^{3/2}}\right]$$

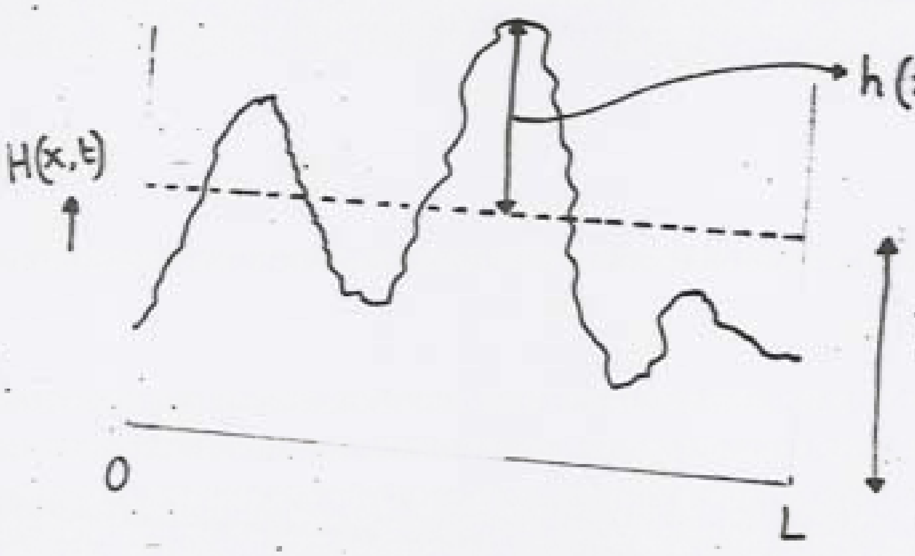
$f(x) \longrightarrow$ AIRY DISTRIBUTION FUNCTION

(FLAJOLET, POBLETE, VIOLA



(P_2)
(Q)
(R_1)
(R_2)
Tree inversions
Graphs
Excursion area
Tree pathlength
Airy functions
AIRY DISTRIBUTION
(FLAJOLET, 2001)
PHYSICS PROBLEM
FLUCTUATING INTERFACES

FLUCTUATING INTERFACES



$h(x,t) = H(x,t) - \overline{H}(t$

↳ FLUCTUATIO

$\overline{H}(t) = \frac{1}{L}\int_0^L H(x,t)\,d$

EDWARDS-WILKINSON:

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \eta(x,t) \longrightarrow \text{GAUSSIAN WHITE NOISE}$$

$\langle \eta(x,t) \rangle = 0$

$\langle \eta(x_1,t_1)\eta(x_2,t_2) \rangle = 2\,\delta(x_1-x_2)\,\delta(t_1-t_2)$

KARDAR-PARISI-ZHANG (KPZ):

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \lambda\left(\frac{\partial H}{\partial x}\right)^2 + \eta(x,t)$$

LUCTUATIONS (NOT THE ABSOLUTE HEIGHT) BECOME STATIONARY AS $t \to \infty$

↓

<u>OUGHNESS</u> '80's + 90's

$$W = \sqrt{\overline{\langle h^2(x,t) \rangle}} = \sqrt{\overline{\langle (H-\overline{H})^2 \rangle}} \underset{1 \ll t \ll L^z}{\sim} t^\beta \longrightarrow \text{GROWING REGIME}$$

$$\underset{t \gg L^z}{\sim} L^\chi \longrightarrow \text{STATIONARY REGIME}$$

$$(L,t) \sim L^\chi\, g\left(\frac{t}{L^z}\right)$$

$\chi \to$ ROUGHNESS EXPONENT $= \frac{1}{2}$ EW & KPZ

$\beta \to$ GROWTH EXPONENT $= \frac{1}{4}$ (EW), $\frac{1}{3}$ (KPZ)

$z \to$ DYNAMICAL EXPONENT $= 2$ (EW), $\frac{3}{2}$ (KPZ)

OTHER OBSERVABLES : LATE 90'S → PRESENT

- FULL DISTRIBUTION OF WIDTH IN THE STATIONARY STATE

$$\frac{1}{L}\int_0^L h^2(x,t)\,dx$$

- STATISTICS OF FIRST PASSAGE EVENTS → PERSISTEN

- DENSITY OF LOCAL MAXIMA & MINIMA OF HEIGHTS

- FULL DISTRIBUTION OF $h(x,t)$ FOR KPZ INTERFACES IN GROWING REGIME $(t \ll L^2)$:

$$\mathrm{PROB}\left[\frac{h}{t^{1/3}} \leq x\right] = F_{TW}(x)$$

↳ TRACY-WIDOM DISTRIBUTION

. . . .

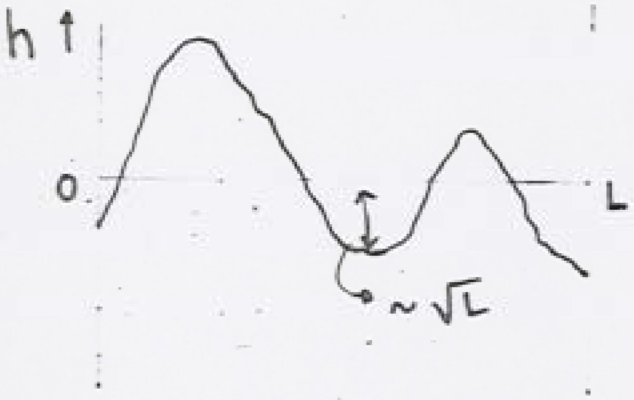
. . . .

. . . .

TODAY'S TOPIC → 'MAXIMAL HEIGHT FLUCTUATION'

STATIONARY REGIME

$$t \gg L^z$$



CORRELATION LENGTH

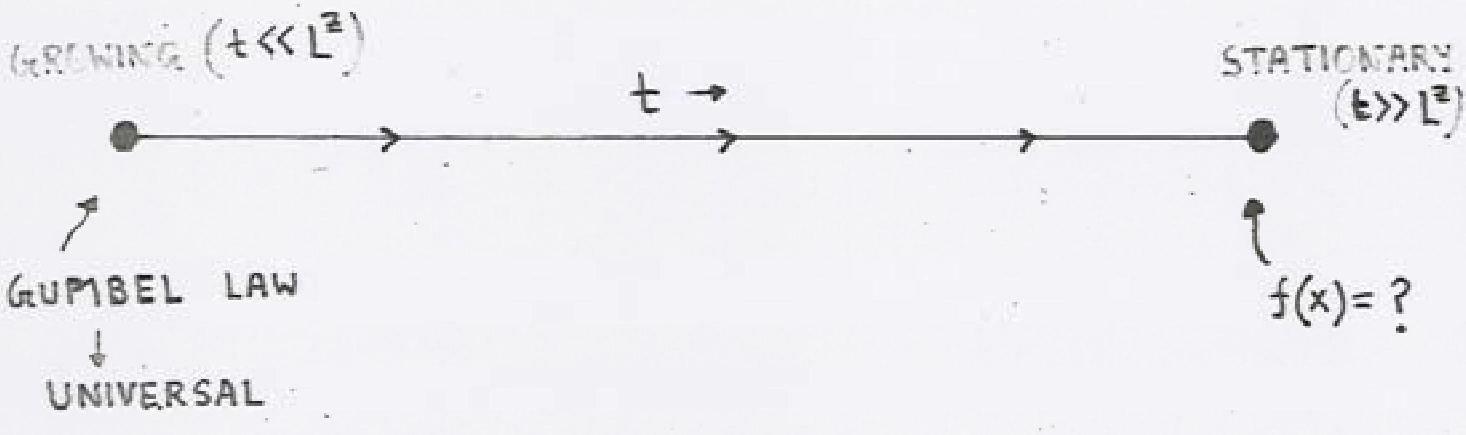
$$\xi(t) \geqslant L$$

$$h_m(t) = \max\left[h(1,t), h(2,t), \dots h(L,t)\right]$$

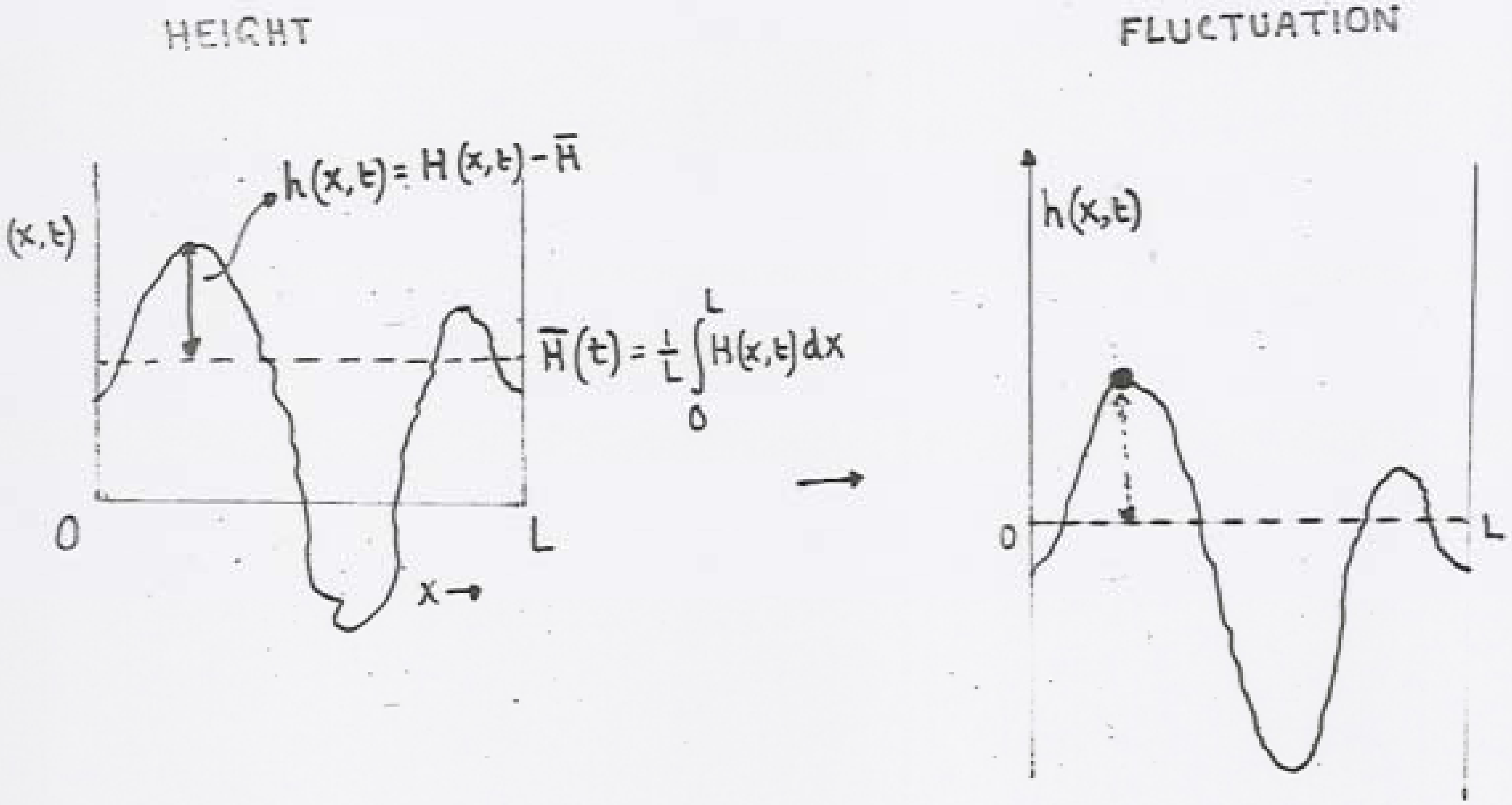
$t \gg L^z$: $h(i,t)$'s $\rightarrow$ STRONGLY CORRELATED

$$h \sim \sqrt{L} \Rightarrow h_m \sim \sqrt{L}$$

$$\boxed{P[h_m, L] \sim \frac{1}{\sqrt{L}}\, f\!\left(\frac{h_m}{\sqrt{L}}\right)}$$



EXTREME FLUCTUATIONS



$$h_m(t) = \max\left[h(1,t), h(2,t), \cdots, h(L,t)\right]$$

↳ GLOBAL MAXIMUM

↳ NOT A SAMPLE AVERAGED QUANTITY

↳ ONE VALUE PER SAMPLE

↳ RARE EVENT

STATISTICS OF $h_m(t)$?

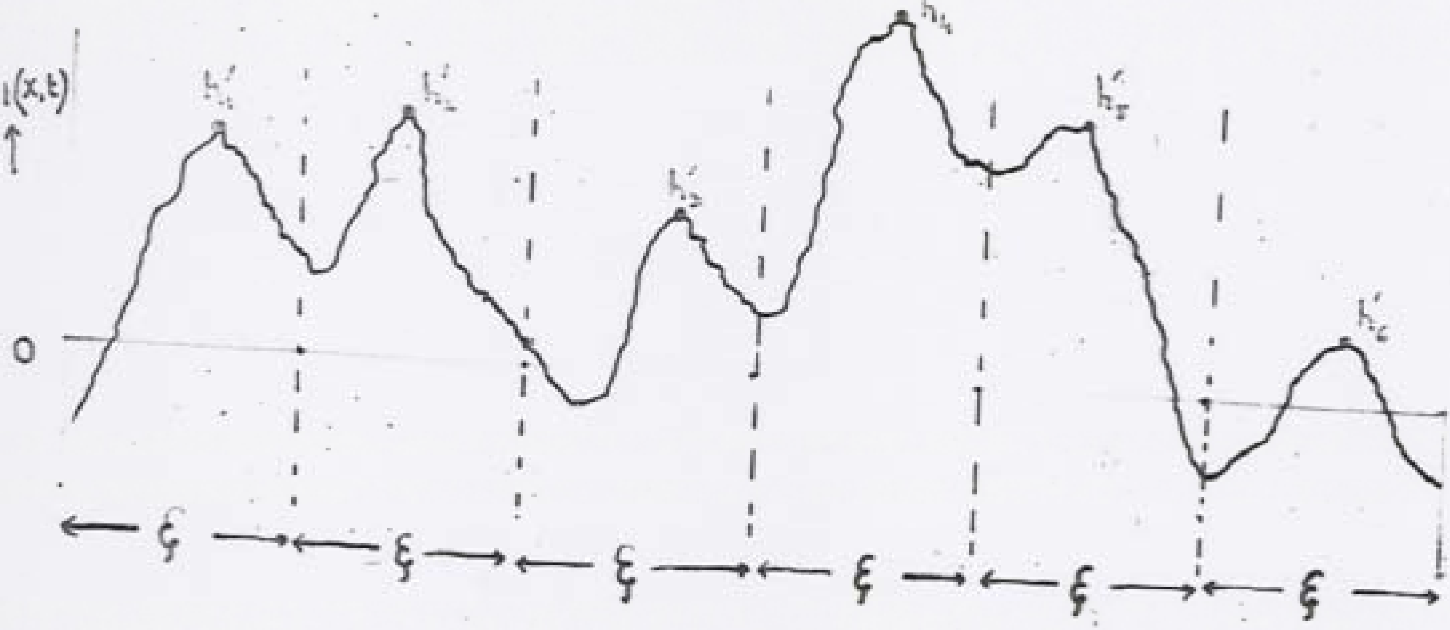
TWO DISTINCT BEHAVIORS:

- GROWING REGIME : $t \ll L^z$
- STATIONARY REGIME : $t \gg L^z$

GROWING REGIME

$t \ll L^z$

GROWING CORRELATION LENGTH : $\xi(t) \sim t^{1/z}$, $h \sim$



$N \sim \dfrac{L}{\xi(t)}$

h''s $\to$ UNCORRELATED $\quad h' \sim t^{\beta}$; $P(h') \sim \frac{1}{t^{\beta}} f\left(\frac{h'}{t^{\beta}}\right)$

$$\max[\{h\}] = \max[h'_1, h'_2, \cdots h'_N]$$

$$\text{PROB}[\max\{h\} < h_m] = \text{PROB}[h'_1 < h_m, h'_2 < h_m, \cdots h'_N < h_m]$$

$$= \left[1 - \int_{h_m/t^{\beta}}^{\infty} f(y)\,dy\right]^N \sim e^{-N\int_{h_m t^{\beta}}^{\infty} f(y)\,dy}$$

Ex: $f(y) = e^{-y}$

$$\text{PROB}[\max\{h\} < h_m] \sim e^{-N e^{-h_m t^{-\beta}}} \sim e^{-e^{-h_m t^{-\beta} + \log(L/t^{1/z})}}$$

$$\text{PROB}\left[\frac{h_m - t^{\beta}\log(L t^{-1/z})}{t^{\beta}} < y\right] \to e^{-e^{-y}} \to \text{GUMBEL LAW}$$

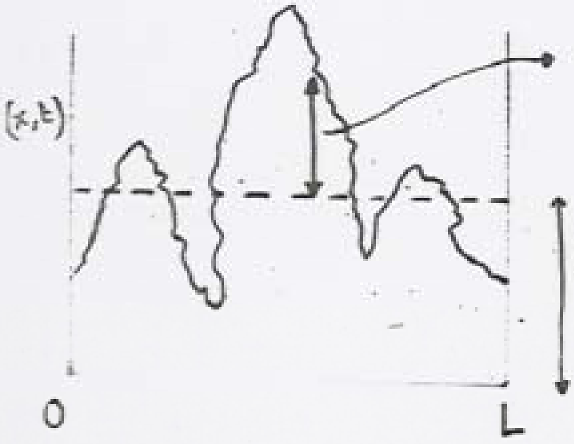
$\downarrow$

UNIVERSAL

"STRONG LAW OF THE LARGEST"

Maximal Height Scaling of Kinetically Growing Surfaces

Subhadip Raychaudhuri,[1] Michael Cranston,[2] Corry Przybyla,[1] and Yonathan Shapir[1]
[1]Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
[2]Department of Mathematics, University of Rochester, Rochester, New York 14627
(Received 7 May 2001; published 5 September 2001)



$$h(x,t) = H(x,t) - \frac{1}{L}\int_0^L H(x,t)\,dx \rightarrow \text{FLUCTUATION}$$

$$\overline{H} = \frac{1}{L}\int_0^L H(x,t)\,dx$$

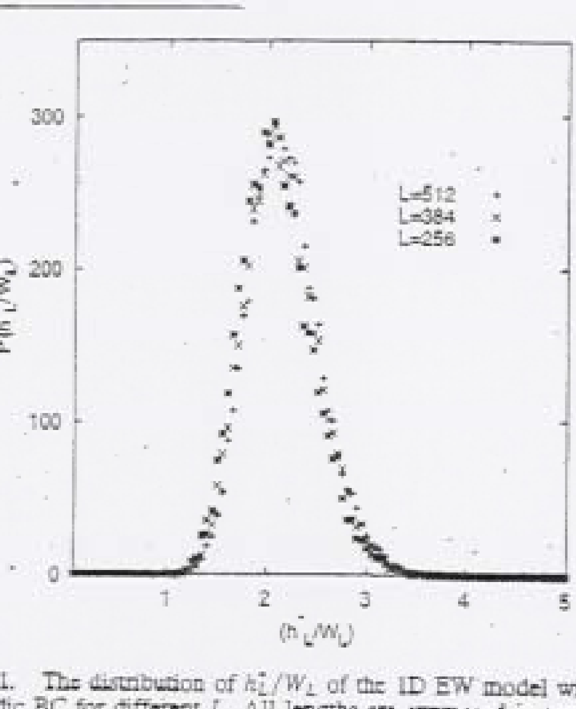
GLOBALLY MAXIMAL HEIGHT FLUCTUATION:

$$h_m = max\left[h(1,t), h(2,t), \cdots h(L,t)\right]$$

IN THE STATIONARY REGIME $(t \gg L^z)$:

$$h_m \sim \sqrt{L}$$

$$P(h_m, L) \sim \frac{1}{\sqrt{L}} f\left(\frac{h_m}{\sqrt{L}}\right) \quad , \quad f(x) = ?$$



→ PERIODIC BOUNDARY CONDITION

FIG. 1. The distribution of h^*_L/W_L of the 1D EW model with periodic BC for different L. All lengths are expressed in terms of the lattice spacing.

OUR MAIN RESULTS :

- $h_m = \max\left[h(1,t),\ h(2,t),\ \cdots\ h(L,t)\right]$

$$\text{PROB}\left[h_m,\ L,\ t\to\infty\right] = P(h_m, L) = \frac{1}{\sqrt{L}}\, f\left(\frac{h_m}{\sqrt{L}}\right)$$

SCALING FUNCTION $f(x)$ → UNIVERSAL EXCEPT FOR THE BOUNDAR
CONDITIO

PERIODIC BOUNDARY CONDITION :

$f(x)$ = AIRY DISTRIBUTION

$$\tilde{f}(s) = s\sqrt{2\pi}\sum_{k=1}^{\infty} e^{-\alpha_k s^{3/2} 2^{-1/3}}$$

EXACT MOMENTS OF h_m

$$\langle h_m^n \rangle = M_n\, L^{n/2}$$

$M_0 = 1$

$M_1 = \sqrt{\frac{\pi}{8}}$

$M_2 = \frac{5}{12}$

$M_3 = \frac{15}{32}\sqrt{\frac{\pi}{8}}$

$\vdots$

FREE BOUNDARY CONDITION :

$f(x) = F_0(x)$ → F-AIRY DISTRIBUTION

$$\tilde{F}_0(s) = s^{3/2}\, 2^{-1/3} \sum_{k=1}^{\infty} C(\alpha_k)\, e^{-\alpha_k s^{3/2} 2^{-1/3}}$$

$$(\alpha_k) = \left[\frac{\int_{-\alpha_k}^{\infty} Ai(x)\,dx}{Ai'(-\alpha_k)}\right]^2$$

EXACT MOMENTS OF h_m

$$\langle h_m^n \rangle = \mu_n\, L^{n/2}$$

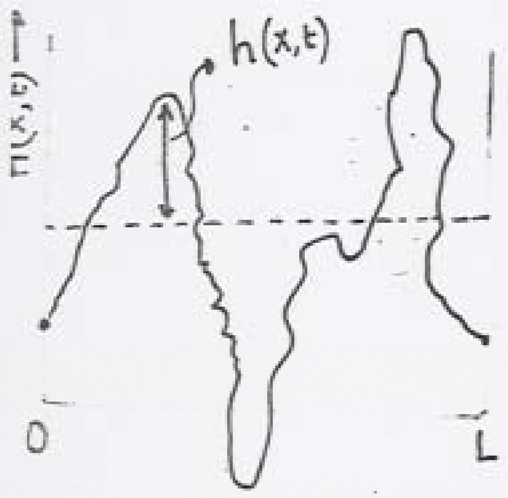
$\mu_0 = 1$

$\mu_1 = \sqrt{\frac{2}{\pi}}$

$\mu_2 = \frac{17}{24}$

$\mu_3 = \frac{123}{[illegible]}\sqrt{\frac{2}{[illegible]}}$

EDWARDS-WILKINSON INTERFACE:



$$\bar{H}(t) = \frac{1}{L}\int_0^L H(x,t)\,dx$$

EVOLUTION EQUATION

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \eta(x,t)$$

↳ WHITE NOISE

$$h(x,t) = H(x,t) - \frac{1}{L}\int_0^L H(x,t)\,dx$$

↳ FLUCTUATION

- $h(x,t) \rightarrow$ GAUSSIAN VARIABLE

$$\langle h(x,t) \rangle = 0 \quad \forall t$$

$$\langle h^2(x,t) \rangle \xrightarrow[t\to\infty]{} \frac{L}{12}$$

SINGLE SITE FLUCTUATION DISTRIBUTION

$$P[h, t\to\infty] = \sqrt{\frac{6}{\pi L}}\, e^{-6h^2/L} \rightarrow \text{STATIONARY DIST.}$$

- STATIONARY HEIGHT-HEIGHT CORRELATION

$$\langle h(x_1,t)\, h(x_2,t) \rangle \xrightarrow[t\to\infty]{} \frac{L}{12}\left[1 - \frac{6x}{L}\left(1-\frac{x}{L}\right)\right], \quad x = |x_1 - x_2|$$

$\longrightarrow$ STRONG CORRELATIONS

- STATIONARY SLOPE-SLOPE CORRELATION

$$\left\langle \frac{\partial h}{\partial x}\Big|_{x_1} \frac{\partial h}{\partial x}\Big|_{x_2} \right\rangle \xrightarrow[t\to\infty]{} \delta(x_1 - x_2) - \frac{1}{L}$$

UNCORRELATED

PBC

$$\int_0^L \frac{\partial h}{\partial x}\,dx = 0$$

- <u>IMPORTANT CONSTRAINT</u>: SINCE $h(x,t) = H(x,t) - \frac{1}{L}\int_0^L H(x,t)\,dx$

$$\boxed{\int_0^L h(x,t)\,dx = 0} \rightarrow \text{TOTAL AREA UNDER } h = 0$$

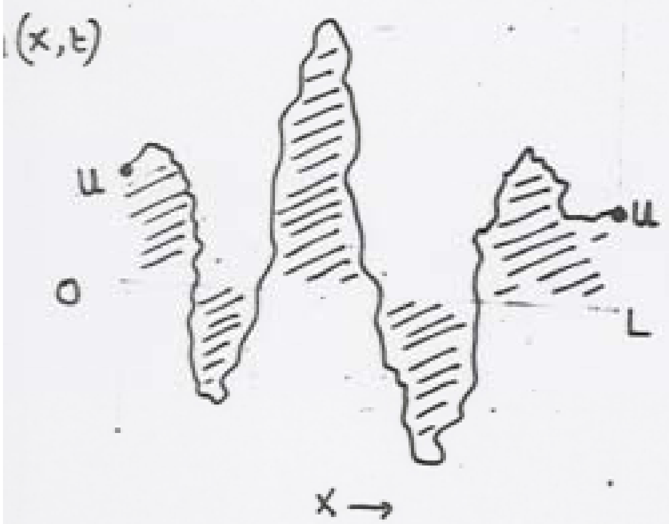
$\rightarrow$ ZERO AREA CONSTRAINT

STATIONARY JOINT DISTRIBUTION OF HEIGHT FLUCTUATIONS

$$P[\{h\}, t] \xrightarrow[t\to\infty]{} P_{st}[\{h\}]$$

$$= C_L \underbrace{e^{-\frac{1}{2}\int_0^L \left(\frac{\partial h}{\partial x}\right)^2 dx}}_{\text{UNCORRELATED SLOPES}} \underbrace{\delta[h(0)-h(L)]}_{\text{PERIODIC BC}} \underbrace{\delta\left[\int_0^L h(x)\,dx\right]}_{\text{AREA CONSTRAINT}}$$

$h(0) = h(L) = u$ → PERIODIC BC

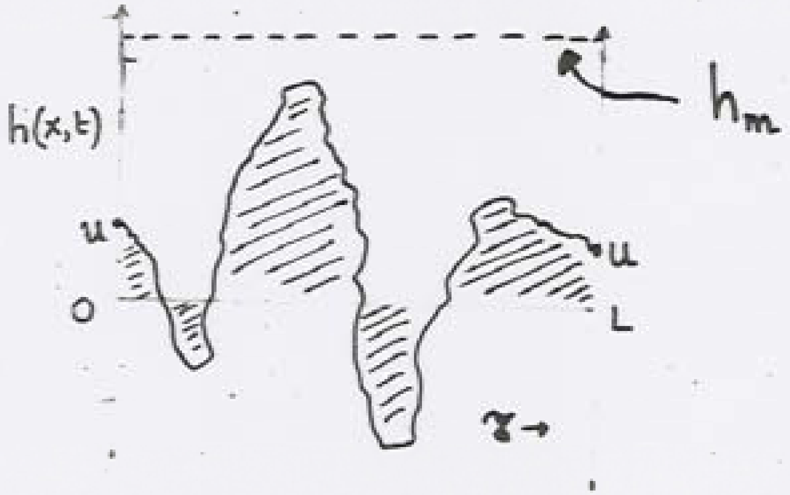


→ BROWNIAN BRIDGE IN SPACE BUT CONSTRAINED TO HAVE AREA = 0

CONSTANT $C_L = \sqrt{2\pi}\, L^{3/2}$; $C_L \int d\vec{h}\, P_{st}[\{h\}] = 1$

STATIONARY MEASURE

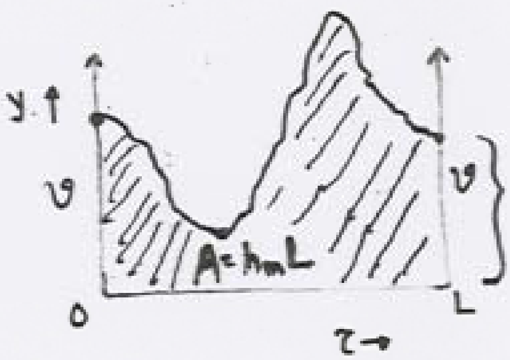
$$P_{St}\left[\{h\}\right] = \sqrt{2\pi}\; L^{3/2}\; e^{-\frac{1}{2}\int_0^L \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau}\;\; \delta\left[h(0)-h(L)\right]\;\; \delta\left[\int_0^L h(\tau)\,d\tau\right]$$



$h(0) = h(L) = u$

$$F(h_m) = \mathrm{PROB}\left[\max\{h\} < h_m\right]$$
$$= \mathrm{PROB}\left[h_1 < h_m,\; h_2 < h_m,\; \cdots\; h_L < h_m\right]$$
$$= \sqrt{2\pi}\; L^{3/2} \int_{-\infty}^{h_m} du \int_{h(0)=u}^{h(L)=u} \mathcal{D}h(\tau)\; e^{-\frac{1}{2}\int_0^L \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau}\;\; \delta\left[\int_0^L h(\tau)\,d\tau\right] \prod_{\tau=0}^{L} \theta\left[h_m - h(\tau)\right]$$

SHIFT $\Big\downarrow$ $\quad y(\tau) = h_m - h(\tau)$
$\quad v = h_m - u$



$$= \sqrt{2\pi}\; L^{3/2} \int_0^{\infty} dv \int_{y(0)=v}^{y(L)=v} \mathcal{D}y(\tau)\; e^{-\frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial \tau}\right)^2 d\tau}\;\; \delta\left[\underbrace{h_m L}_{h_m L = A} - \int_0^L y(\tau)\,d\tau\right] \prod_{\tau=0}^{L} \theta\left[y(\tau)\right]$$

$$F(h_m) = \mathcal{F}(A)$$

$$F(A) = \sqrt{2\pi}\, L^{-3/2} \int_0 dv \int_{y(0)=v} \mathcal{D}y(\tau)\; e^{-\frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial \tau}\right)^2 d\tau}\; \delta\left[\int_0^L y(\tau)\,d\tau - A\right] \prod_{\tau=0}^{L} \theta[\;$$

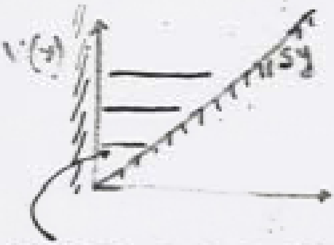
LAPLACE TRANSFORM:

$$\int_0^\infty F(A)\, e^{-As}\, dA = \sqrt{2\pi}\, L^{3/2} \int_0^\infty dv \int_{y(0)=v}^{y(L)=v} \mathcal{D}y(\tau)\; e^{-\frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial \tau}\right)^2 d\tau - s\int_0^L y(\tau)\,d\tau} \prod_{\tau=0}^{L} \theta[y$$

$$= \sqrt{2\pi}\, L^{3/2} \int_0^\infty dv \left\langle v \right| e^{-\hat{H} L} \left| v \right\rangle$$

$$= \sqrt{2\pi}\, L^{3/2}\, \mathrm{Tr}\left[e^{-\hat{H} L}\right] = \sqrt{2\pi}\, L^{3/2} \sum_k e^{-E_k L}$$

$$\hat{H} \equiv -\frac{1}{2}\frac{\partial^2}{\partial y^2} + V(y) \quad ; \qquad V(y) = s\,y \quad \text{for } y > 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\quad = \infty \quad \text{for } y \leq 0$$



DISCRETE EIGENVALUES

ENFUNCTIONS:

$$\Psi_E(y) = \mathcal{N}\, Ai\left[(2s)^{1/3}\left(y - \frac{E}{s}\right)\right]$$

ENVALUES:

$$Ai\left[-E\,\frac{2^{1/3}}{s^{2/3}}\right] = 0 \Rightarrow E_k = \alpha_k\, s^{2/3}\, 2^{-1/3}$$

$$P(h_m, L) = \frac{1}{\sqrt{L}}\, f\left(\frac{h_m}{\sqrt{L}}\right)$$

$$\tilde{f}(s) = s\sqrt{2\pi} \sum^{\infty} e^{-\alpha_k s^{2/3} 2^{-1/3}}$$

EW-PBC

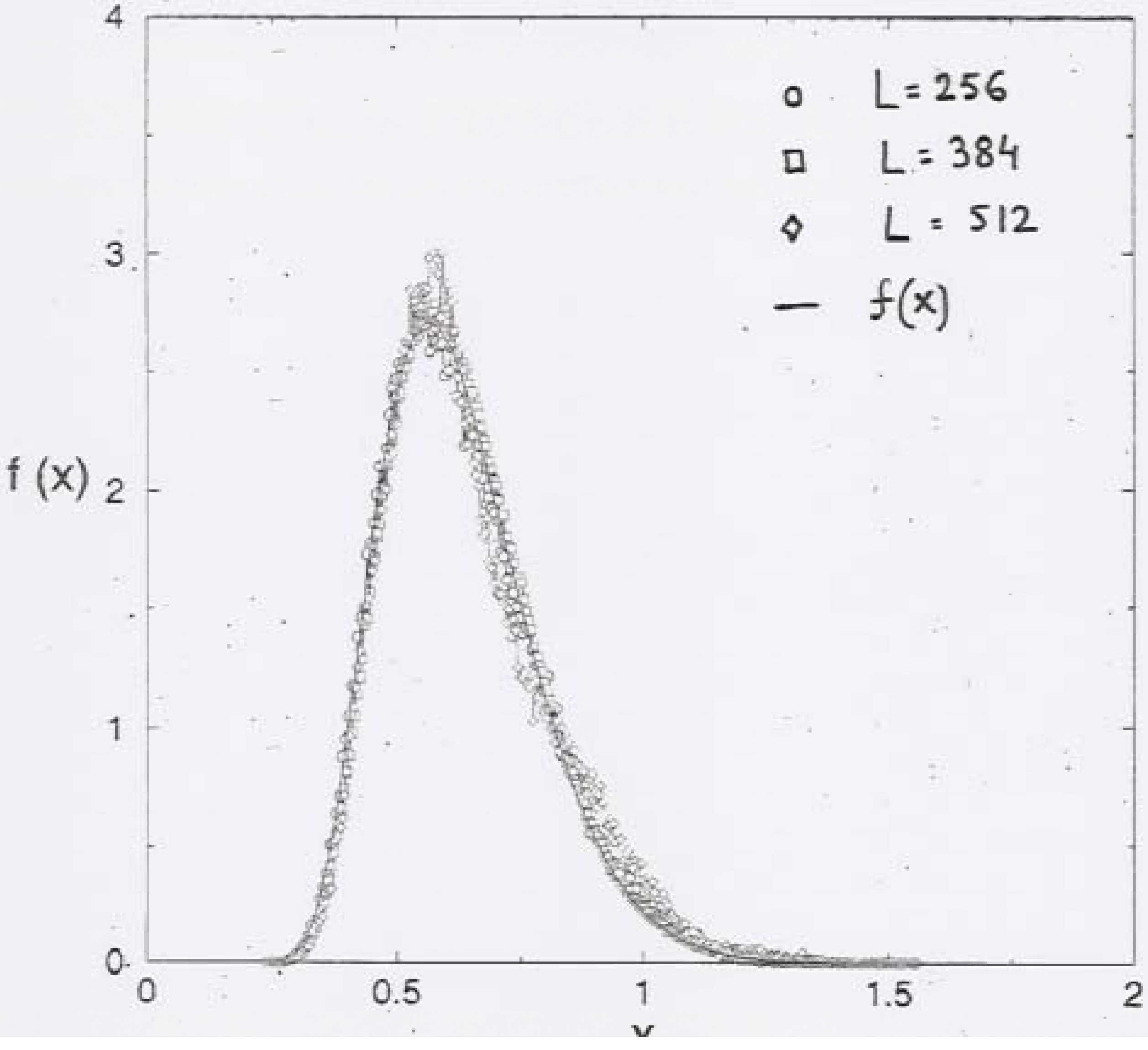
$$P(h_m, L) = \frac{1}{\sqrt{L}}\, f\left(\frac{h_m}{\sqrt{L}}\right)$$

$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-\frac{b_k}{x^2}}\, b_k^{2/3}\, U\left(-\frac{5}{6}, \frac{4}{3}, \frac{b_k}{x^2}\right)$$

$$b_k = \frac{2\gamma_k^3}{27}$$

PLOTTED USING MATHEMATICA

EDWARDS-WILKINSON SIMULATION (PERIODIC BC)



KPZ EQUATION :

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \lambda \left(\frac{\partial H}{\partial x}\right)^2 + \eta(x,t)$$

FLUCTUATION :

$$h(x,t) = H(x,t) - \frac{1}{L}\int_0^L H(x,t)\,dx$$

$$\Rightarrow \int_0^L h(x,t)\,dx = 0 \rightarrow \text{SAME AREA CONSTRAINT}$$

ONE CAN SHOW → SAME STATIONARY JOINT DISTRIBUTION (1+1)-d

$$P_{st}\left[\{h\}\right] = \sqrt{2\pi}\, L^{3/2}\, e^{-\frac{1}{2}\int_0^L \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau}\, \delta\left[h(0)-h(L)\right]\, \delta\left[\int_0^L h(\tau)\,d\tau\right]$$

$$P(h_m, L) = \text{PROB}\left[\max\{h_1, \cdots h_L\} = h_m\,,\, L\right] = \frac{1}{\sqrt{L}}\, f\left(\frac{h_m}{\sqrt{L}}\right)$$

$$f(x) \rightarrow \text{AIRY DISTRIBUTION}$$

NUMERICAL CHECK → NONTRIVIAL

DISCRETIZATION OF THE $\underbrace{\text{NON-LINEAR}}_{\lambda\left(\frac{\partial H}{\partial x}\right)^2}$ TERM → PROBLEMATIC

$$\left(\frac{\partial H}{\partial x}\right)^2 \rightarrow \frac{1}{4}\left(h_{i+1} - h_{i-1}\right)^2 \longrightarrow \text{BAD}$$

LAM & SHIN, '98

$$\left(\frac{\partial H}{\partial x}\right)^2 \rightarrow \frac{1}{3}\left[\left(h_{i+1}-h_i\right)^2 + \left(h_{i+1}-h_i\right)\left(h_i - h_{i-1}\right) + \left(h_i - h_{i-1}\right)^2\right] \longrightarrow \text{GOOD SCHEME}$$

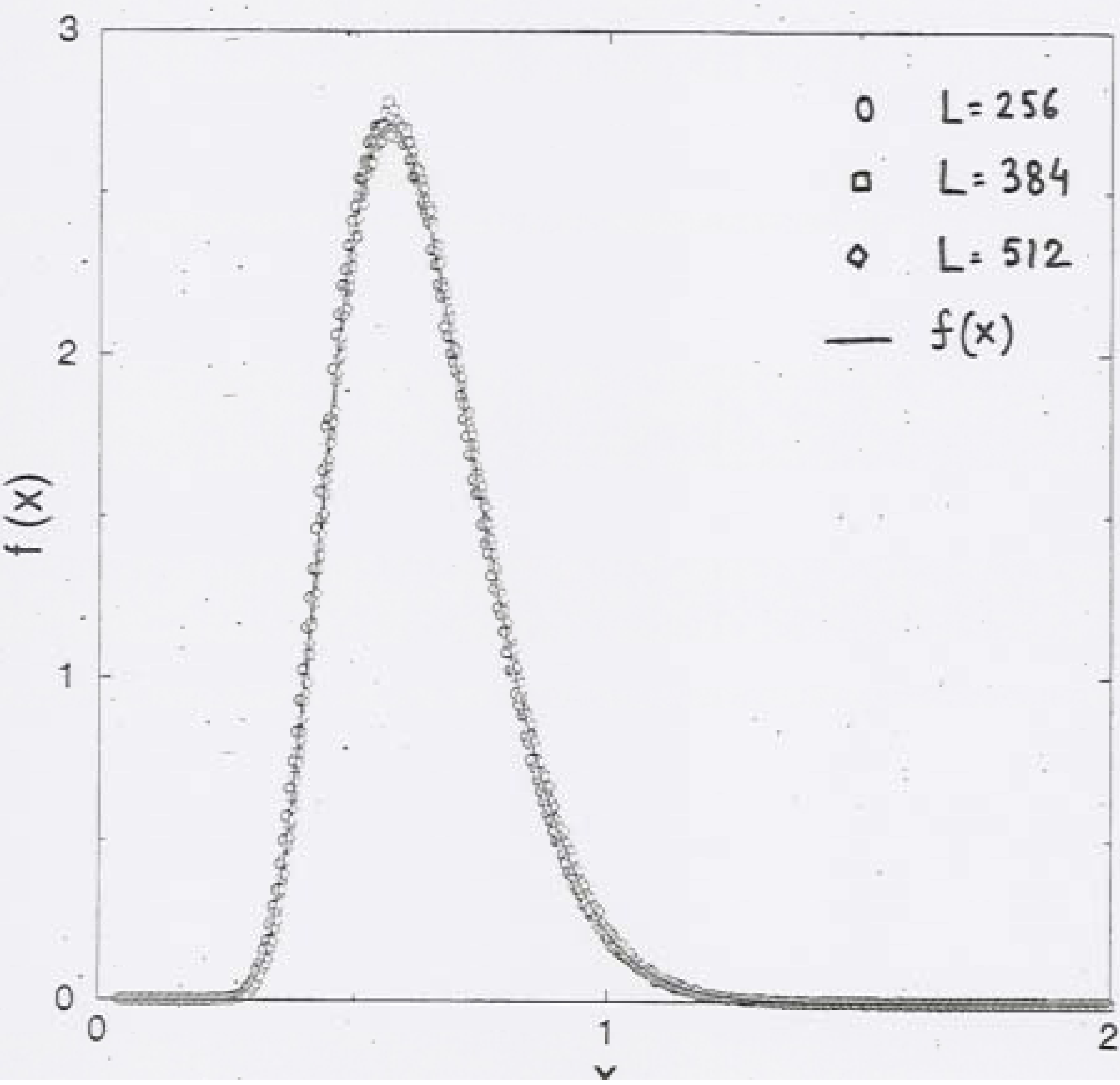
$$P(h_m, L) = \frac{1}{\sqrt{L}}\, f\left(\frac{h_m}{\sqrt{L}}\right)$$

$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-\frac{b_k}{x^2}}\, b_k^{2/3}\, U\left(-\frac{5}{6}, \frac{4}{3}, \frac{b_k}{x^2}\right)$$

$$b_k = \frac{2\alpha_k^3}{27}$$

PLOTTED USING MATHEMATICA

KPZ SIMULATION (PERIODIC BC)



WITH PERIODIC BOUNDARY CONDITION, THE DISTRIBUTION OF MAXIMAL HEIGHT FLUCTUATION IN THE STATIONARY STATE IS **UNIVERSAL**

$$P(h_m, L) = \frac{1}{\sqrt{L}} f\left(\frac{h_m}{\sqrt{L}}\right)$$

WHERE THE SCALING FUNCTION

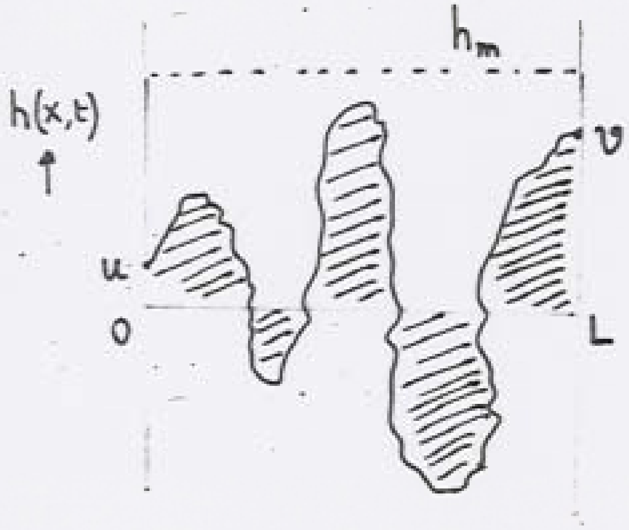
$$f(x) = f_{AD}(x) \rightarrow \text{AIRY DISTRIBUTION FUNCTION}$$

FREE BOUNDARY CONDITION:

STATIONARY JOINT DISTRIBUTION

$$P_{st}\left[\{h\}\right] = L \, e^{-\frac{1}{2}\int_0^L \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau} \quad \delta\left[\int_0^L h(\tau)\,d\tau\right]$$

↳ AREA CONSTRAINT



$$F(h_m) = \text{PROB}\left[\max\{h\} < h_m\right] = \text{PROB}\left[h_1 < h_m,\ h_2 < h_m,\ \cdots\ h_L < h_m\right]$$

$$F(h_m) = F(A), \quad A = h_m L.$$

$$\int_0^\infty F(A)\, e^{-SA}\, dA = L \int_0^\infty du \int_0^\infty dv \, \langle u | e^{-\hat{H} L} | v \rangle$$

$\Longrightarrow$

$$P(h_m, L) = \frac{1}{\sqrt{L}} F_0\left(\frac{h_m}{\sqrt{L}}\right)$$

$$\boxed{\begin{aligned} \widetilde{F}_0(s) &= s^{2/3}\, 2^{-1/3} \sum_{k=1}^{\infty} C(\alpha_k)\, e^{-\alpha_k s^{2/3} 2^{-1/3}} \\ C(\alpha_k) &= \left[\frac{\int_{-\alpha_k}^{\infty} Ai(x)\,dx}{Ai'(-\alpha_k)}\right]^2 \end{aligned}}$$

EXACT ASYMPTOTIC BEHAVIORS:

$$F_c(x) \xrightarrow{x \to 0} \frac{2\sqrt{2}}{27\sqrt{\pi}}\, C(\alpha_1)\, \alpha_1^{7/2}\, x^{-4}\, e^{-\frac{2\alpha_1^3}{27 x^2}}$$

$\alpha_1 = 2.3381\ldots$

$C(\alpha_1) = 3.30278\ldots$

$$F_c(x) \xrightarrow{x \to \infty} B\, e^{-\frac{3}{2} x^2}$$

MOMENTS CAN BE DETERMINED EXPLICITLY

$$\mu_n = \int_0^\infty F_c(x)\, x^n\, dx$$

$$\mu_0 = 1$$

$$\mu_1 = \sqrt{\frac{2}{\pi}}$$

$$\mu_2 = \frac{17}{24}$$

$$\mu_3 = \frac{123}{140}\sqrt{\frac{2}{\pi}}$$

$$\vdots$$

$$P(h_m, L) = \frac{1}{\sqrt{L}} F_0\left(\frac{h_m}{\sqrt{L}}\right) \leftarrow \text{FREE BOUNDARY CONDITION}$$

$$F_0(x) = \frac{x^{-7/3}}{2^{1/3}\sqrt{3\pi}} \sum_{k=1}^{\infty} \alpha_k \, C(\alpha_k) \left[U\left(\tfrac{1}{6}, \tfrac{4}{3}, \tfrac{b_k}{x^2}\right) + 2\, U\left(-\tfrac{5}{6}, \tfrac{4}{3}, \tfrac{b_k}{x^2}\right)\right] e^{-\frac{b_k}{x^2}}$$

$$b_k = \frac{2\alpha_k^3}{27}, \qquad C(\alpha_k) = \left[\frac{\int_{-\alpha_k}^{\infty} Ai(x)\,dx}{Ai'(-\alpha_k)}\right]^2$$

$C(\alpha_1) = 3.30279..$

$C(\alpha_2) = 1.01282...$

$C(\alpha_3) = 1.78218...$

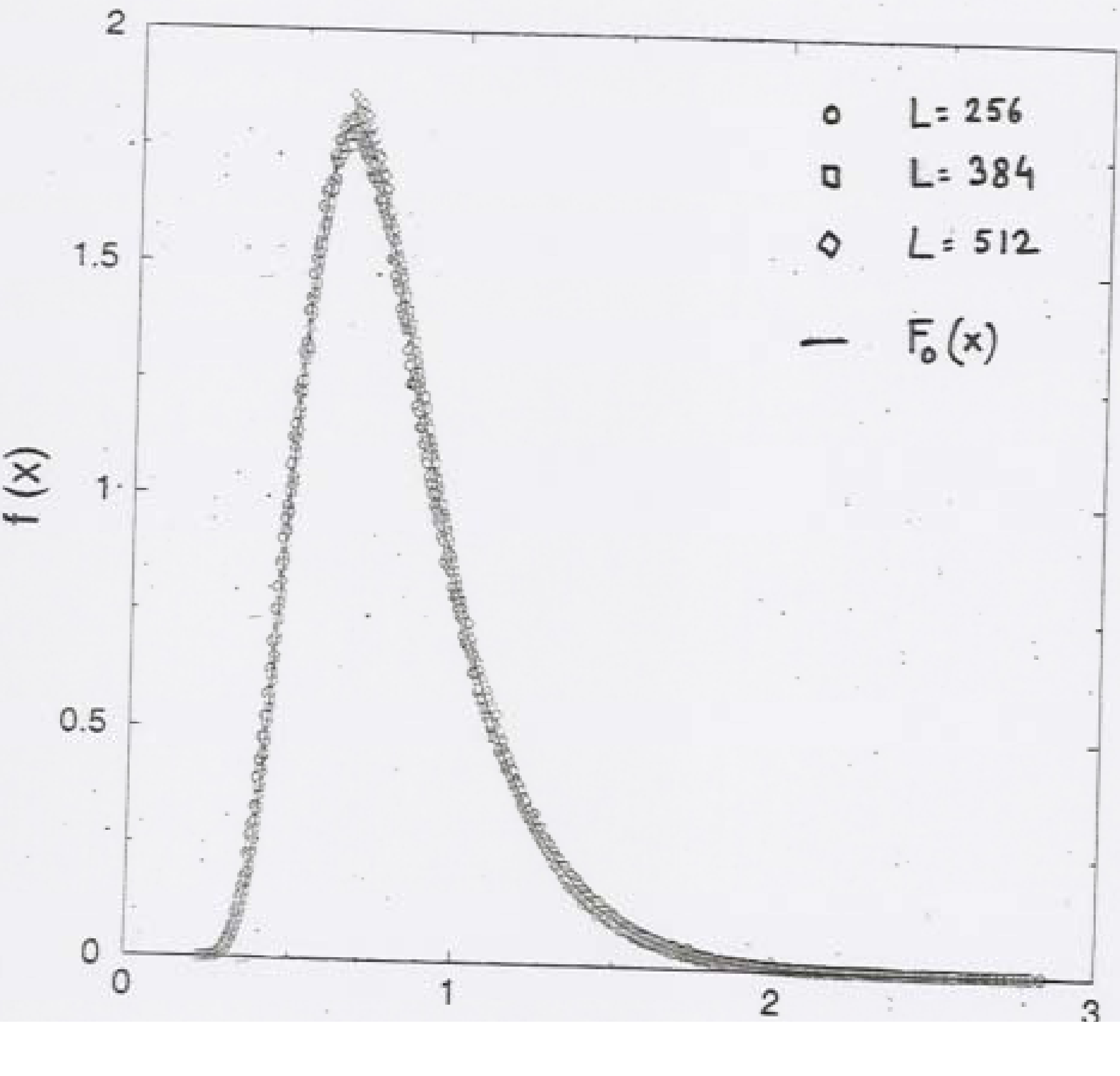
$\vdots$

$C(\alpha_{10}) = 0.73725$

$\downarrow$

PLOTTED USING MATHEMATICA

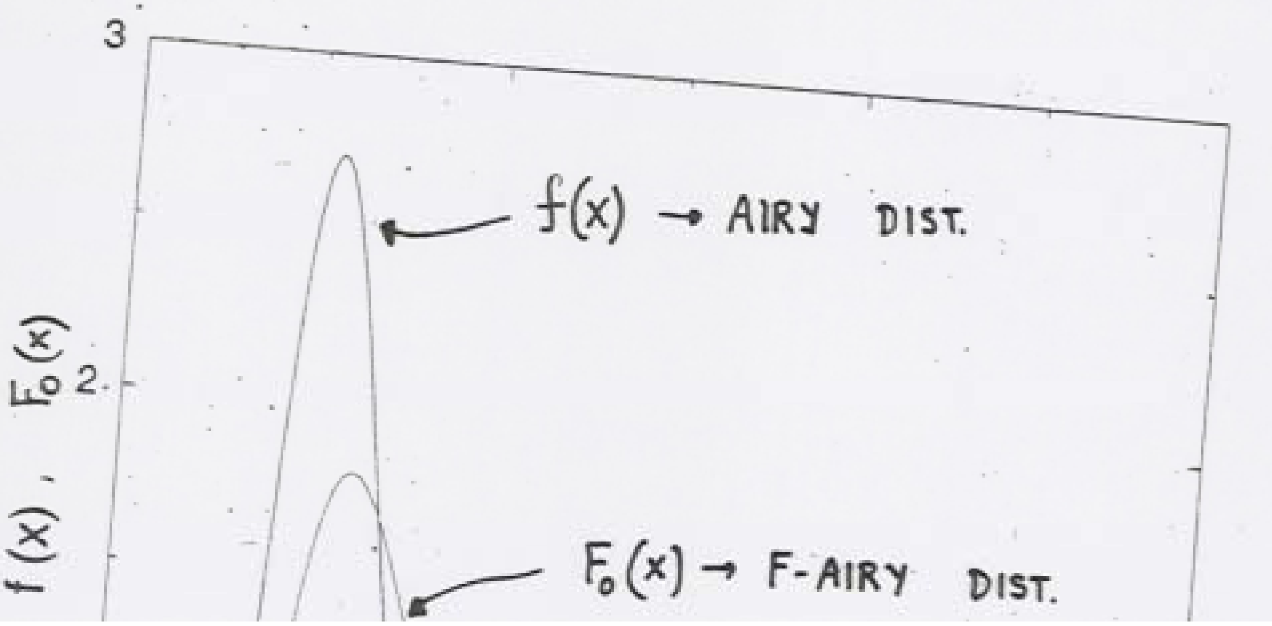
EDWARDS-WILKINSON SIMULATION (FREE BOUNDARY CONDITION)



$P(h_m, L) = \frac{1}{\sqrt{L}} f\left(\frac{h_m}{\sqrt{L}}\right)$, $f(x) \to$ AIRY DISTRIBUTION

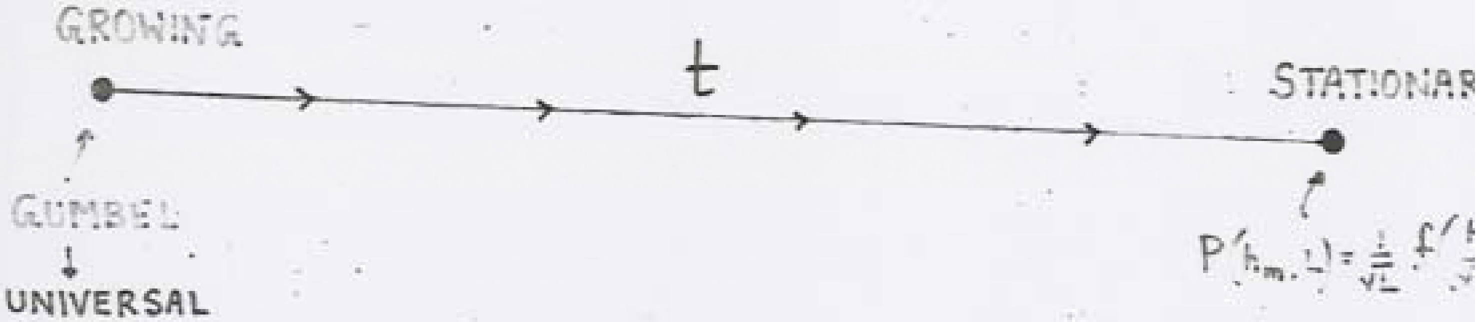
FREE : $P(h_m, L) = \frac{1}{\sqrt{L}} F_0\left(\frac{h_m}{\sqrt{L}}\right)$. $F_0(x) \to$ F-AIRY DISTRIBUTION

PLOTTED USING MATHEMATICA



SUMMARY & CONCLUSION

- 'EXTREME' HEIGHT FLUCTUATIONS IN (1+1)-d INTERFACES



$f(x)$ → AIRY → PERIODIC
$f(x)$ → F-AIRY → FREE

↳ UNIVERSAL

- 'EXACTLY SOLVABLE' CASE FOR THE DISTRIBUTION OF EXTREMUM OF A SET OF 'STRONGLY CORRELATED' RANDOM VARIABLES.

APPLICATIONS

- FLUCTUATING STEPS ON Si(111)-Al SURFACES
- DISPLACEMENT OF NON-MAGNETIC PARTICLES IN DIPOLAR CHAINS AT LOW MAGNETIC FIELD
- DISPLACEMENT OF BEADS IN A POLYMER CHAIN WITH HARMONIC INTERACTION (ROUSE MODEL)

↳ (1+1)-d EDWARDS-WILKINSON EQ.

OPEN QUESTIONS:

EXTREME HEIGHT FLUCTUATIONS FOR (d+1)-dim SURFACES