Spectra of Complex Networks

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Talk Outline

- Short introduction to networks
  - Time line
  - Some concepts
- Spectra of several networks
- General theory
- Application to Internet
- Conclusions

Time Line on Networks

1736	1941	1951	1956	1960	1967	1972
Euler	Flory-Stockmayer	Solomon-Rappaport	Simon	Erdos-Renyi	Milgram	Bollobás
7 bridges	Polymers	Birth of graph theory	Word analysis (power-law)	Graph theory	Social Networks	More complex graphs

Time Line on Networks

1998 1999 1999 2000….

Lots of physicists enter in the game…

Vatts-Strogatz
Small-world

Barabási-Albert
Scale-free model

Real Networks:
Biology
Internet
WWW
Economy
etc…

Faloutsos-Faloutsos-Faloutsos
Power law
Internet

Albert-Jeong-Barabási
Power-law
WWW

Dorogovtsev-Mendes
Exact solution SW

Dorogovtsev-Mendes-Samukhin
Krapivsky-Redner
Exact solution SF

Network complexity

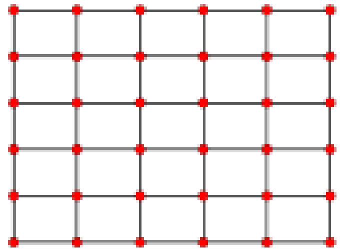
- Structure: *different types of topology*
- Network evolution: *grow of the # links and # nodes*
- Links diversity: *links can have direction, weight, etc*
- Node diversity: *different types of nodes*
- Dynamical complexity: *to each node we can assign a set of dynamical rules (spins, agents, etc)*

Some topological properties

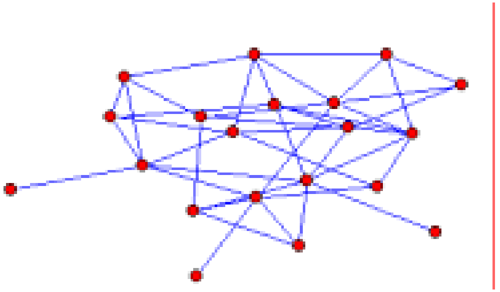
- Degree & degree distribution
- Clustering
- Correlations
- Shortest paths
- Betweenness
- Spectrum (*we will focus attention on it!*)

What should we expect?

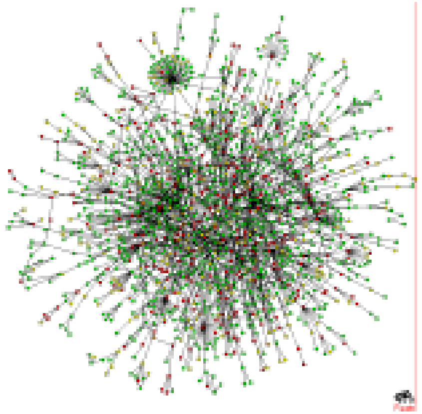
- In regular lattices all nodes are identical (same degree)



- In random networks the majority of nodes have approximately the same degree ($k_i \sim \langle k \rangle$)



- Real-world networks: the distribution has a power tail



$$P(k) \approx k^{-\gamma}$$ “scale-free” networks

Classical Random Graph (*CRG*)

- Maximal random networks under the constrain that the mean degree is $<k>$.

- Erdos-Renyi model (1959)
  - Statistical ensemble of graphs with N vertices and L edges, where each member has equal probability of realization ($<k> = 2L/N$).

- Gilbert model (1959)
  - Each pair of N vertices is connected with probability p

In the thermodynamic limit both models are equivalent:

$$<k> = p(N-1)$$

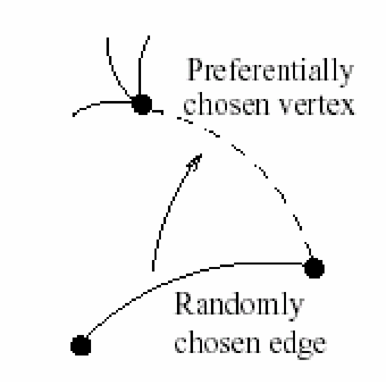
The degree distribution of these *CRG* is Poisson like.

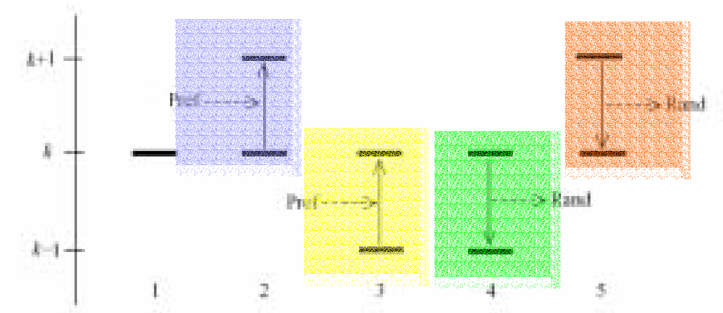
What types of networks can we have?

- Equilibrium
  - N, L fixed; $<k> = 2L/N = const.$
- Non-equilibrium (growing)
  - Linear: $L \sim N \sim t$; $<k> \sim const.$
  - Non linear: $N \sim t$, $L \sim t^{a+1} \Rightarrow <k> \sim t^{a}$

How we built and equilibrium network?

- The construction of Erdos-Renyi is restricted, it only produce Poissons!
- We can use different ensembles (microcanonical, canonical, …)
- Can equilibrium networks have a fat tailed distribution?
  - In principle yes!
  - Canonical procedure:
    - (*set*) Set of all graphs with L edges.
    - (*rules*) At each step of evolution, one of the ends of a randomly chosen edge is rewired to a preferentially chosen vertex of degree k. The rate of this process is $f(k)$.





$$\langle N(k,t+1)\rangle = \langle N(k,t)\rangle - \frac{f(k)}{N\langle f(k)\rangle}\langle N(k,t)\rangle + \frac{f(k-1)}{N\langle f(k)\rangle}\langle N(k-1,t$$

$$-\frac{k}{N\bar{k}}\langle N(k,t)\rangle + \frac{k+1}{N\bar{k}}\langle N(k+1,t)\rangle .$$

$$N\frac{\partial P(k,t)}{\partial t} = \frac{1}{\langle f(k)\rangle}[-f(k)P(k,t) + f(k-1)P(k-1,t)]$$
$$+\frac{1}{\bar{k}}[-kP(k,t) + (k+1)P(k+1,t)] .$$

Stationary solution:

$$\frac{f(k)}{\langle f(k)\rangle}P(k) - \frac{k+1}{\bar{k}}P(k+1) = \text{const} = 0$$

We set const=0, given kP(k) ->0, then:

$$f(k) = \frac{(k+1)P(k+1)}{P(k)}\frac{\langle f(k)\rangle}{\bar{k}}$$

Fat-tailed: $f(k)\bar{k}/\langle f(k)\rangle = k + o(k)$

A fat-tailed degree distribution can be realized in an equilibrium network only at a single value (critical) of the average degree, $<k>_c$ given by the condition:

$$<k> = <f(k)>$$

If we give, $P(k) \sim k^{-\gamma}$, what will be $f(k)$?

$$f(k) \cong k + 1 - \gamma + O(k^{-1})$$

Uncorrelated networks

- In the case of *CRG* correlations are absent:
  - The degree of connected vertices are uncorrelated, and *loops* are not essential in the large network limit.
  - The fact that *loops* are not essential in the thermodynamic limit implies that any finite neighborhood of a vertex has a *tree-like structure*.

What are complex networks?

- Are nets with a more complex structure than *CRG*'s.
  - Degree distribution
  - Correlations
  - Real nets are complex networks, with fat-tailed degree distribution, usually with strong correlations and loops.

Degree-degree correlations

- In the case of uncorrelated, one end of an edge is attached to a vertex of degree k with probability $kP(k)/\langle k \rangle$.

- If correlations exist, the joint distribution $P(k,k')$ differs from the uncorrelated case:

$$P(k,k') = \frac{kP(k)k'P(k')}{\langle k \rangle^2}$$

- For the citation graph

$$\text{for} \quad k \gg k' \gg 1 \quad \text{we have} \quad P(k,k') \sim k^{-(\gamma-1)} k'^{-2}$$

Reasons for the fat-tailed distributions

- Self-organization
- Optimization processes involving many agents
- Multiplicative stochastic processes
- Interconnection of geographically close vertices
- Secondary effects…

Preferential linking

- The most popular mechanism of self-organization is *preferential attachment*.
  - Nodes with high degree attract new connections with higher probability
  - Probability to become attached to a node of degree k, is proportional to some function of k, $f(k)$.
  - Scale-free degree distributions appear when:

$$f(k) = \frac{k + A}{\langle k \rangle + A}$$

  - The exponent is between 2 and infinity.

- Models based on this concept were presented by:
  - G. Yule (1925)
  - H.E. Simon (1955)
  - D.J. de Price (1976)
  - A.-L. Barabási and R. Albert (1999)
    - A new vertex connects to already existing one with probability proportional to the degree of such vertex.

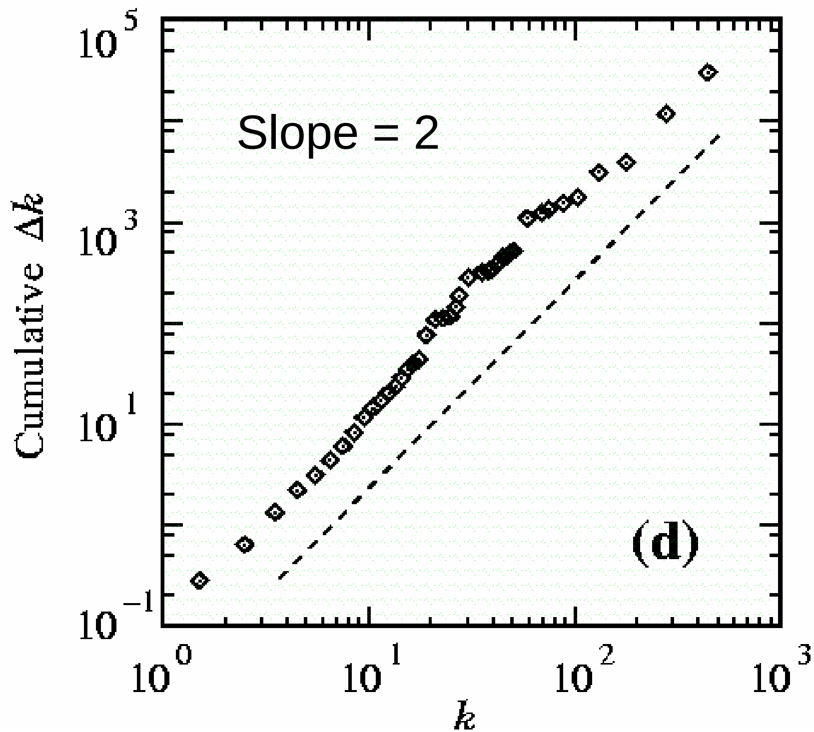
Evidence *preferential linking*…

Maps taken at different times (Δt = 6 months)

Measure $\Delta k(k)$, change of the # of links for a vertex with k edges

(S.-H. Yook et al.)

$$\boxed{\Delta k(k) \sim k}$$



internet

Other types of linking $\Pi(k_i) \sim k_i^{\alpha}$

does not produce power laws
(Krapivsky & Redner, 2000)

BA model (Barabási&Albert, Science **286**, 509(1999))

Rules:

- Network growth
  - Network evolves in size over time.
- Preferential attachment
  - Probability that a newly added node will attach to node, i

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

- Degree distribution

$$P(k) \sim k^{-\gamma}$$

- Exact solution (DMS, PRL **85**, 4633 (2000))

$$P(k) = \frac{(1+a)\Gamma((m+1)a+1)\Gamma(q+ma)}{\Gamma(ma)\Gamma(k+2(m+1)a)} \sim k^{-(2+a)}$$

Condensation of edges

- Only in growing networks (non-equilibrium) the mechanism of preferential linking gives rise to scale-free nets.
- In equilibrium nets (fixed N) only above a critical value of mean degree the preferential linking produce fat-tailed degree distributions.
- Condensation appears, i.e. a big fraction of edges get connected to a vanishingly small fraction of nodes (or even to a single vertex).
- The degree distribution of the rest of vertices is fat-tailed.

Clustering coefficient

- Local clustering (# loops of length 3): $C_i = \dfrac{n_i}{k_i(k_i-1)/2}$
- Degree dependent local clustering, *C(k)*: average of C_i over nodes of degree *k*.
  - Probability that two nearest neighbors of a vertex has degree *k*.
- Mean clustering: $$\langle C \rangle \equiv \langle C_i \rangle = \sum_k P(k) C(k)$$
- Clustering coefficient: $$C \equiv \frac{\langle n_i \rangle}{\langle k_i(k_i-1)/2 \rangle} = \frac{\sum_k P(k)\langle n(k) \rangle}{\left(\langle k^2 \rangle - \langle k \rangle^2\right)/2}$$

Note: $\langle C \rangle \neq C$ if the local clustering is degree-dependent.

For the CRG: $C(k) = C = \langle C \rangle \sim \langle k \rangle / N$.

Small-world effect

- Is the mean length of the shortest path between two vertices.

- For the *CRG* model: $$l = \frac{\ln N}{\ln\langle k \rangle}$$

- For a d-dimensional lattice: $$l \sim N^{1/d}$$

- When the growth of $l(N)$ is slower than any positive power law of *N*, is called *small-world effect.*

Ultra small-world effect

- In a correlated network the mean degree of a *nn* of a vertex is $\langle k^2 \rangle / \langle k \rangle$ which is greater than the mean degree of a vertex.

- This implies a change on the formula for the small-world

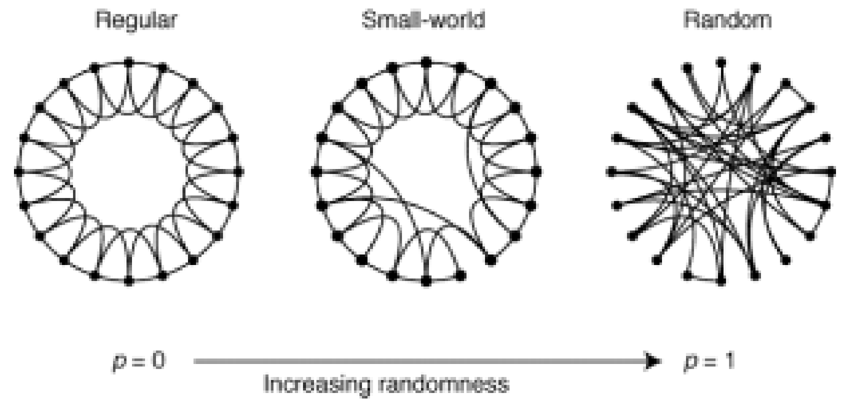
$$l \approx \frac{\ln N}{\ln\left[\left(\langle k^2 \rangle / \langle k \rangle\right) - 1\right]}$$

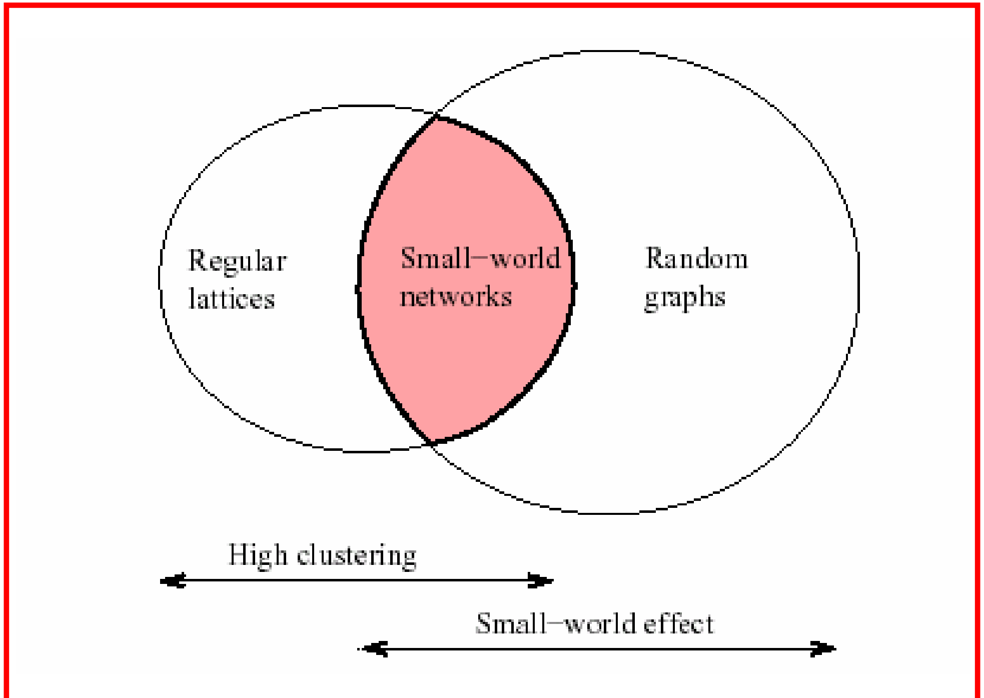
- If the second moment of the degree distribution diverges (infinite network), the average number of nn of a vertex approach infinity. Previous formula is not valid.

- In this case $l(N)$ grows with *N* slower than $\ln N$ *, its called "ultra small-world" effect.*

What happens in real networks?

- The clustering coefficient is much larger than it is in an equivalent random network





Watts & Strogatz,

Nature **393**, 440 (1998)

Betweenness (correlated with P(k))

- Let $B(i,j)$ be the total number of shortest paths between vertices i and j.
- And, $B(i,j;m)$ the set of them that pass through vertex m (set of matrices, one for each vertex).
- The betwenness of vertex m, is:

$$b(m) = \sum_{i \neq j} \frac{B(i,j;m)}{B(i,j)} \qquad 0 \leq B_{ij}^{k} \leq 1$$

- This is the probability that a shortest path between a pair of vertices passes through vertex m.
- Vertices with high betweenness play an important role.
- Real cases ~ k^{-2}!

Cut-off of degree distribution

- In finite nets, vertices with infinitely large degree are absent.
- No “perfect” scale-free degree distributions.
- In finite nets, it obstructs the observation of fat-tailed distributions.
- In a scale-free network of size N,

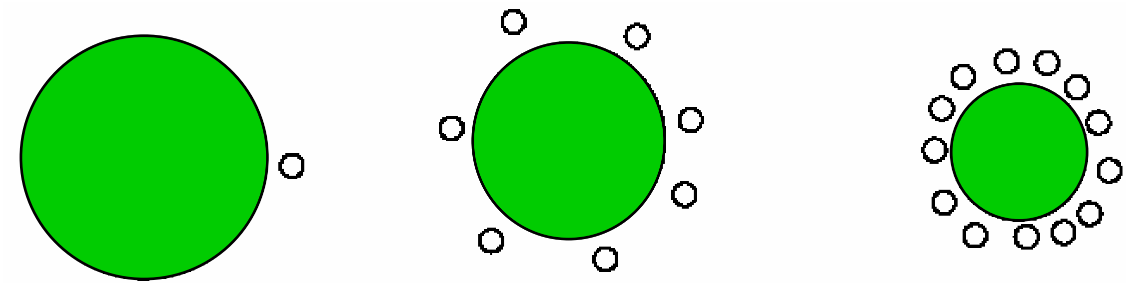
$$k_{cut} \sim N^{1/(\gamma-1)}$$

Ultra-resilience

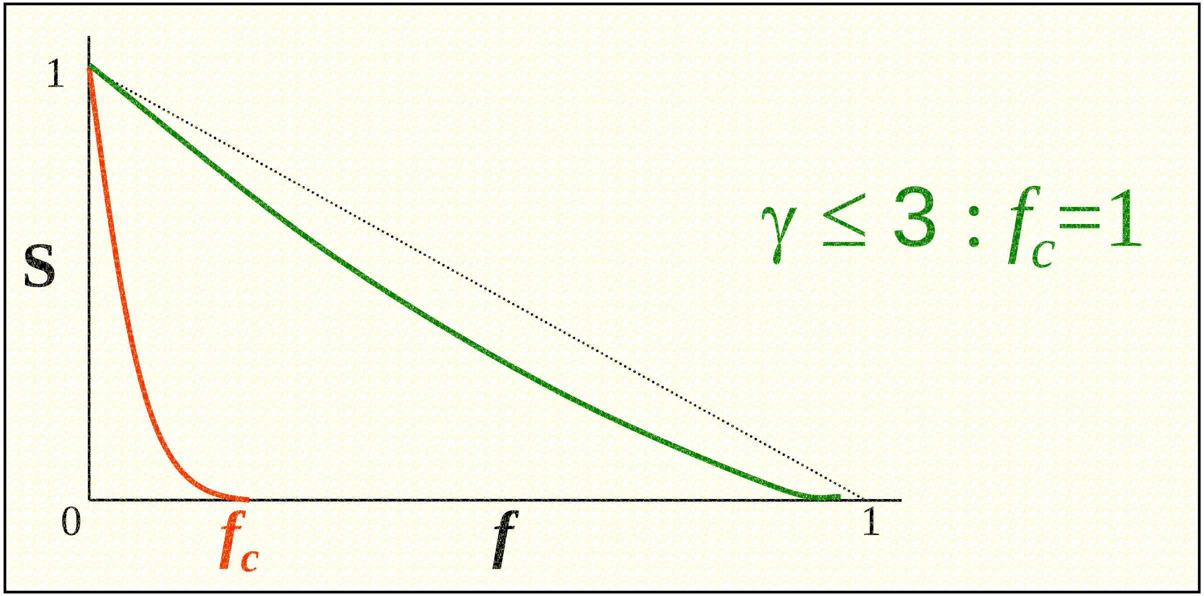
- If the second moment of the degree distribution diverges (this happens when $\langle k^n \rangle \equiv \sum_k k^n P(k) \to \infty \quad \text{for} \quad n \geq \gamma - 1$) the average degree of the *nn* of a vertex also diverges.
  - This means that a vertex in the infinite network, in average, has an infinite number of second *nn's*.
  - Equivalent to the existence of a giant connected component on the net.

- Remove a finite fraction of vertices (random failure), the average number of second *nn's* still infinity (does not eliminate the giant component).
- The same is not true in the case of intentional attach (remove vertices with high degree).

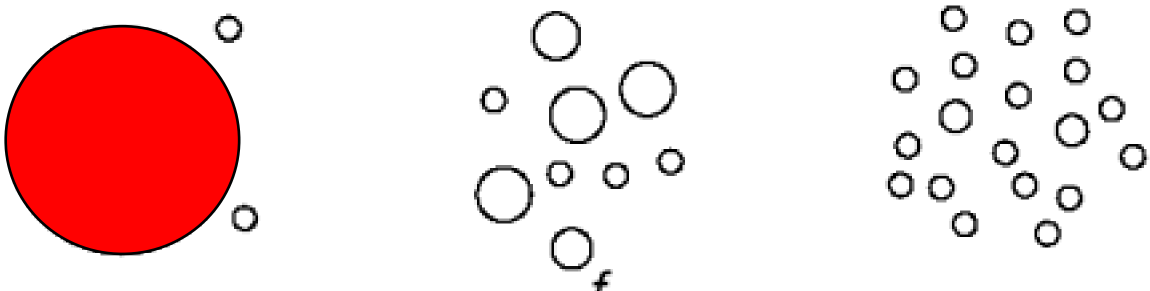
Robustness of scale-free networks



Failures



Attacks



Critical behavior of cooperative models on nets

- Networks are infinite dimension objects
  - Critical fluctuations in cooperative models are absent
  - Critical behavior is described by mean field theories
- In the case of equilibrium nets with degree distribution with a well defined scale, standard mean-field theory (with standard exponents) is valid.
- If the degree distribution is fat-tailed, critical behavior is non-standard.
  - Unusual critical exponents
  - Order of the phase transition is high or even infinite
  - No critical fluctuations
  - Mean-field still works, but is non-standard, because the presence of highly connected vertices
  - This is valid for various cooperative phenomena: percolation, magnetic phase transitions, synchronization, etc and in various networks: correlated and uncorrelated, small-worlds, etc.

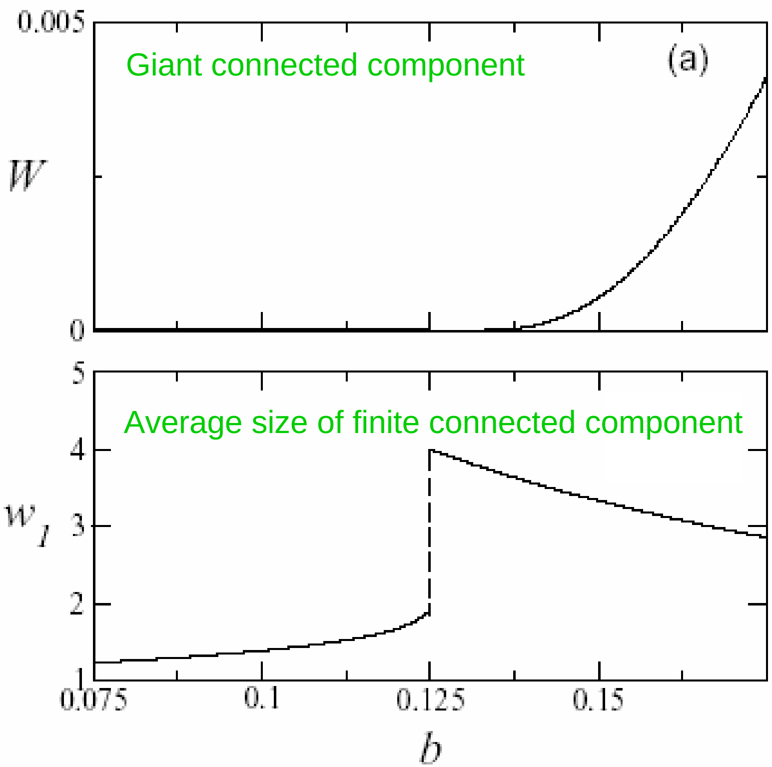
Ising model on a scale-free net

	M	$\delta C(T<T_c)$	χ	T_c
$\gamma > 5$, $\langle k^4 \rangle < \infty$	$\propto \sqrt{T_c - T}$	jump at T_c decreases as $\langle k^4 \rangle$ grows		
$\gamma = 5$, $\langle k^4 \rangle = \infty$, $\langle k^2 \rangle < \infty$	$\propto \sqrt{(T_c - T)/\ln(T_c - T)^{-1}}$	$\propto 1/\ln(T_c - T)^{-1}$	$\propto \lvert T_c - T \rvert^{-1}$	$2J/\ln \frac{\langle k^2 \rangle}{\langle k^2 \rangle - 2k}$
$3 < \gamma < 5$, $\langle k^4 \rangle = \infty$, $\langle k^2 \rangle < \infty$	$\propto (T_c - T)^{1/(\gamma-3)}$	$\propto (T_c - T)^{(5-\gamma)/(\gamma-3)}$		
$\gamma = 3$, $\langle k^2 \rangle = \infty$	$\propto e^{-2T/\bar{k}}$	$\propto T^2 e^{-4T/\bar{k}}$	J/T	$\propto \bar{k} \ln N$
$2 < \gamma < 3$, $\langle k^2 \rangle = \infty$	$\propto T^{-1/(3-\gamma)}$	$\propto T^{-(\gamma-1)/(3-\gamma)}$		$\propto \bar{k} N^{(3-\gamma)/(\gamma-1)}$

Dorogovtsev, Gotsev and Mendes, cond-mat/0203227

Percolation transition

- Exception maybe the case of non-equilibrium networks where a “non-mean-field” phase transition appears of the type *Berezinskii-Kosterlitz-Thouless singularity.*



Size of the giant connected component versus the ratio of creation of new edges.

All derivatives of $W(b)$ are zero at critical point (infinite order phase transition).

$$W(b) = 0.590\ldots\exp\left(-\frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{b-b_c}}\right)$$

Adjacency matrix. What can we get from it?

$Tr\,(\boldsymbol{A}) = 0$

$Tr\,(\boldsymbol{A}^2) = 2m$ (m = # edges)

$Tr\,(\boldsymbol{A}^3) = 6t$ (t = # triangles)

- if $tr(\boldsymbol{A}^3) = 0$ means the graph has no cycles (null clustering)

$(\boldsymbol{A}^n)_{ik}$ = # of paths of lenght n netween i and k

- If the adjacency matrix is symmetric and real => eigenvalues are real and the largest is not degenerate

- Largest eigenvalue: related with the maximum degree on the net (k_{cut})

- Second largest: related to the diameter of the graph

$$D(G) \leqslant \frac{\log(N-1)}{\log(1/\lambda_2)}$$

Adjacency Matrix

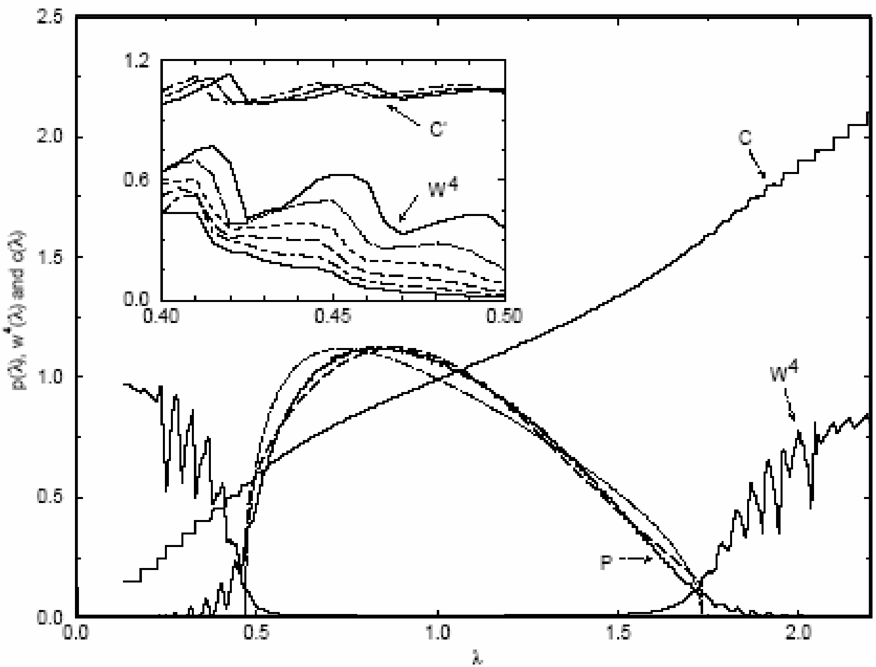
- If the adjacency matrix (NxN) is random (random entries), N-> ∞, it satisfies the Wigner theorem, and the density of states converges to the semi-circular law

$$\rho(\lambda) = \begin{cases} (2\pi)^{-1}\sqrt{4-\lambda^2}, & \text{if } |\lambda| < 2\sigma; \\ 0, & \text{otherwise.} \end{cases}$$

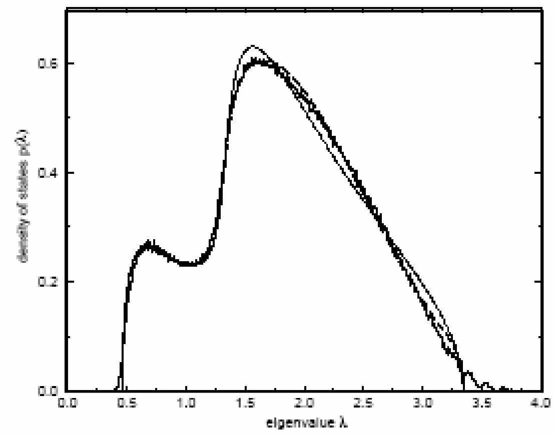
E.P. Wigner, Ann. Math. 62, 548 (55)
Ann. Math. 65, 203 (57)
Ann. Math. 67, 325 (58)

Spectra: small-world

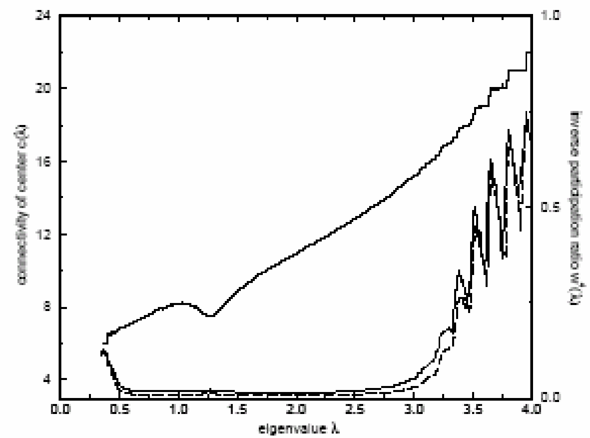
G. Biroli, R. Monasson, *J. Phys. A: Math. Gen.* **32**, L255 (1999);
R. Monasson, *Eur. Phys. J. B* **12**, 555 (1999).



Density of states *p*(l), inverse participation ratio *w*4(l) and connectivity of the centers *c*(l) (divided by *q*) averaged over 2000 samples for *q* = 20, *N* = 800.



Small-world network *K* = 3, *q* = 5. Density of states from numerics, EMA and SDA approximation.



Inverse participation ratio for N = 256 and N = 512 averaged over 1000 samples.

Spectra: sparse random matrix

Rodgers & Bray, PRB 37, 3557 (88)

Bond occupation probability = $\boldsymbol{p}\,/\boldsymbol{N}$ ($\boldsymbol{N}$ = # nodes)

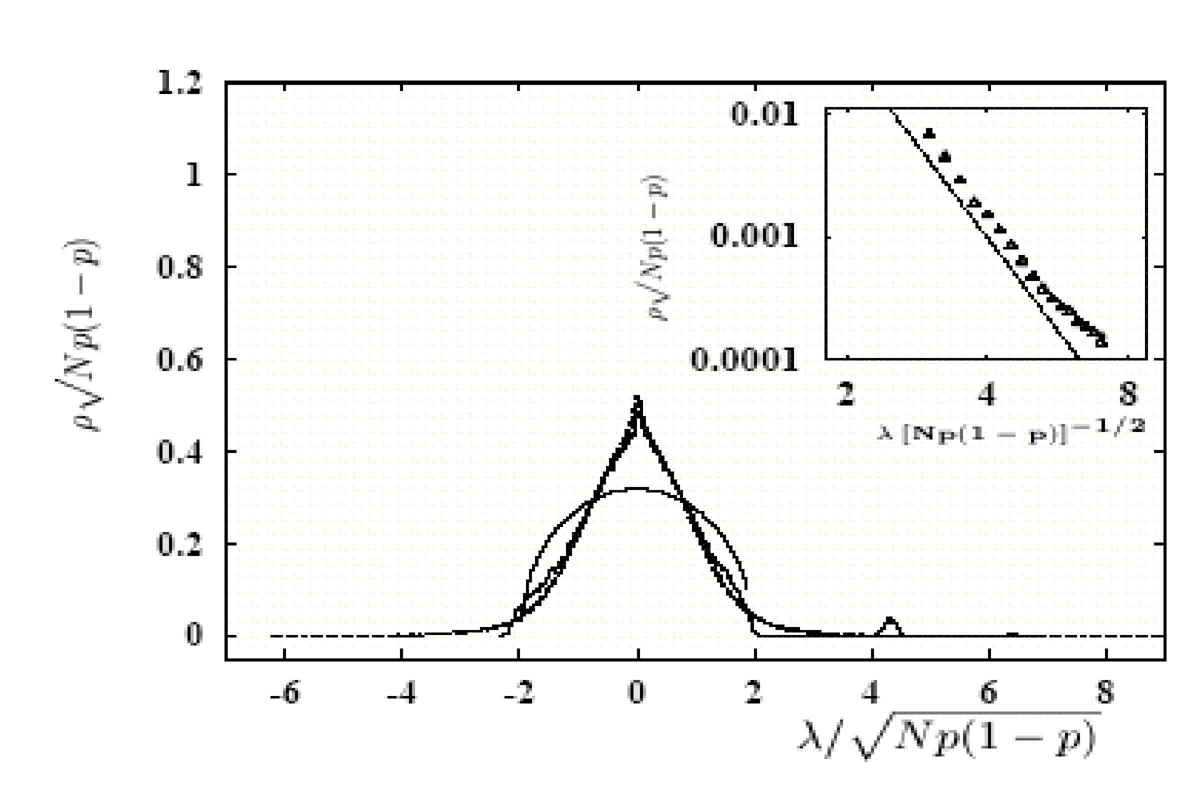
($\boldsymbol{p}$ is like the mean number of non zero elements per row)

$$\rho(\lambda) = \underbrace{\frac{2}{\pi\lambda_c}(\lambda_c^2 - \lambda)^{1/2}}_{\text{Semi-circular law}}\left\{1 + \frac{1}{p}\left(1 - \frac{4\lambda^2}{\lambda_c^2}\right) + \dots\right\}$$

tail:

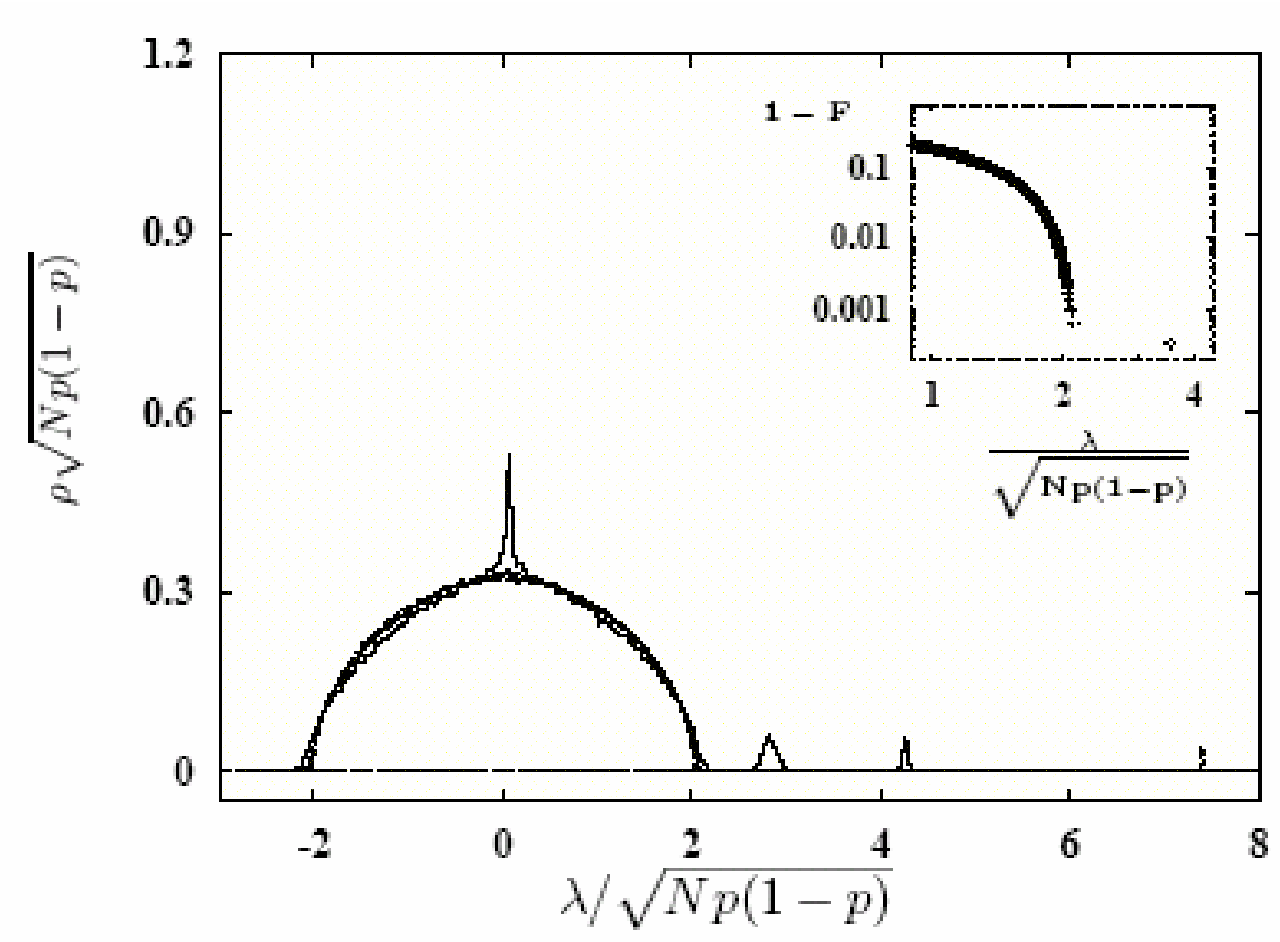
$$\rho(\lambda) \cong e^{-\lambda^2 \ln(\lambda^2/ep)};\ \lambda^2 >> \lambda_c^2$$

Spectra: scale-free network



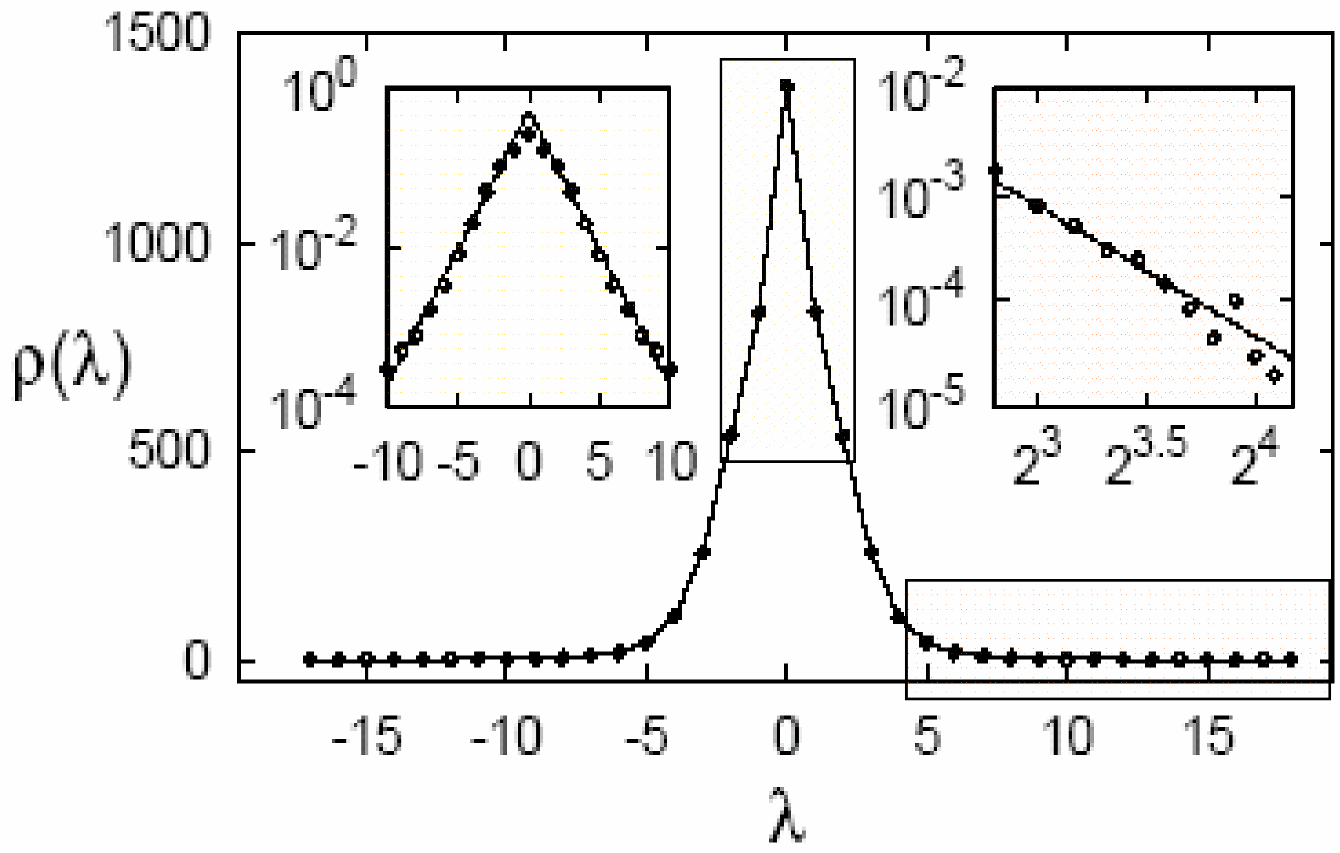
I. Farkas et al., PRE 64, 026704(2001)

Spectra: random graph



I. Farkas et al., PRE 64, 026704(2001)

Spectra: scale-free network



K.-I. Goh et al., PRE 64, 051903 (2001)

We propose:

Study the spectra of random *Bethe* lattices as a standard spectra for complex networks

- Why?

  - they mimic very well the main topological properties of complex networks

  - their tree-like structure is described completely by the degree distribution and degree correlations.

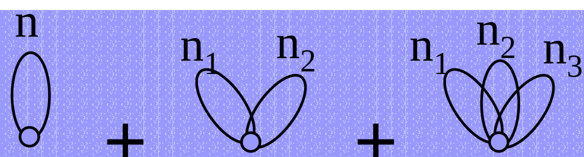
  - there are exact methods to study the spectra of random walks on tree-like graphs.

  - the clustering coefficient is equal to zero, contrarily to real nets.

General theory…

Adjacency matrix; $\boldsymbol{A}=a_{wv}$, NxN and symmetric.

Method: RW's on a tree like graph

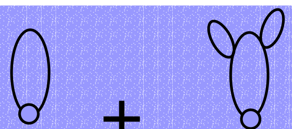
$\rho_v(n) =$ 

$+ \ldots$

$$= q_n + q_{n_1} q_{n_2} \delta_{n_1+n_2,n} + q_{n_1} q_{n_2} q_{n_3} \delta_{n_1+n_2+n_3,n} + \ldots$$

$$R(z) = \frac{1}{N}\sum_{v=1}^{N}\sum_{n=0}^{\infty}\rho_v(n) z^n$$

Generating function of the number of walks, $\rho_v(n)$ of length n starting in v and end in v.

$q_v(n) =$  $+ \ldots$

If $q_v(n)$, is the number of walks of length n starting in v and end in v for the first time.

$$Q_v(z) = \sum_{n=0}^{\infty} q_v(n) z^n$$

The previous relations are related by:

$$R(z) = \frac{1}{N}\sum_{v=1}^{N}\frac{1}{1-Q_v(z)}$$

$$t_{w,v}^{(m)}(n) \equiv$$ Number of paths of length n starting in w and ending in v for the first time

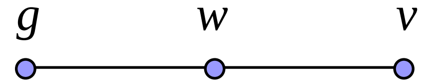
Defining the respective **generating function**:
$$T_{w,v}^{(m)} = \sum_{v=0}^{\infty} t_{wv}^{(m)}(n) z^n$$

$$q_v(n) = t_{wv}(n-1); \quad t(0) = 0; \quad q_0 = q_1 = 0$$

On can prove:

$$Q_v(z) = z \sum_{w} T_{wv}^{(1)}(z),$$
$$T_{wv}^{(m)}(z) = T_{wg_1}^{(1)}(z) T_{g_1 g_2}^{(1)}(z) \ldots T_{g_{m-1} v}^{(1)}(z)$$

where: $w \to g_1 \to g_2 \to \ldots \to g_{m-1} \to v$



$$T_{wv}^{(1)}(z) = z + z \sum_{g} T_{gv}^{(2)}(z)$$
$$= z + z \sum_{g} T_{gw}^{(1)}(z) T_{wv}^{(1)}(z)$$

The node g is the neighbor of w and a second neighbor of the node v.

Solving the previous recurrence equation, we can find $T_{wv}^{(1)}(z)$ and $Q_v(z)$

Let's change $z \to z^{-1}$, we obtain: $$\tilde{T}_{wv}^{(1)}(z) = \frac{1}{z - \sum_g \tilde{T}_{gw}^{(1)}(z)}$$

$$B(z) \equiv z^{-1}R(z^{-1}) \qquad z = \lambda + i\varepsilon \quad (\varepsilon > 0)$$

$$\rho(\lambda) = -\frac{1}{\pi}\mathrm{Im}\langle B(\lambda + i\varepsilon)\rangle$$

This equations are valid both for correlated and uncorrelated tree-like graphs.

Example: k-regular graph

$\tilde{T}_{wv}^{(1)}(z) \equiv T(z)$ and $Q_v(z) \equiv Q(z)$ then $zT(z)\text{-}(k\text{-}1)T^2(z) = 1$, solving this eqation :

$$\rho(\lambda) = \frac{\mathrm{k}}{2\pi}\frac{\sqrt{4(k-1)-\lambda^2}}{k^2-\lambda^2}$$

Continuous spectra of extended states with eigenvalues: $|\lambda| < 2\sqrt{k-1}$

Spectra of uncorrelated graph

Distribution function of $\tilde{T}^{(1)}_{wv}(z)$: $F_\lambda(x) = \left\langle \exp(-ix\tilde{T}^{(1)}_{wv}(\lambda + i\varepsilon)) \right\rangle$

The average is over the ensemble of uncorrelated graphs with given P(k). The statistical independence of the k-1 random parameters

$$\tilde{T}^{(1)}_{wv}(\lambda + i\varepsilon) \equiv T_i \ \ \text{with}\ i = 1,2,\ldots,k\text{-}1,\ k = k_w$$

$$F_\lambda(x) = \left\langle \exp(-ix\tilde{T}^{(1)}_{wv}(\lambda + i\varepsilon)) \right\rangle = \left\langle \exp\left(-\frac{ix}{\lambda + i\varepsilon - \sum_{i=1}^{k-1} T_i} \right) \right\rangle = 1 - \sqrt{x} \int_0^\infty \frac{dy}{\sqrt{y}} J_1(2\sqrt{xy}) e^{i\lambda y} \Phi_1(F_\lambda(y))$$

$$\rho(\lambda) = -\frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{i\lambda y} \Phi(F_\lambda(y)) dy$$

$$\Phi_1(x) \equiv \sum_{k=1}^\infty k P(k) x^{k-1} / \langle k \rangle$$

$$\Phi(x) \equiv \sum_{k=1}^\infty P(k) x^k$$

If we can solve the previous auto-consistent equation, we get the density of states, but …

We need some approximation!

Effective medium approximation (*EMA*)

Neglect fluctuations of T around $<T>$.

If we use: $F_\lambda(x) \cong e^{-ixT(\lambda)}$ we get:

$$T(\lambda) = \frac{1}{\langle k \rangle} \sum_k \frac{kP(k)}{\lambda + i\varepsilon - (k-1)T(\lambda)}$$

and

$$\rho(\lambda) = -\frac{1}{\pi} \sum_k \frac{kP(k)\,\mathrm{Im}\,T(\lambda)}{(\lambda - k\,\mathrm{Re}\,T(\lambda))^2 + k^2\,\mathrm{Im}\,T(\lambda)^2}$$

Tail behavior:

$$(|\lambda| >> 1)$$

$$T(\lambda) \cong \frac{1}{\lambda} - i\pi \frac{|\lambda|^3 P(k)}{\langle k \rangle}$$

$$\rho(\lambda) \cong 2|\lambda| P(\lambda^2)$$

For scale-free net…

$$P(k) = P_0 k^{-\gamma}$$

$$\rho(\lambda) = 2P_0 \left|\lambda\right|^{-(2\gamma-1)} \cong \lambda^{-\delta}$$

$$\Rightarrow \boxed{\delta = 2\gamma - 1}$$ (exact result)

- A classical random graph has the Poisson degree distribution

$$P(k) = e^{-\langle k \rangle} \langle k \rangle^k / k!.$$

- The tail is given by:

$$\rho(\lambda) \sim \lambda^{-2(\lambda^2+a)} \exp[(1+\ln\langle k \rangle)\lambda^2]$$

Adjacency matrix

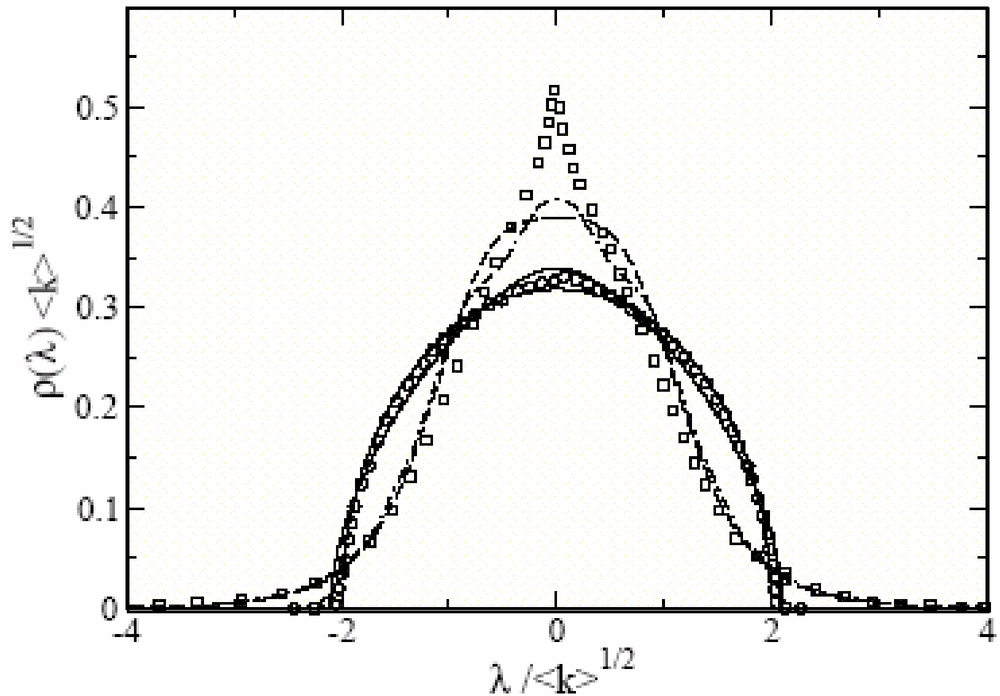


FIG. 1. Density of eigenvalues of the adjacency matrices of two networks. (i) The classical random graph (the Erdős-Rényi model) with the average degree $\langle k \rangle = 10$: the effective medium (EM) approach (the solid line) and numerical calculations for the graphs of 20 000 vertices [13] (the open circles).

Simulations:
I. Farkas et al., PRE64, 026704 (0

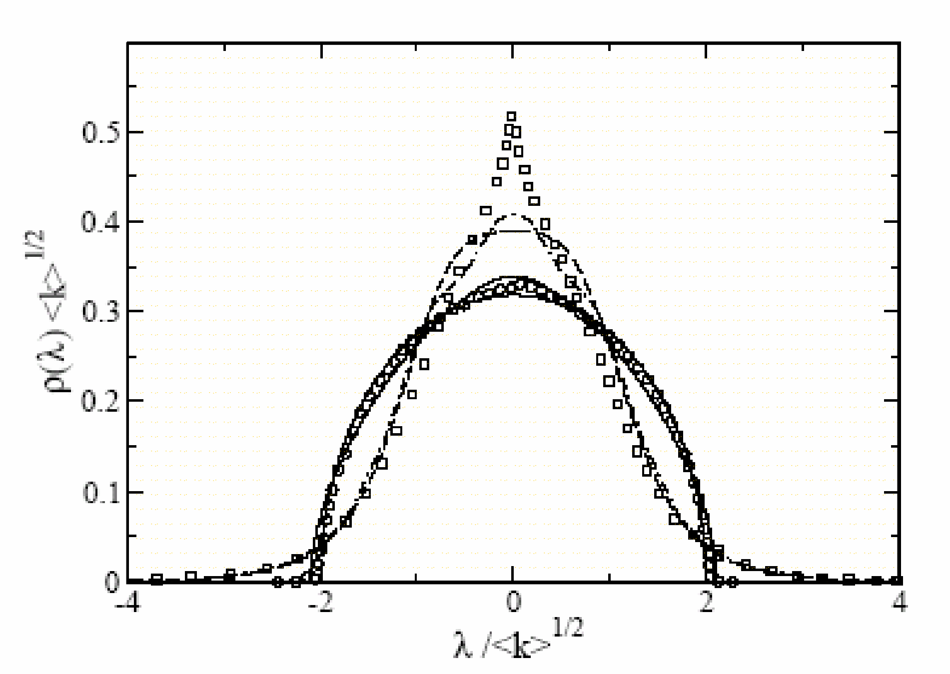
$$N \to \infty,\ pN = \text{const.}$$

$$\mathrm{N} = 20000,\ \ \mathrm{L} = 100000$$

$$\Rightarrow \langle k \rangle = \frac{2L}{N} = 10$$

Scale-free network

The spectra differs strongly from the semi-circular law



The BA model has:

- Tree like structure
- No correlations
- Negligibly small clustering

Random uncorrelated Bethe lattice mimic well this net!

Improve the EM approximation:

$$F_\lambda(x) = \left[1 + a(\lambda)x^2\right] e^{-ixT(\lambda)}$$

Power-law tail

Density of eigenvalues for $\lambda >> 1 \;\rightarrow\; \rho(\lambda) \propto \lambda^{-\delta} \qquad (k_\lambda = \lambda^2)$

Our prediction agrees with simulation results by Farkas *et al*. and Goh *et al.*

Internet (AS level) $\gamma \cong 2.1$

$\lambda_i \propto i^{\varepsilon}$ (i = order of eigenvalue)

Multi data set $\rightarrow \varepsilon \approx$ -0.447

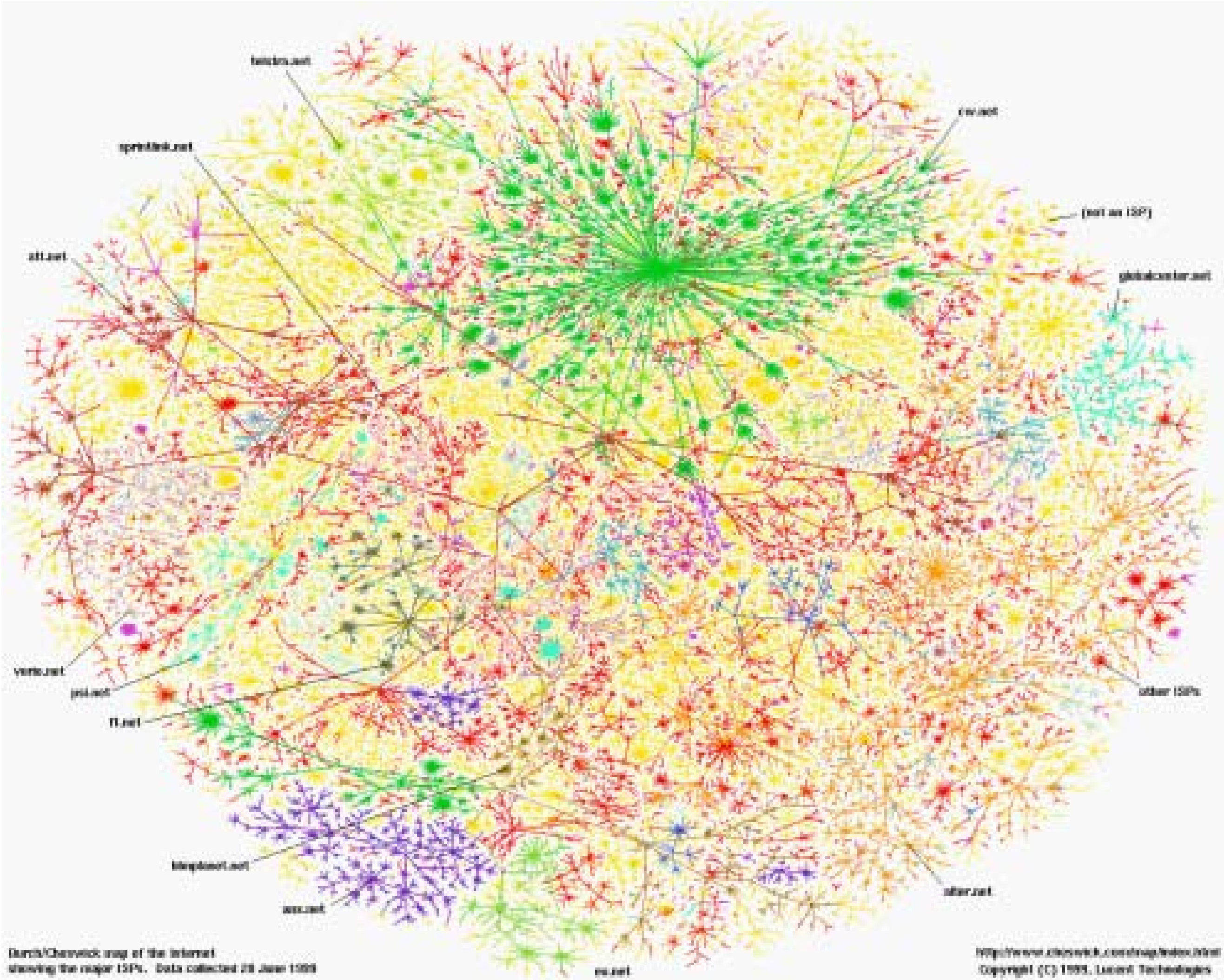
Oregon data set $\rightarrow \varepsilon \approx$ -0.477

$$\rho(\lambda) = -\mathrm{Im}\sum_i \frac{1}{\lambda - \lambda_i + i\varepsilon} = \sum_i \delta(\lambda - \lambda_i) \propto \lambda^{-1+\frac{1}{\varepsilon}}$$

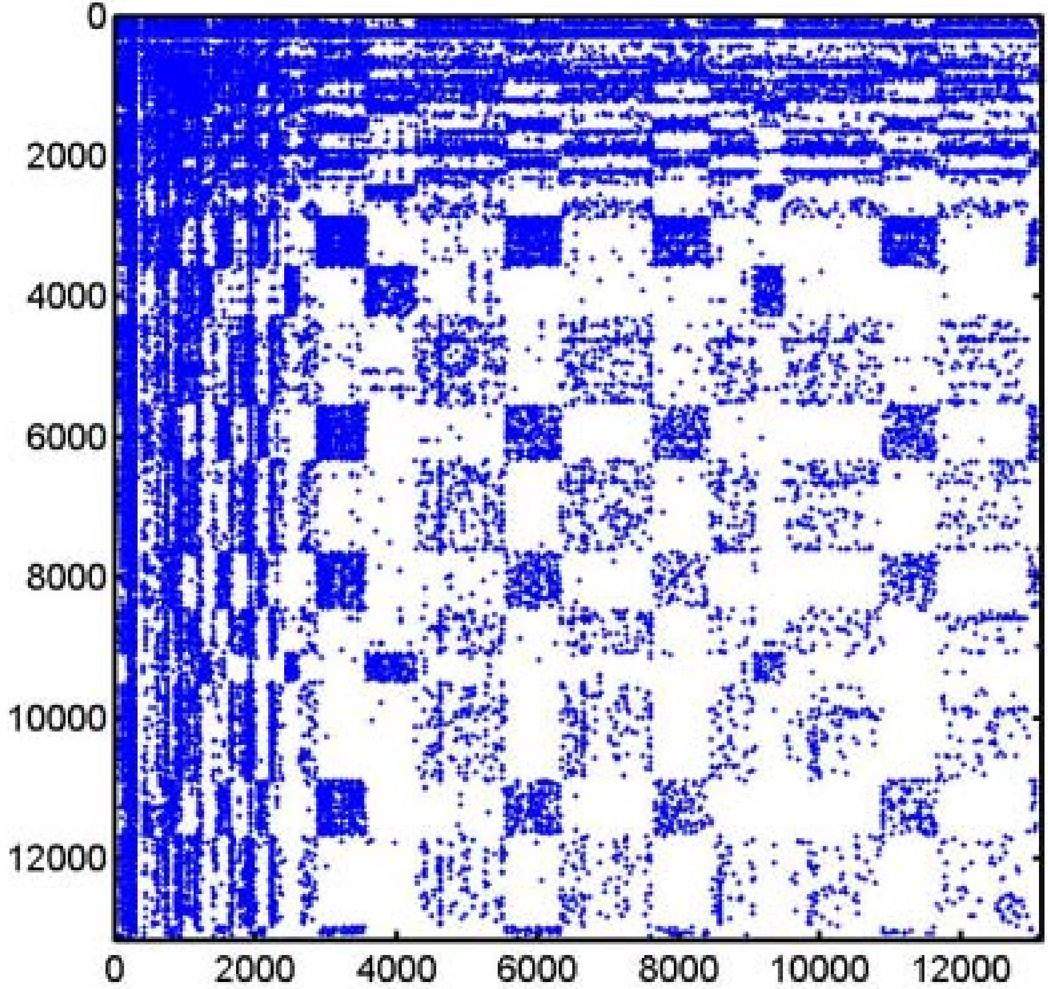
$$1 - \frac{1}{\varepsilon} \approx \begin{cases} 3.2 & \text{(Multi)} \\ 3.1 & \text{(Oregon)} \end{cases}$$

From our result: $\delta = 2\gamma - 1 \cong 3.2$

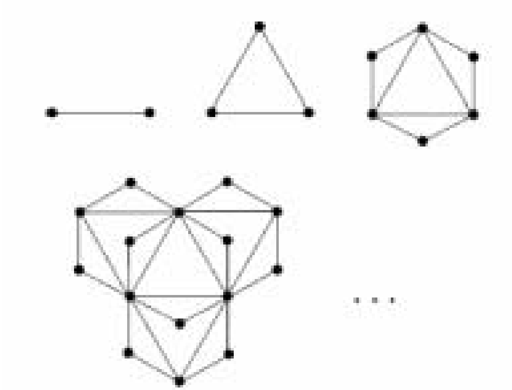
Internet (map): Lumeta



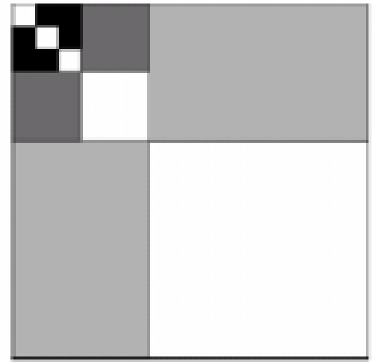
Respective adjacency matrix…



Pseudo fractal graph (hierarchical network)

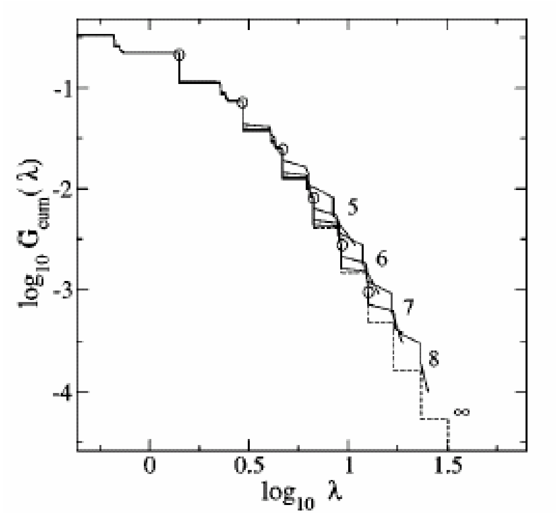


Adjacency matrix →



$$P(k) \approx k^{-\gamma}$$

$$\gamma = 1 + \frac{\ln 3}{\ln 2} \approx 2.58$$



Log-log plot of the **cumulative distribution** of eigenvalues of the adjacency matrix, G_{cum}. The curves show the spectra for t = 5,6,7,8.

$$G_{cum}(\lambda) \approx \lambda^{-(\delta-1)}$$

$$\delta = 4.575 \pm 0.015 \approx 4.6 \quad \text{(by numerical diagonalization}$$

$$\text{Ansatz}: \delta = 2 + \gamma$$

$$\text{Using } \ \delta = 2\gamma - 1 \ \text{ we get } \ \approx 4.2$$

Why this disagreement?

C=0.8 (large!)

C(k)=k $^{-1}$

there are long range correlations

Weakly connected nodes

- What is the effect of nodes with degree $1 \le k \le 5$ on the spectra of a graph with degree distribution $P(k)=P_0 k^{-\gamma}$?

  For $k_0 \ge 5$, the “form” is the same as BA model.

  For $k_0 \le 4$, two peaks $(\lambda \ne 0)$ appear.

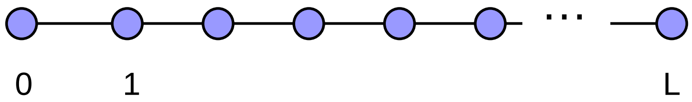
- What is the origin of the peaks?

$<k>$ close to k_0

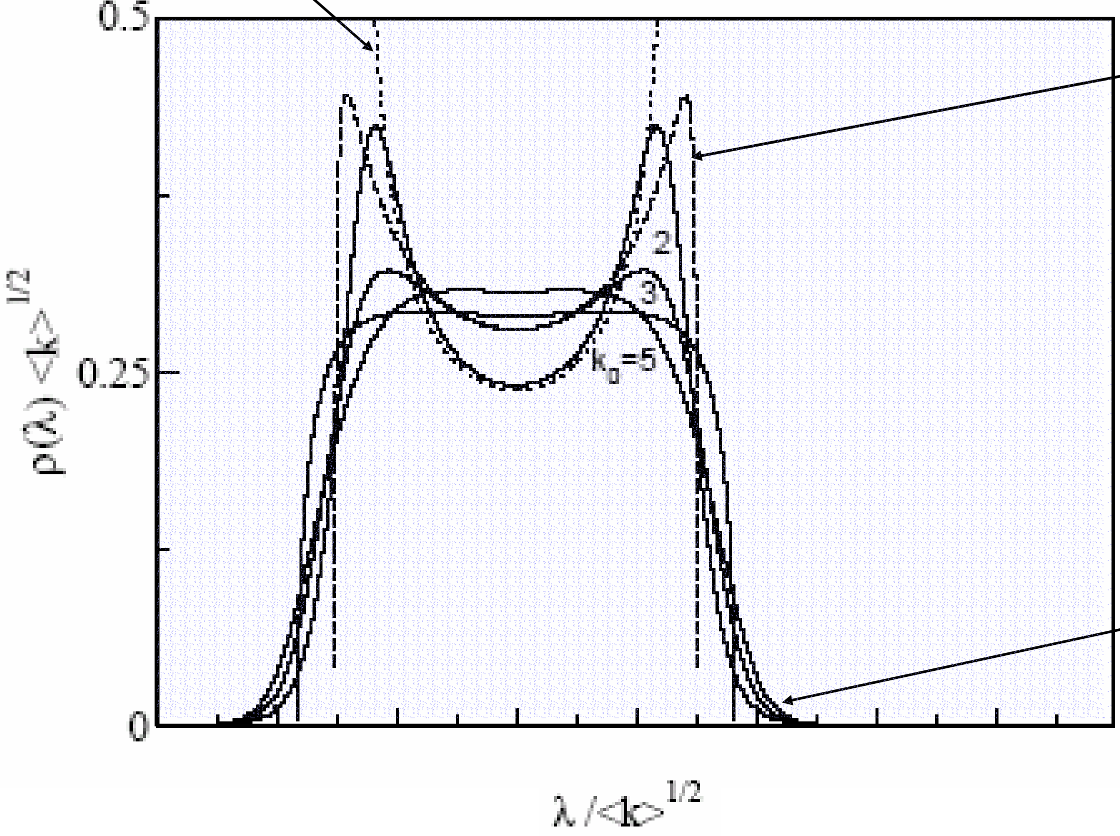
At $k_0 = 3$, $<k> \sim 3.49$ => probability to find nodes with 3 links is larger than fo nodes with $k \ge 4$

Large parts of the net have local k=3-regular structure

Linear chain:



$$\rho(\lambda) = (4-\lambda^2)^{-1/2}; \qquad \rho(\lambda)_{|\lambda \to \pm 2} \to \infty$$



k=3-regular lattice

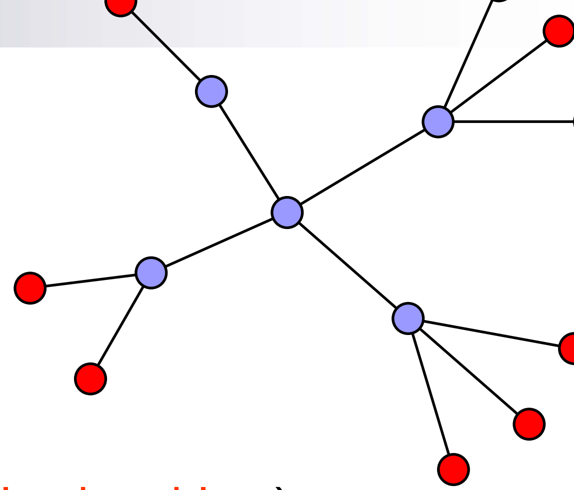
$$\rho(\lambda) = \frac{k}{2\pi}\frac{[4(k-1)-\lambda^2]^{1/2}}{k^2-\lambda^2}$$

$$\text{for}: \quad |\lambda| < 2(k-1)^{1/2}$$

$$\rho(\lambda) \sim \lambda^{-\delta}; \qquad \delta = 2\gamma -$$

Dead end nodes

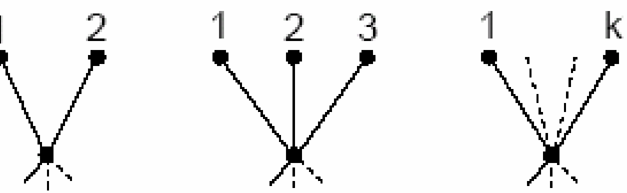
(they produce a peak at $\lambda = 0$)



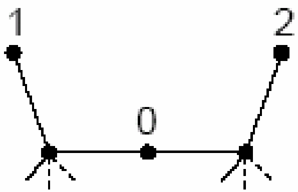
This effect is also present in classical random graphs.

The peak at $\lambda = 0$ is related with localized states (unsolved problem)

D. Vukadinovi´c, P. Huang, and T. Erlebach, Lect. NotesComput. Sci. 2346, 83 (2002).

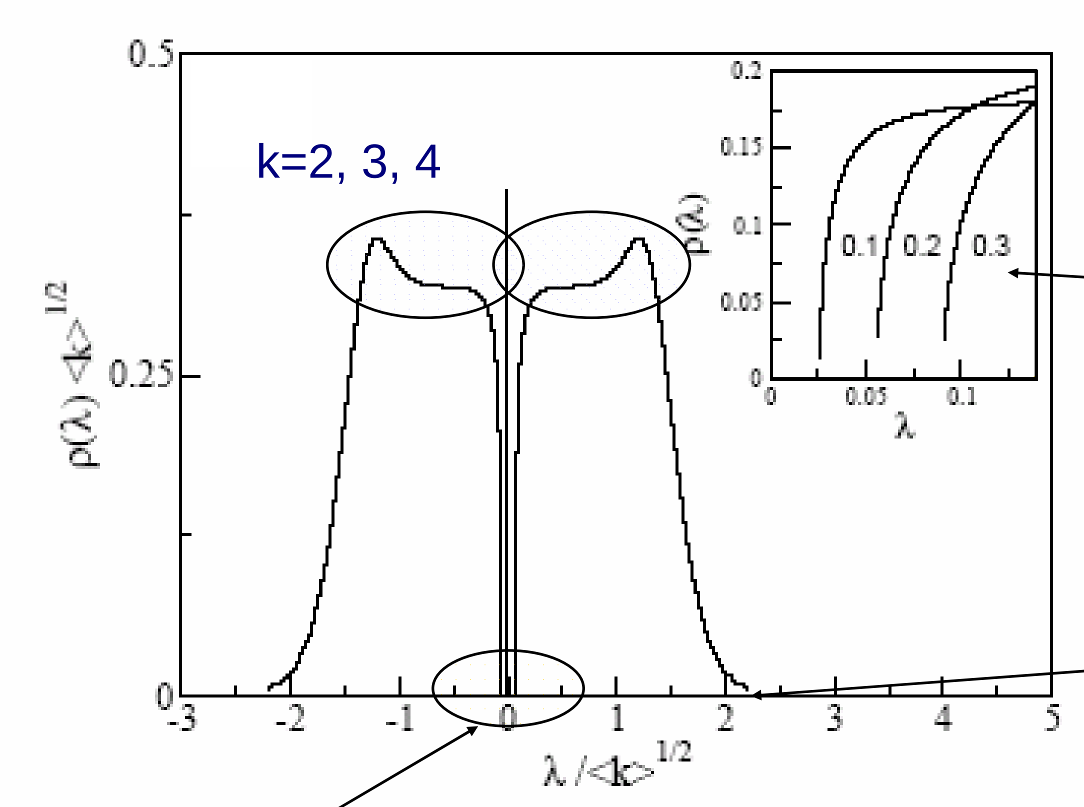


These vertices produce eigenstates with eigenvalue $\lambda = 0$



eigenvector is localized at vertices 0, 1and 2

$$\gamma = 5; P(1) = 0.3$$



Effect of P(1)

Tail as before:

$$\rho(\lambda) \sim \lambda^{-\delta}$$

Emergence of a dip (EMA)

Accuracy of the EM approximation

$$M_n \equiv \langle T^n \rangle = \frac{1}{(n-1)!\, i^n} \int_0^\infty dy\, y^{n-1} e^{iy\lambda} \Phi_1(F_\lambda(y))$$

Define: $q_n \equiv M_n / T^n(\breve{\lambda})$

The function $F(x) = e^{-ixT(\lambda)}$ would be an exact solution if $q_n = 1$ for all $n > 1$. Note that at $n = 1$ we have $q_1 = 1$, this equality is the basic equation in the framework of the *EM* approximation.

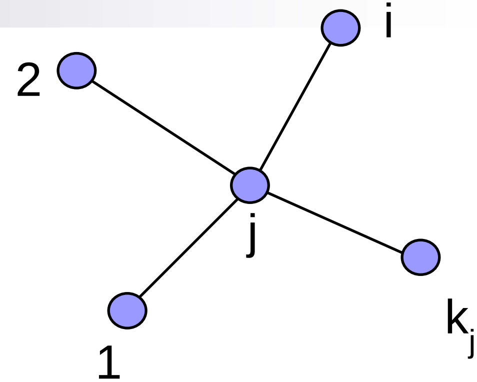
For $P(k)=P_0 k^{-\gamma}$ and $\lambda>>1$ in the leading order of $1/\lambda$:

$$\begin{aligned} q_n &\cong 1 - Ank_0\lambda^{-2} && \text{at } \gamma > 3\,, \\ &\cong 1 - Ank_0\lambda^{-2}\ln\lambda && \text{at } \gamma = 3\,, \\ &\cong 1 - Ank_0^{\gamma-2}\lambda^{-2(\gamma-2)} && \text{at } 2 < \gamma < 3\,, \end{aligned}$$

For $k_0^{1/2} << \lambda << k_0^{1/2} N^{1/(2(\gamma-1))}$ the EM is asymptotically exact!

The problem is for $\lambda \approx k_0^{1/2}$!

Diffusion



Transition matrix:

$$P_{ij} \equiv prob.(j \to i) \qquad \left(\sum_{i=1}^{k_j} P_{ij} = 1 \right) \qquad P_{ij} = \frac{\mathrm{a}_{\mathrm{ij}}}{\mathrm{k}_{\mathrm{j}}}; \quad a_{ij} \text{ is the adjacency matrix}$$

$$g_i(t); \qquad g_0(t=0) = 1$$

The temporal evolution of the probability of occupation of node *i* ($g_i(t)$)

$$\frac{\partial g_i(t)}{\partial t} = \sum_l P_{il} g_l(t) - g_i(t) \sum_k P_{ki} = -\sum_l (\delta_{il} - P_{ij}) g_l(t)$$

The Laplacian:

$$L_{ij} \equiv \delta_{il} - P_{ij}$$

Spectrum of transition matrix

Probability to go from a node v to a $n.n. = 1/k_v$

Transition matrix: $P(w,v)=a_{wv} /k_v$

Laplacian of the graph:

$$L_{v,w} = \begin{cases} 1 & \text{if } v = u \\ -a_{v,w}/\sqrt{k_v k_w} & \text{otherwise} \end{cases}$$

$$\widehat{P} = \widehat{D}^{1/2}(1 - \widehat{L})\widehat{D}^{-1/2}$$

Let $\rho(\lambda)$ the density of eigenvalues of P, and $1=\lambda_1>\lambda_2>\ldots>\lambda_N$

Diameter of the graph is related with the second eigenvalue: $D(G) \leqslant \dfrac{\log(N-1)}{\log(1/\lambda_2)}$

To find the spectra for $\lambda \leqslant \lambda_2$

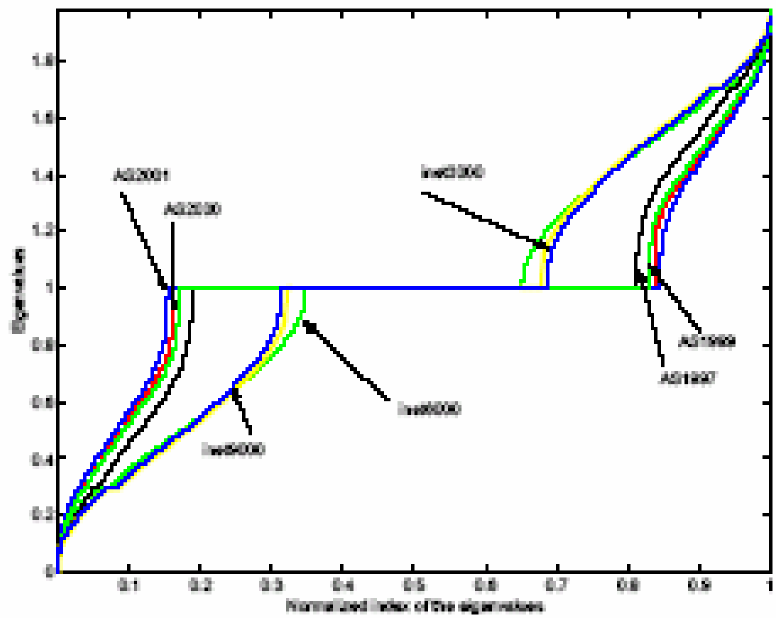
$$T(\lambda) = \frac{1}{\langle k \rangle} \sum_k \frac{kP(k)}{k\lambda + i\varepsilon - (k-1)T(\lambda)} \qquad \rho(\lambda) = -\frac{1}{\pi} \operatorname{Im} \frac{1}{\lambda - T(\lambda)}$$

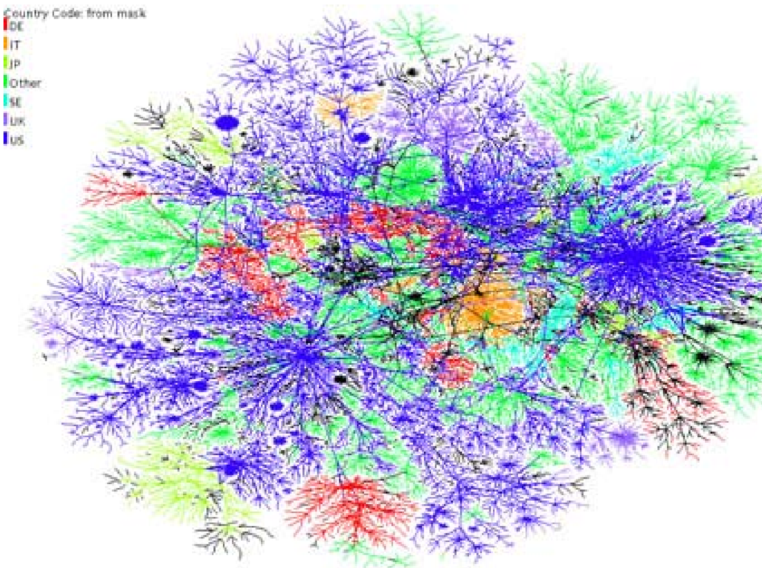
For a k-regular graph

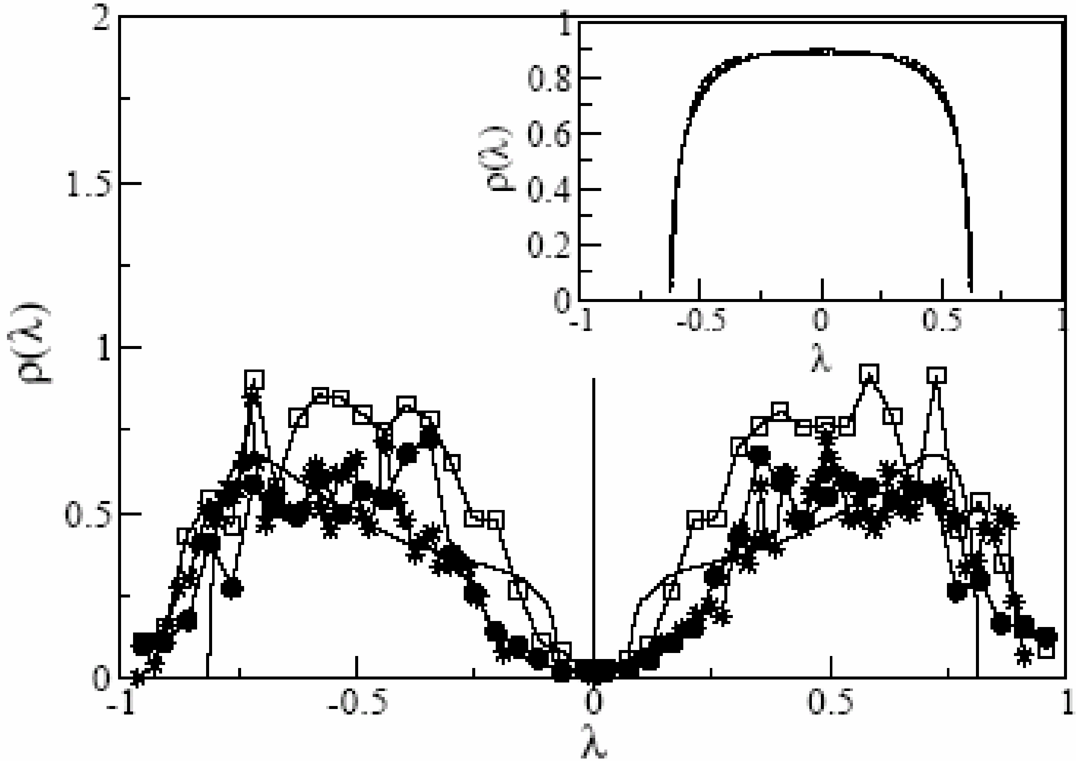
$\lambda_2 = 2\sqrt{k-1}/k.$ and $\rho(\lambda) = \dfrac{k}{2\pi} \dfrac{\sqrt{4(k-1)/k^2 - \lambda^2}}{1 - \lambda^2}$

D. Vukadinovi´c, P. Huang, T. Erlebach, Lect. NotesComp. Sci. 2346, 83 (2002)

$$\mathcal{L}(G)(u,v) = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0, \\ -\frac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent}, \\ 0 & \text{otherwise}. \end{cases}$$



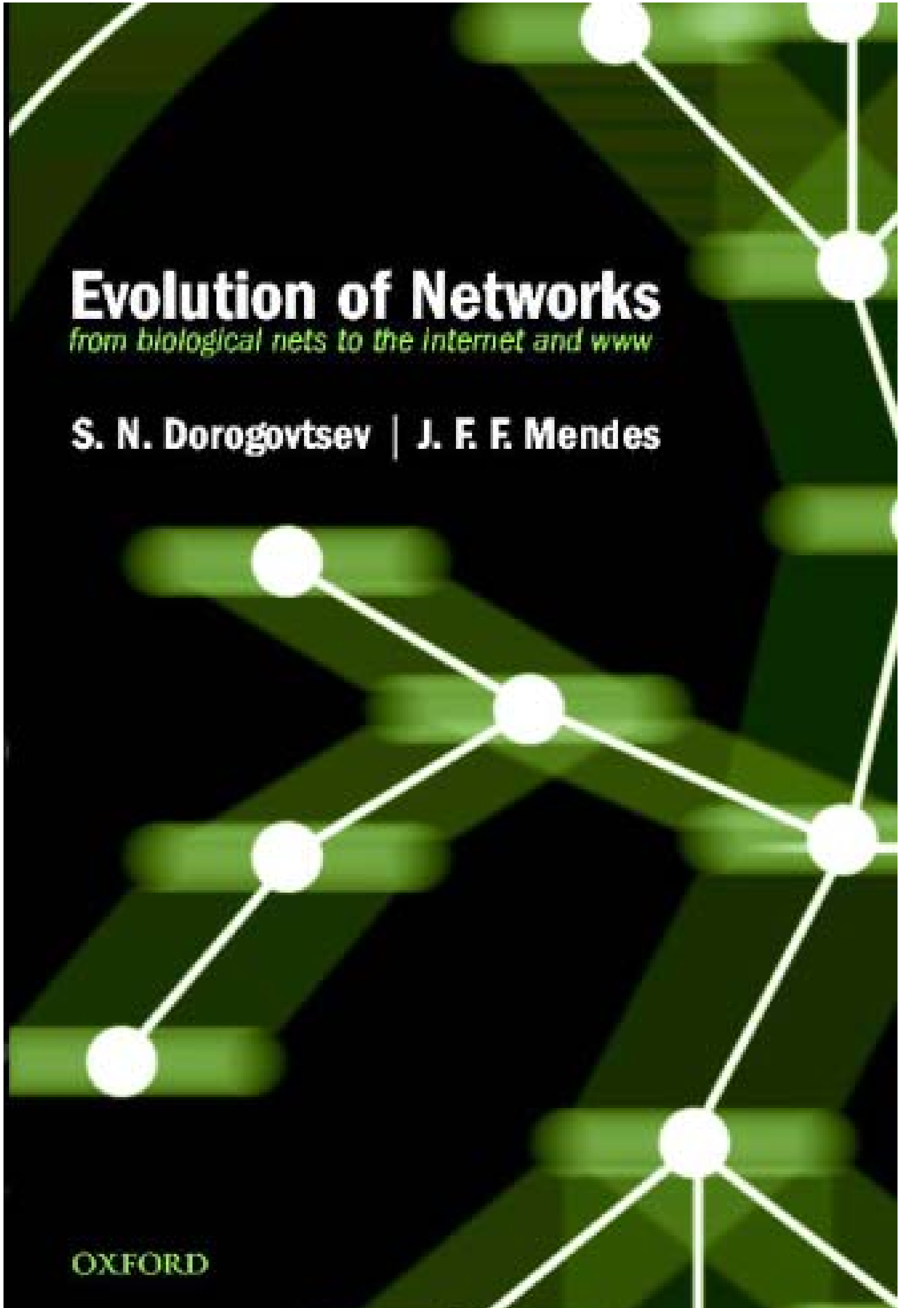




2
1.5
1
0.5
0
ρ(λ)
-1
-0.5
0
0.5
1
λ
1
0.8
0.6
0.4
0.2
0
ρ(λ)
-1
-0.5
0
0.5
1
λ

Conclusions

- We derived exact equations that describe the spectra of random tree-like graph (correlated & uncorrelated)
- We proposed a simple approximation (effective medium approx.)
- We confirm that the spectra of *SF* & *CRG* do not follow the semi-circular law
- $\rho(\lambda) \propto |\lambda|^{-\delta}$ with $\delta = 2\gamma - 1$
- Large eigenvalues are produced by the highly connected nodes ($k=\lambda^2$)
- Dead end nodes play a special role, they produce localized eigenstates with $\lambda=0$ and a dip around the central peak.
- EM approximation gives good results for: $k_0 \le \lambda^2 \le k_{cut}$
- Results for the Internet are in good agreement with theory for random like-tree, because although the $c = 0.2$, the local clustering coefficient decreases rapidly with the increasing of the degree of a node.


Evolution of Networks
from biological nets to the internet and www
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