Avalanche dynamics in complex networks: Application of branching process

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VI. Summary

Avalanche:

The sequence of consecutive events triggered by the change in the state of a single element.

e.g.,

- cascading failure of power transmission system,
- congestion propagation in the Internet,
- economic crisis,
- sandpile avalanche,
- information cascade; fad spreading, etc.

In many cases, the avalanche size distribution follows a power law at criticality.

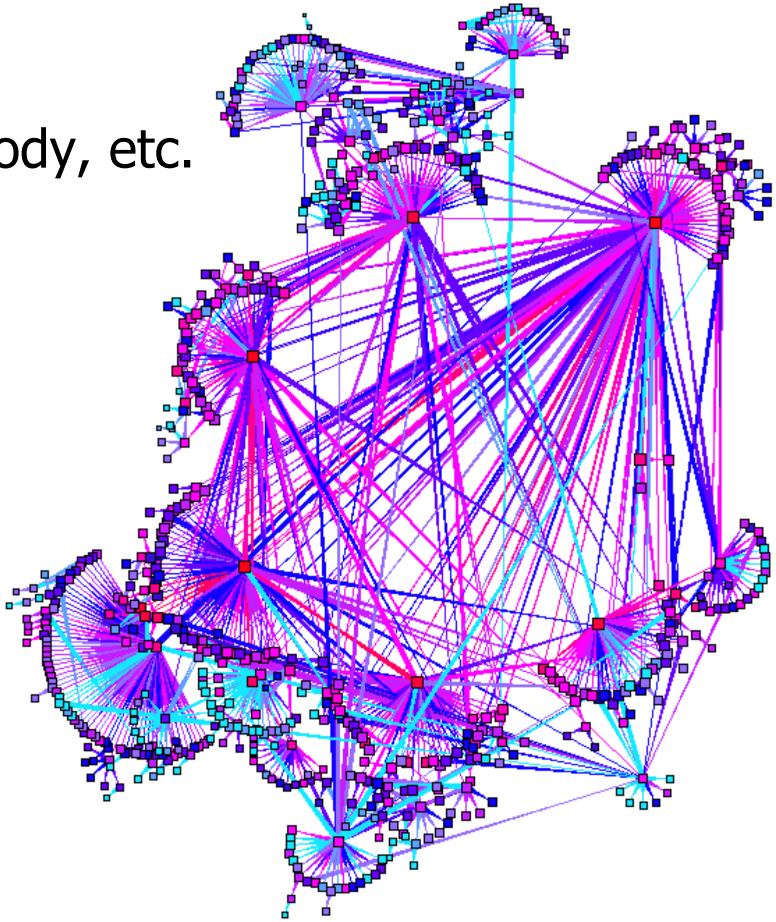
How would the critical behavior be affected as the underlying substrate becomes complex networks?

Scale-free networks:

The networks whose degree distribution $p(k)$ follows a power law for large k, $p(k) \sim k^{-\gamma}$.

e.g.,

- the Internet and the WWW,
- the metabolic networks inside body, etc.



How would the avalanche size distribution be affected by the heterogeneity of the scale-free network?

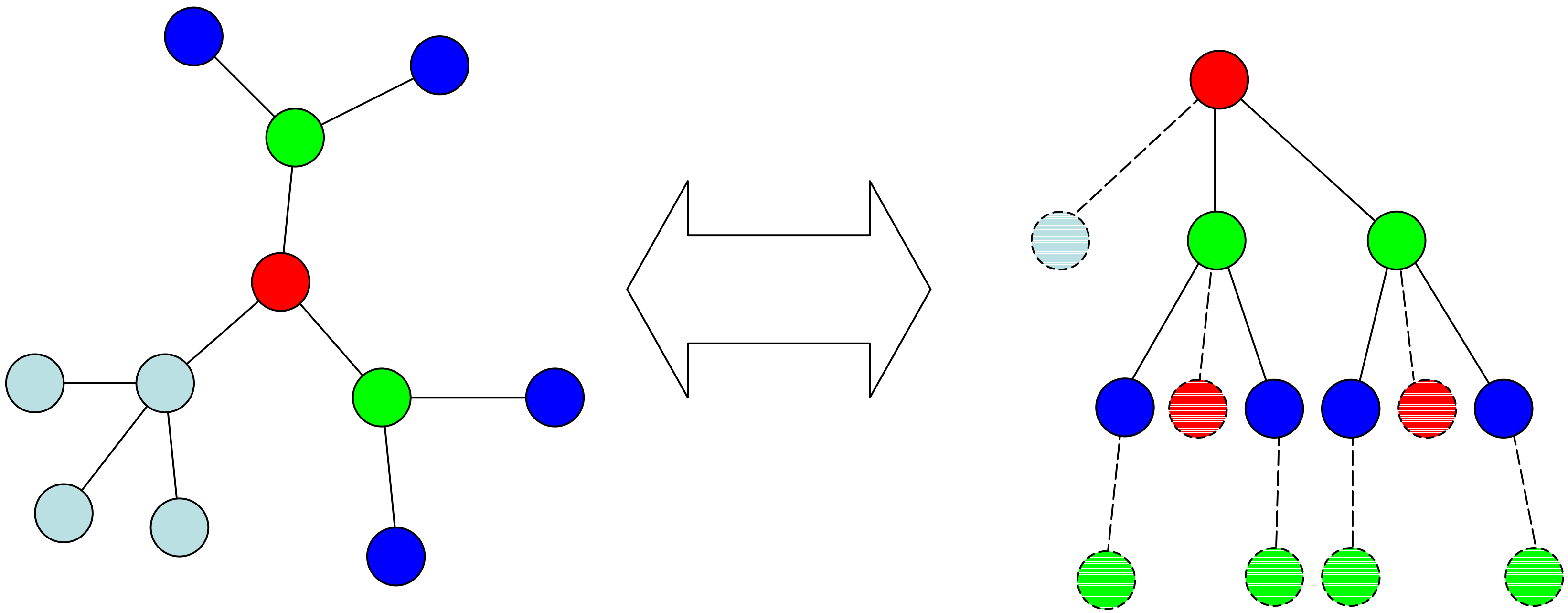
cf. Erdős-Rényi random graph

$\Leftrightarrow$ Poisson degree distribution

$\Leftrightarrow$ Mean-field result

General framework

avalanche   branching tree



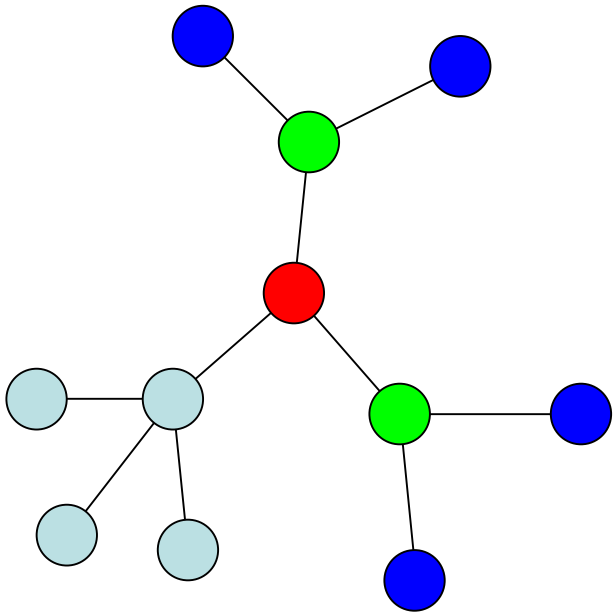
avalanche size distribution
avalanche duration distribution

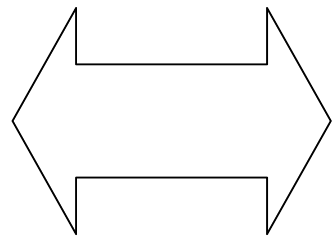
tree size distribution
tree lifetime distribution

e.g. BTW sandpile model above the upper critical dimension.
The avalanche size distribution is solved as $p(s) \sim s^{-3/2}$ [Alstrom '88] .

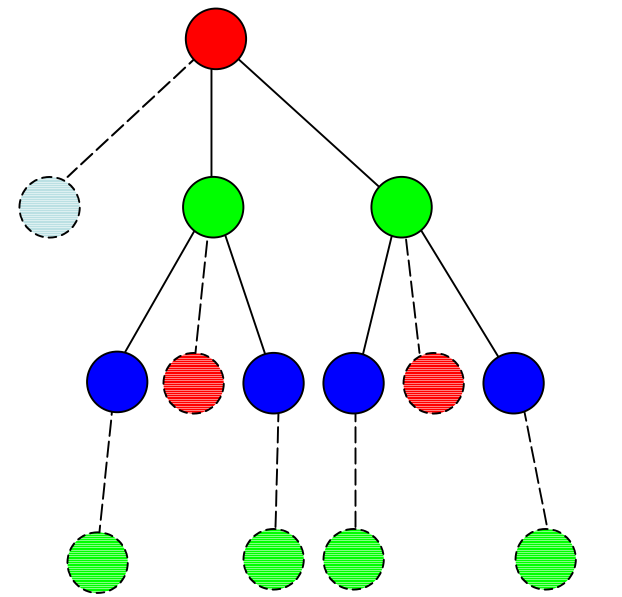
General framework

avalanche

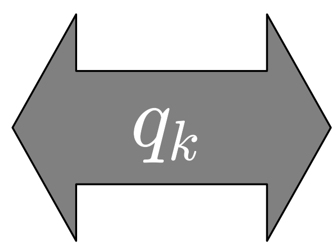




branching tree



the prob. that a site with degree k will topple

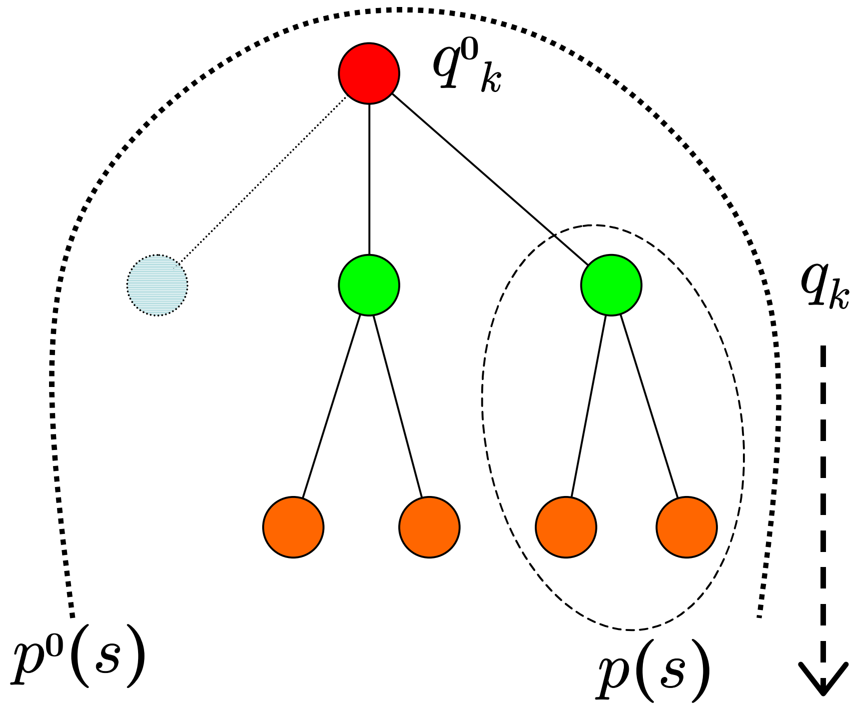


the prob. that the branching into k-branch will occur

We have to know q_k to calculate $p(s)$.

Basic assumption: Avalanche rarely forms loops during propagation.

Tree size distribution of the branching process with given q_k.



$$\mathcal{Q}(\omega) = \sum_{k=0}^{\infty} q_k \omega^k,$$

$$\mathcal{P}(z) = \sum_{s=1}^{\infty} p(s) z^s$$

$$\mathcal{P}(z) = z\mathcal{Q}(\mathcal{P}(z)),$$

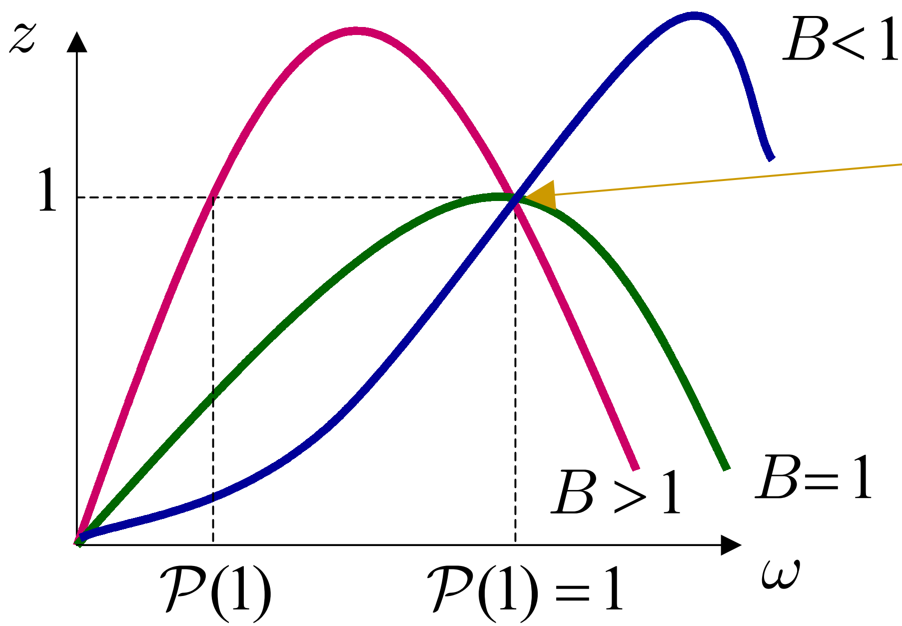
$$\mathcal{P}^0(z) = z\mathcal{Q}^0(\mathcal{P}(z))$$

$$p^0(s) = \frac{1}{s!} \frac{\mathrm{d}^s}{\mathrm{d}z^s} \mathcal{P}^0(z) \Big|_{z=1}$$

$$\mathcal{P}(z) = zQ(\mathcal{P}(z))$$

$$\omega = \mathcal{P}(z) \qquad z = \mathcal{P}^{-1}(\omega) = \frac{\omega}{Q(\omega)}$$

$$B = \sum_{k=0}^{\infty} k q_k$$



$$\left.\frac{dz}{d\omega}\right|_{\omega=1} = 1 - B$$

$\mathcal{P}(1) = 1$ for $B \leq 1$

$\mathcal{P}(1) < 1$ for $B > 1$

$\mathcal{P}(1) < 1$: Prob. that a tree is infinitely large is $1 - \mathcal{P}(1)$

At criticality B=1, $\mathcal{P}(z)$ becomes singular at z=1 and expanded as $\mathcal{P}(z) \cong 1 - b(1-z)^{\phi}$, leading to $p(s) \sim s^{-\tau}$ with $\tau = \phi + 1$.

$$(1-z)^{\phi} = \sum_{s=0}^{\infty} a_s z^s, \qquad a_s = \frac{\Gamma(s-\phi)}{\Gamma(s+1)\Gamma(-\phi)} \sim s^{-(\phi+1)}$$

If $\mathcal{Q}(\omega) = 1 - B(1-\omega) + C(1-\omega)^2 + C'(1-\omega)^{\phi} + \cdots$

ϕ : non-integer

At criticality ($B=1$),

$$z = \mathcal{P}^{-1}(\omega) = \frac{\omega}{\mathcal{Q}(\omega)} \simeq 1 - c(1-\omega)^a, \quad a = \min[2, \phi]$$

$$\mathcal{P}(z) = \omega \simeq 1 - \tilde{c}(1-z)^{1/a} \qquad \Rightarrow p(s) \sim s^{-(1+1/a)}$$

Off criticality ($B \neq 1$) and $C' \neq 0$,

$$z = \frac{\omega}{\mathcal{Q}(\omega)} \simeq \omega[1 + B(1-\omega)]$$

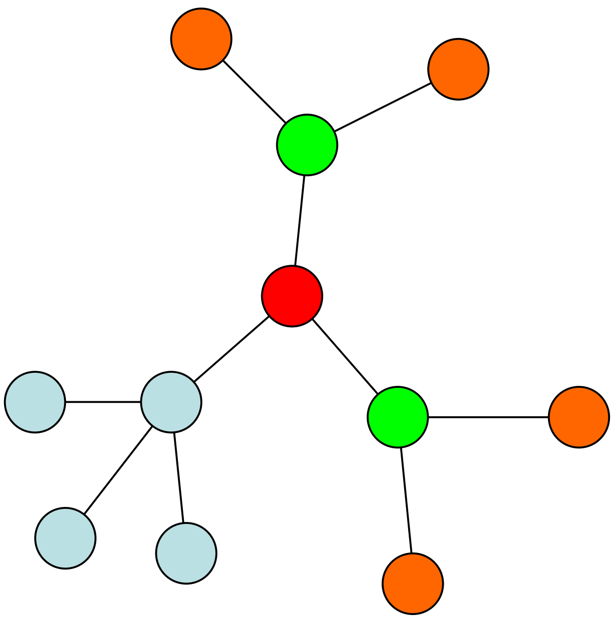
$$\mathcal{P}(z) = z\mathcal{Q}(\omega) \simeq z\mathcal{Q}(z) = \widetilde{C}(1-z)^{\phi} + \text{ analytic terms}$$

$$\Rightarrow p(s) \sim s^{-(1+\phi)}$$

If $C'=0$, $\mathcal{P}(z)$ becomes analytic. Thus $p(s)$ decays no slower than the exponential.

See also, Saichev et al. cond-mat (2003)

Bak-Tang-Wiesenfeld sandpile model with heterogeneous threshold $z_i = k_i$



q_k is decomposed into two terms q_k' and q_k'':

$$q_k = q_k' q_k''$$

Prob. that the vertex with degree k receives a grain from toppling

Prob. that the vertex actually topples, *i.e.*, height of the vertex is z_i-1, before toppling.

Assumption:
No correlation between degrees of adjacent vertices.

$$q_k' = \frac{kp(k)}{\sum_k kp(k)}, \qquad q_k'' = \frac{1}{k}$$

$$q_k = \frac{k^{-\gamma}}{\zeta(\gamma-1)} \qquad (k \geq 1)$$

$$B = \sum_{k=0}^{\infty} k q_k = 1$$

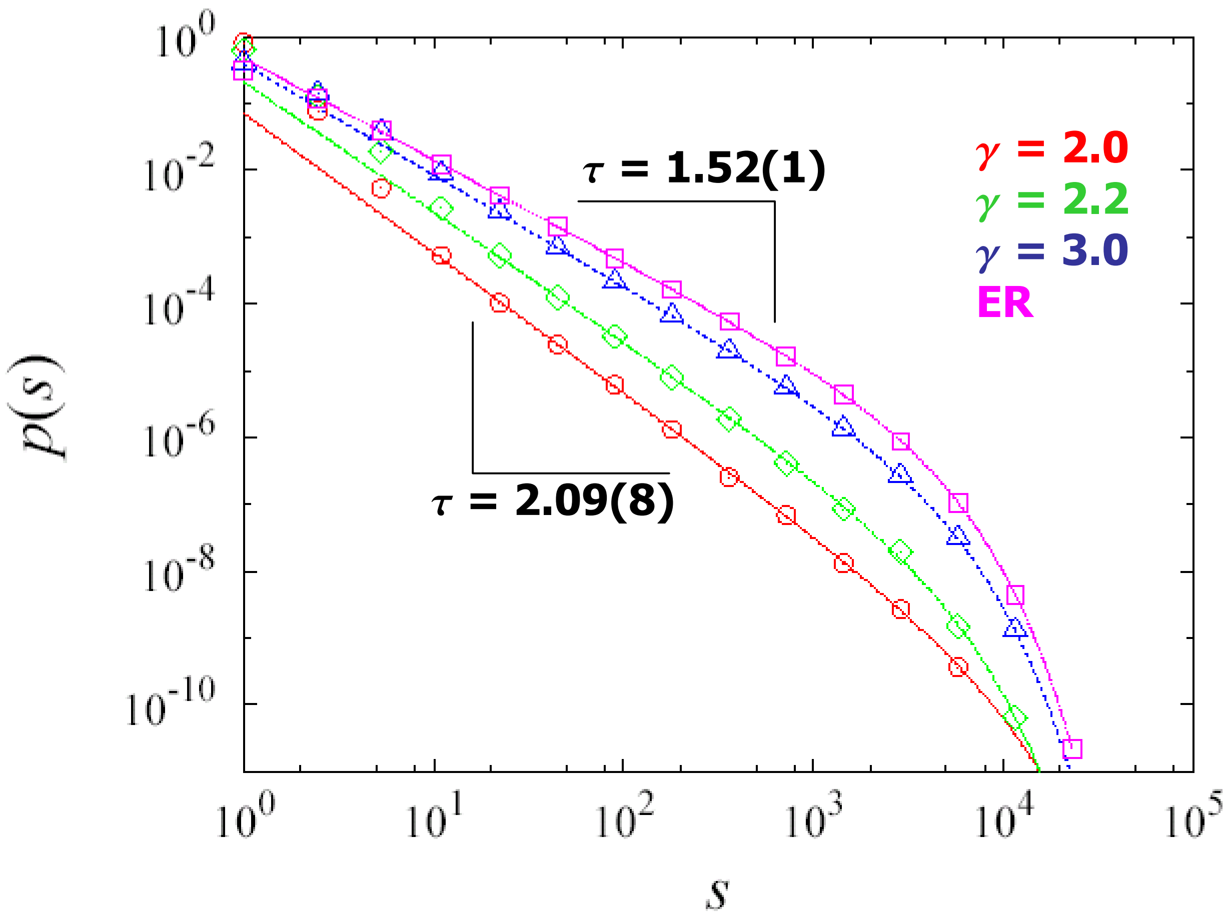
SOC

$$q_0 = 1 - \sum_{k=1}^{\infty} q_k = 1 - \frac{\zeta(\gamma)}{\zeta(\gamma-1)}$$

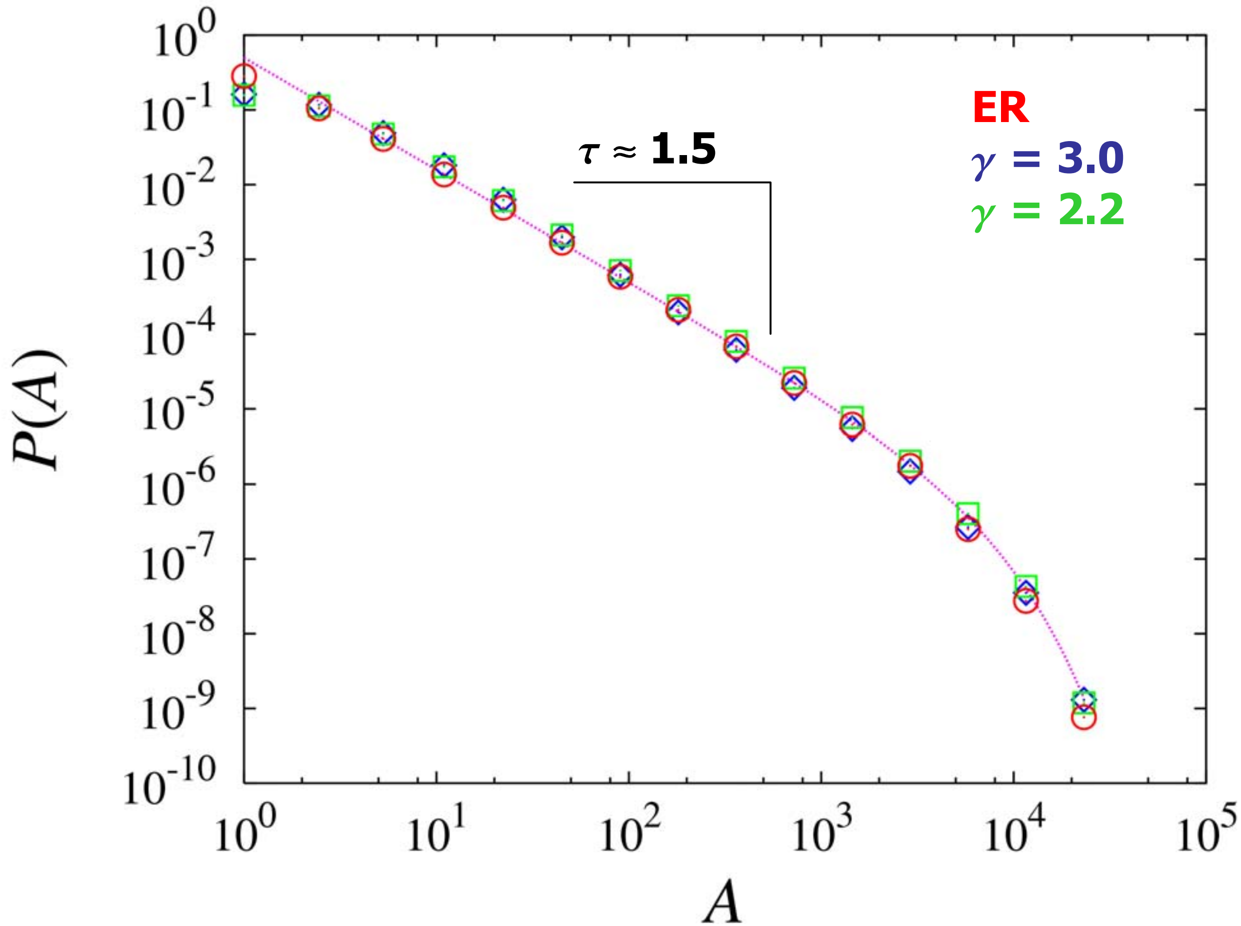
$$\mathcal{Q}(\omega) \simeq 1-(1-\omega)+\begin{cases} (1-\omega)^{\gamma-1} & (2<\gamma<3), \\ -(1-\omega)^2 \ln(1-\omega) & (\gamma=3), \\ (1-\omega)^2 & (\gamma>3). \end{cases}$$

$$p(s) \sim \begin{cases} s^{-\gamma/(\gamma-1)} & (2<\gamma<3), \\ s^{-3/2}(\ln s)^{-1/2} & (\gamma=3), \\ s^{-3/2} & (\gamma>3). \end{cases}$$

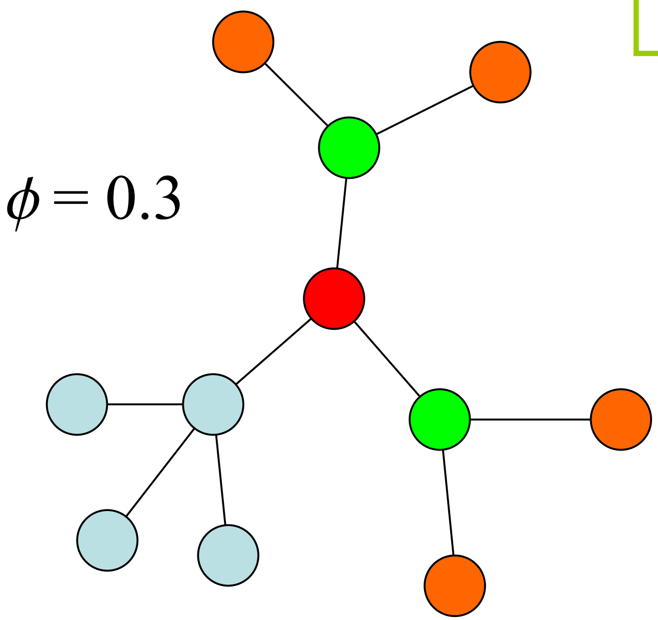
Power-law degree distribution can change the critical behavior of the sandpile



Critical behavior resumes the mean-field one when the threshold becomes uniform : z_i = const = 2



The Watts model for information cascade
[Watts, PNAS 2002]



Evolution rule:
The vertex v changes its state into 1 if the states of the fraction ϕ or more of its neighbors are 1.

- Initially all vertices are in the state 0.
- Invert the state of a seed vertex to 1 and see how the avalanche propagates, and measure avalanche size s.
- Repeat the above for all seed vertices.

$$q_k = \frac{(k+1)p_d(k+1)}{\sum_k k p_d(k)} \quad \text{for} \quad k+1 \le \left\lfloor \frac{1}{\phi} \right\rfloor,$$

The model is not SOC

$$q_k = 0 \quad \text{otherwise}$$

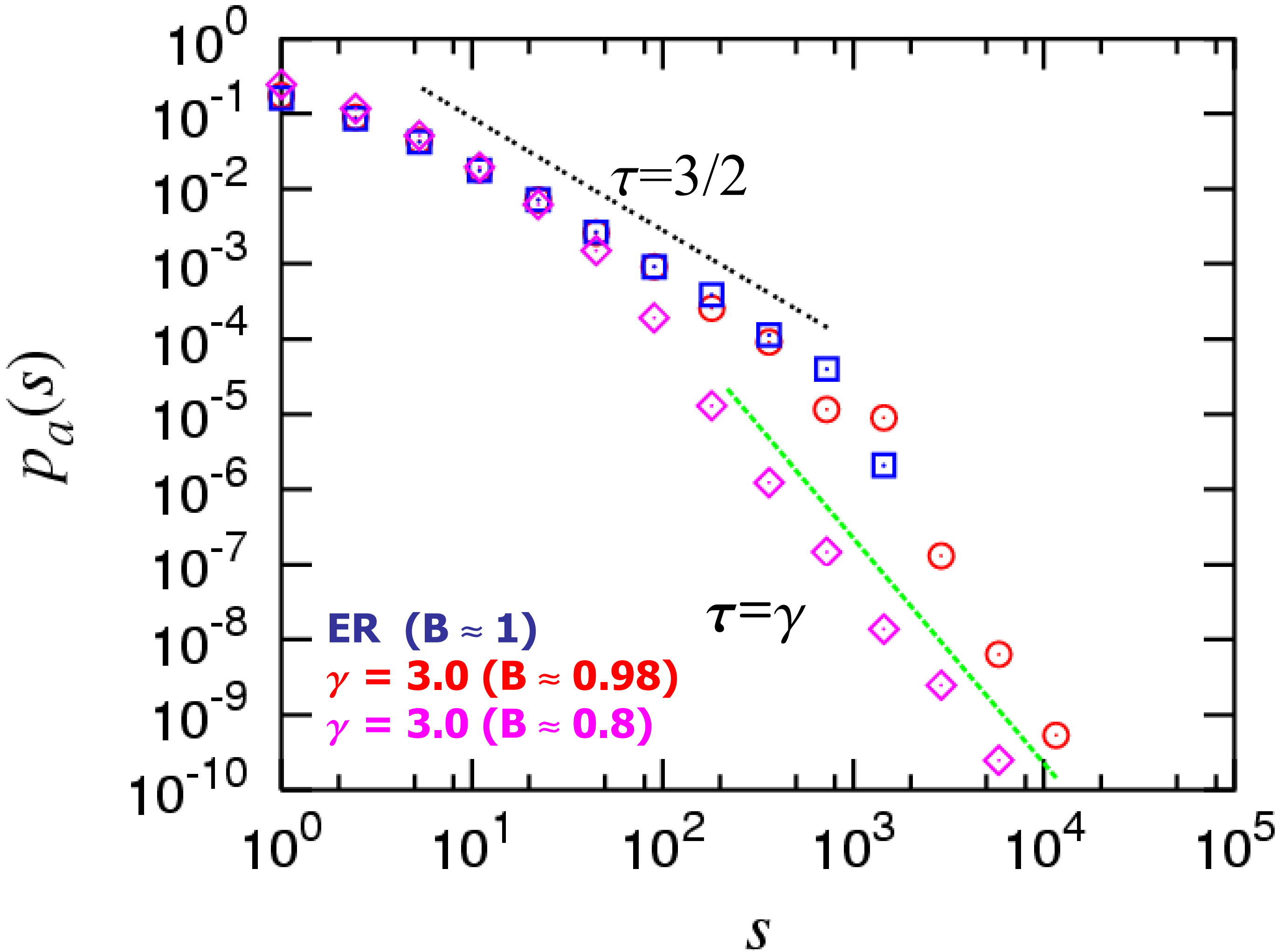
Since q_k is cut at finite k, $\mathcal{Q}(\omega)$ is analytic at ω=1, so that $p(s)$~$s^{-3/2}$ at criticality.

Off the critical point, however, the singularity arising from $q_k^0 = p_d(k)$ becomes relevant such that $\mathcal{P}^0(z)$ = 1 – A(1 – z)$^{(\gamma-1)}$ + analytic terms.

$$p(s) \sim \begin{cases} s^{-3/2} & s \ll s_c \\ s^{-\gamma} & s \gg s_c \end{cases}$$

$$s_c \sim |1-B|^{-2}$$

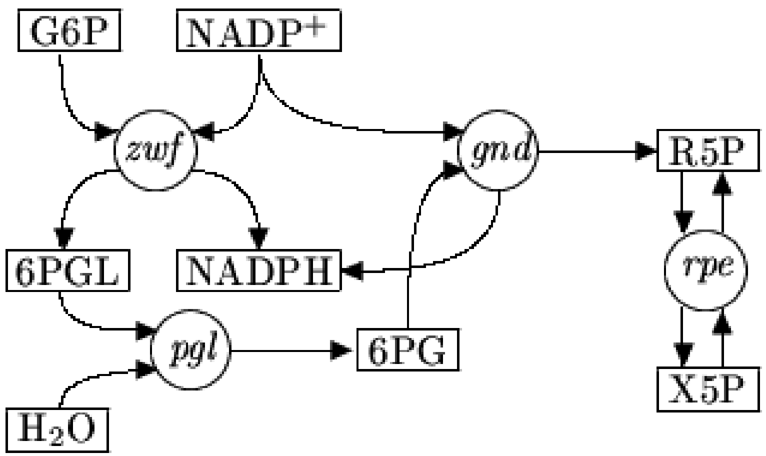
Numerical simulation supports the analytic prediction.



(a)

$$1\ \mathrm{G6P} + 1\ \mathrm{NADP^+} \xrightarrow{zwf} 1\ \mathrm{6PGL} + 1\ \mathrm{NADPH}$$
$$1\ \mathrm{6PGL} + 1\ \mathrm{H_2O} \xrightarrow{pgl} 1\ \mathrm{6PG}$$
$$1\ \mathrm{6PG} + 1\ \mathrm{NADP^+} \xrightarrow{gnd} 1\ \mathrm{R5P} + 1\ \mathrm{NADPH}$$
$$1\ \mathrm{R5P} \overset{rpe}{\longleftrightarrow} \mathrm{X5P}$$

(b)

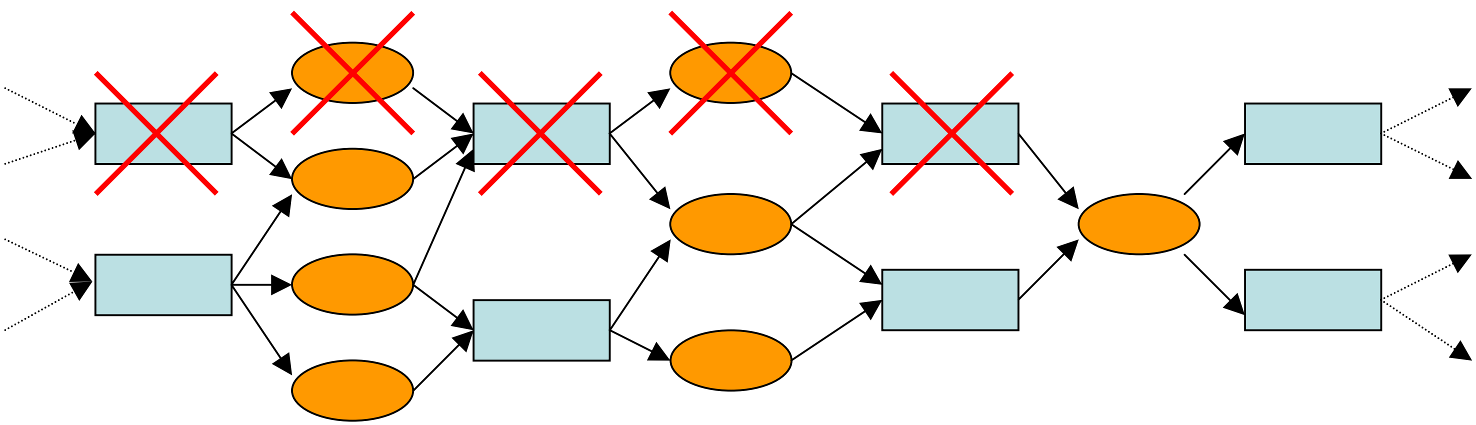


Metabolic network is a directed bipartite network which is scale-free. [Jeong et al. 2000]

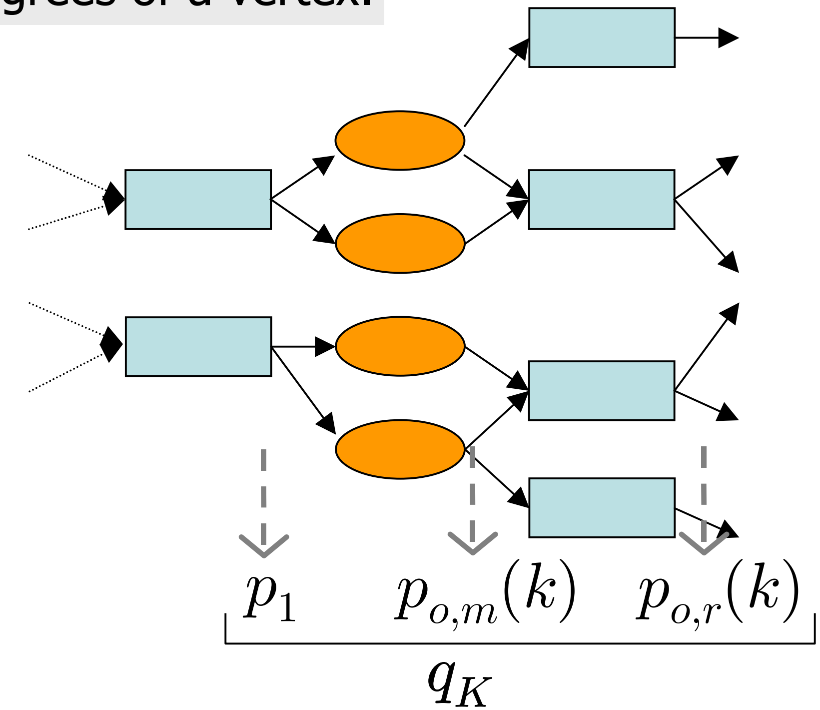
How many reactions will be blocked by a single gene knockout?

Model of reaction blockade in the metabolic networks [Lemke et al., 2004]

- The metabolic network is a directed bipartite network consisting of metabolite nodes and reaction nodes.
- Reaction node is an AND gate and metabolite node an OR gate.



Further assumption:
No correlation between in- and out-degrees of a vertex.



critical regime

$$p(s) \sim s^{-\tau}$$

$$\tau = 3/2 \quad \gamma > 3$$

$$\tau = \gamma/(\gamma - 1)$$

for $2 < \gamma < 3$.

$$p(s) \sim s^{-\gamma}$$

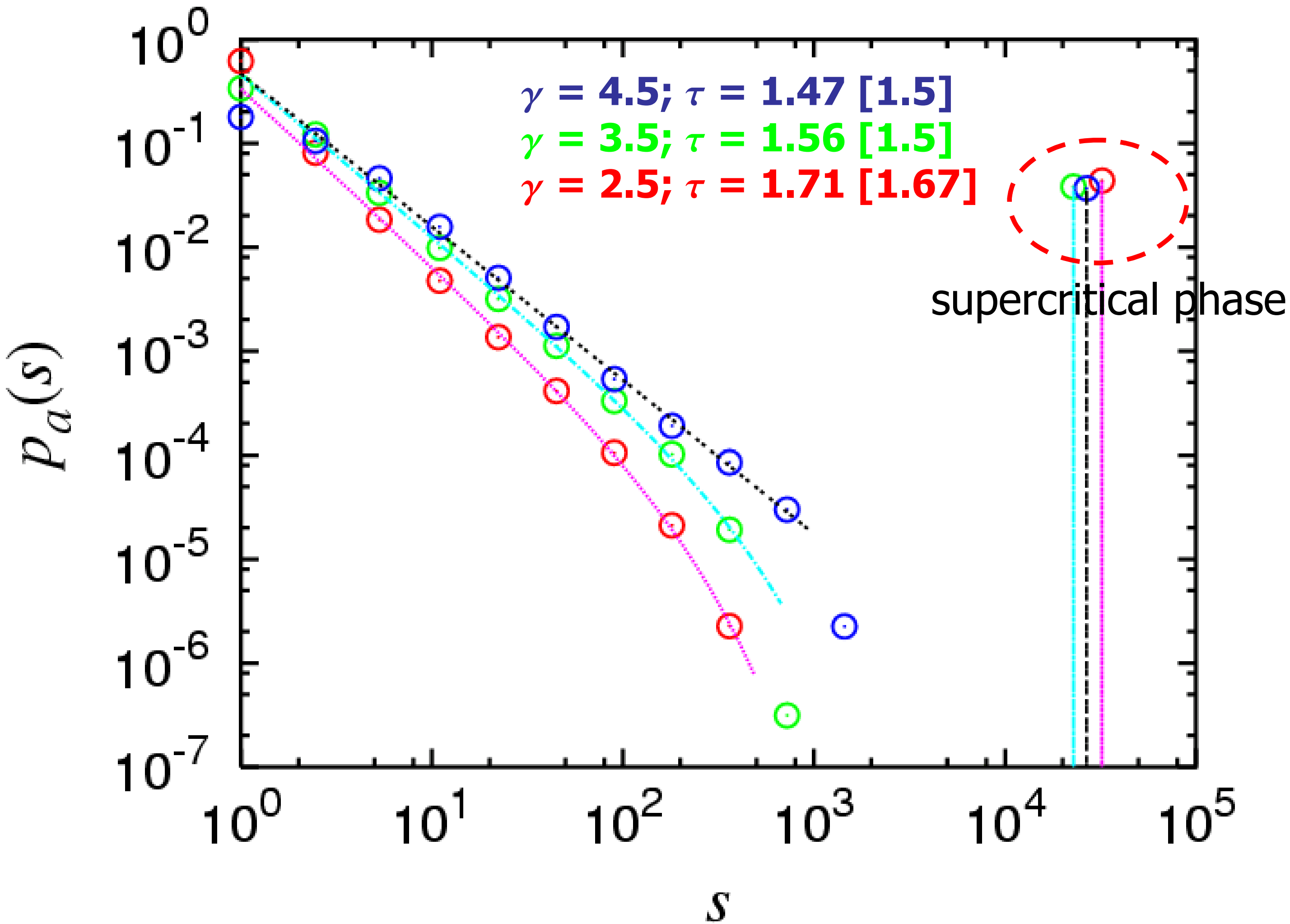
subcritical regime

$$Q(\omega) = (1 - p_1) + p_1 Q_m(Q_r(\omega))$$

$$\simeq 1 - A(1-\omega) + \begin{cases} B(1-\omega)^{\gamma-1} & (2 < \gamma < 3), \\ B'(1-\omega)^2 & (\gamma > 3). \end{cases}$$

$\gamma = \min[\gamma_m, \gamma_r]$ and $A=1$ when the criticality condition is fulfilled.

Numerical simulation supports the analytic prediction.



- Critical behavior of avalanche propagation of various models on scale-free networks has been investigated through the mapping to corresponding branching processes.
- The avalanche size distribution becomes dependent on the degree exponent of the underlying networks when $\gamma<3$, i.e., the heterogeneity is high.
- For $\gamma>3$, the critical behavior is reduced to the conventional mean-field one.
- The effect of the degree distribution can also act off the criticality.
- Avalanche duration distribution can be obtained in a similar way.

More generally, if $z_i = k_i^{\beta}$ $(0 \le \beta \le 1)$

$$C = \sum_{k=0}^{\infty} k q_k = 1$$

Self-organized criticality

$$\tau = (\gamma + 2\beta - 2)/(\gamma + \beta - 2)$$

for $2 < \gamma < \gamma_c$ where $\gamma_c = \beta + 2$

conventional mean-field values are recovered for $\gamma > \gamma_c$

In silico experiment on *E. coli* metabolic network