Nonequilibrium Phase Transitions in Asymmetrically Coupled Directed Percolation Systems

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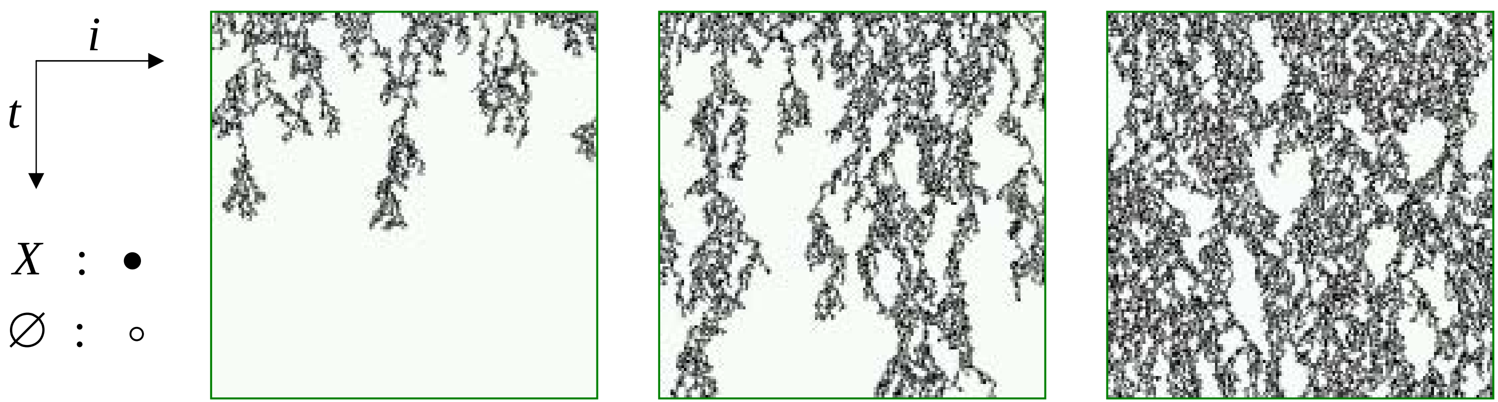
Absorbing Phase Transition in Contact Process

- Contact Process : model for an epidemic spreading on lattices
  - Two state problem : $S_i = 0(\varnothing,\ \text{healthy})$ or $1(X,\ \text{ill})$
  - Dynamics

    spontaneous healing (annihilation) $X \xrightarrow{1} \varnothing$

    infection (branching) $\varnothing \xrightarrow{n\lambda} X$

    no spontaneous creation (n : number of infected neighbors)
  - What is happening



$\lambda < \lambda_c$ extinguished

$\lambda = \lambda_c$ critical

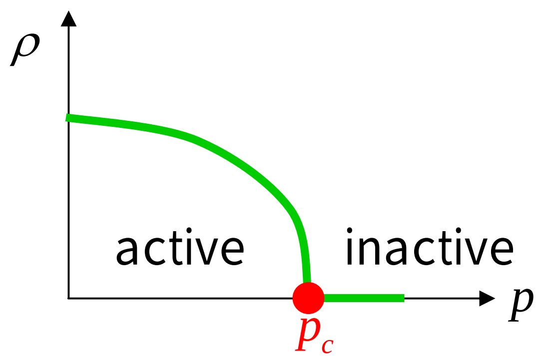
$\lambda > \lambda_c$ everlasting

Absorbing Phase Transition

- Absorbing phase transition : Non-eq. Phase transition between

frozen inactive phase	⟺	fluctuating active phase

- Absorbing state(s) { Ω } transition probability out of Ω is zero.
- Survival probability P_{suv} (t)⇒ $P_{suv} \equiv \lim_{t\to\infty} P_{suv}(t) = \begin{cases} \text{zero} & \text{in the inactive phase} \\ \text{nonzero} & \text{in the active phase} \end{cases}$
- Order parameter ρ : density of particles ("active" sites)



– order parameter $\rho \sim |p-p_c|^{\beta}$

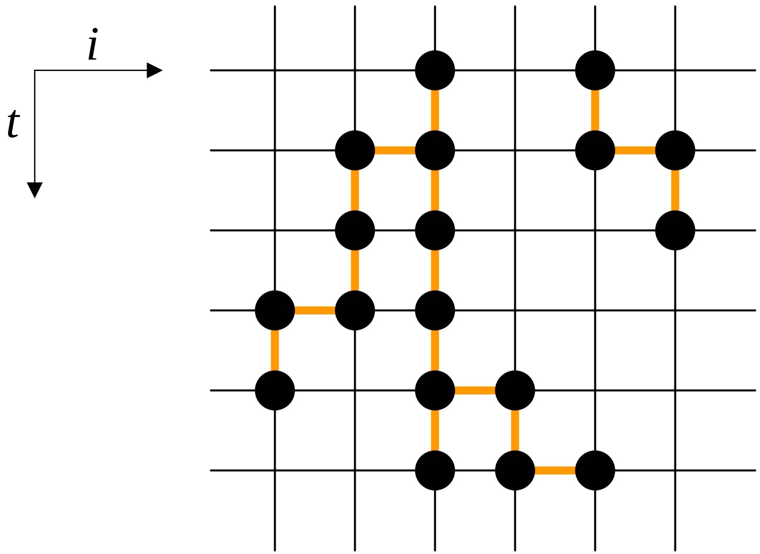
– correlation length $\xi_x \sim |p-p_c|^{-\nu_x}$

– correlation time $\xi_t \sim |p-p_c|^{-\nu_t}$

- Critical scaling at p_c

$$\rho(t) \sim t^{-\beta/\nu_t}, \quad \rho(L) \sim L^{-\beta/\nu_x}, \quad \tau(L) \sim L^{z} \quad (z = \nu_t/\nu_x : \text{dynamic exponent})$$

Contact Process as Directed Percolation



branching X → XX

⟷ presence of directed bond

spontaneous annihilation X → ∅

⟷ absence of directed bond

Absorbing transition ⟷ Percolation transition

CP in d dimension ⟷ DP in (d+1) dimension

DP Universality class

symmetry, conservation, ……

Continuum description of DP

- Langevin equation

$$X \xrightarrow{1} \varnothing$$

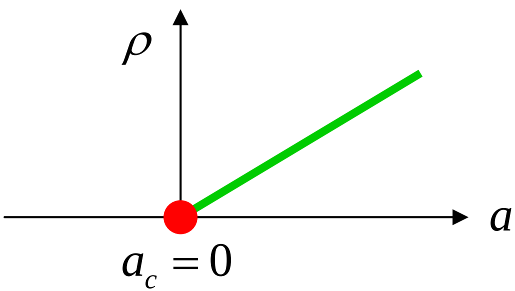
$$X\varnothing \xrightarrow{\lambda} XX$$

$$\frac{d\rho(x,t)}{dt} = -\rho + \lambda\rho(1-\rho) + \cdots\cdots$$

$$= \nabla^2\rho + a\,\rho - b\,\rho^2 + (\text{higher order terms}) + \eta(x,t)$$

$$\langle \eta(x,t)\eta(x',t')\rangle = D\,\rho\,\delta^{(d)}(x-x')\delta(t-t')$$

- Mean field theory (d ≥ d_c = 4)



$$\beta = 1 \qquad \nu_x = 1/2 \qquad \nu_t = 1$$

$$\Rightarrow$$

$$\rho \sim t^{-1} \qquad \tau \sim L^2$$

- For d < d_c

  Perturbative RG, simulations,…

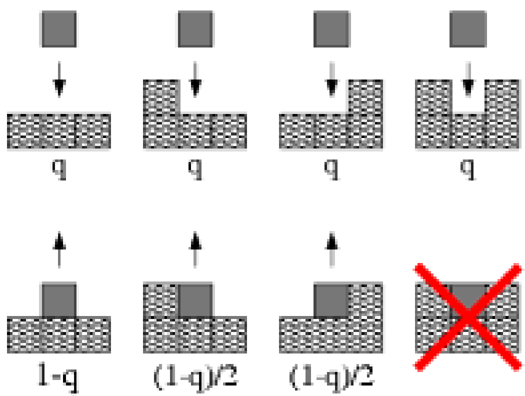
  In 1D, $\beta \approx 0.2765,\ \nu_x \approx 1.097,\ \nu_t \approx 1.734,\ z \approx 1.581$

CDP in 1D : Interface Roughening Model

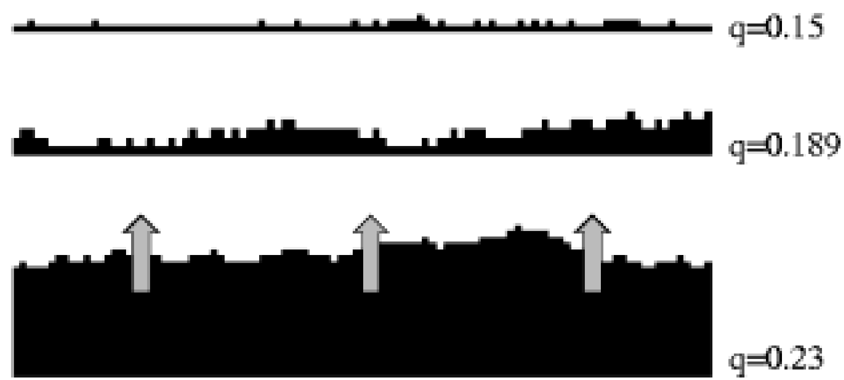
- "no-digging" model for (1+1)D interface roughening [Alon et al, PRL `96]
  Restricted Solid-on-Solid interface $(|h_i - h_{i+1}| = 0,1)$

depo.

evap.



no-digging



flat (active)

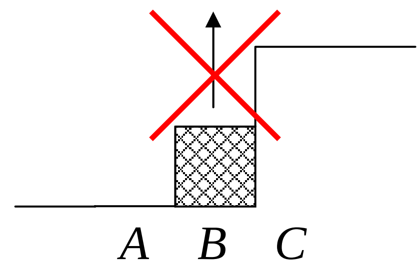
rough (inactive)

[Hinrichsen `00]

- Mapping to CDP

$$A \quad \textit{if} \quad h = 0$$
$$B \quad \textit{if} \quad h = 0,1$$
$$C \quad \textit{if} \quad h = 0,1,2$$
$$\vdots$$

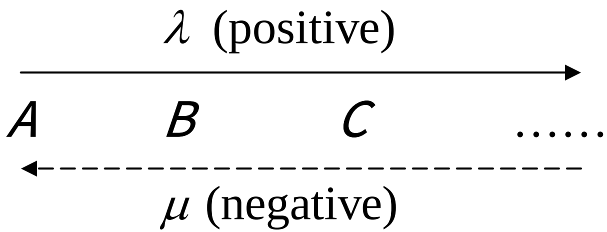
$$X \xrightarrow{\textit{deposition}} \varnothing \; : \textit{annihilation}$$
$$X \xrightarrow{\textit{evaporation}} XX \; : \textit{branching}$$
$$A \xrightarrow{\infty} AB, \quad B \xrightarrow{\infty} BC, \cdots$$



branching is hindered

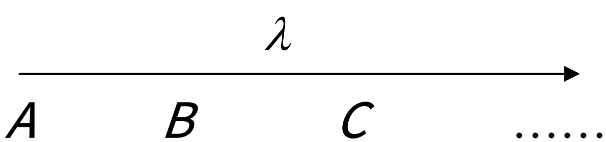
Unidirectionally Coupled DP

- Coupled DP



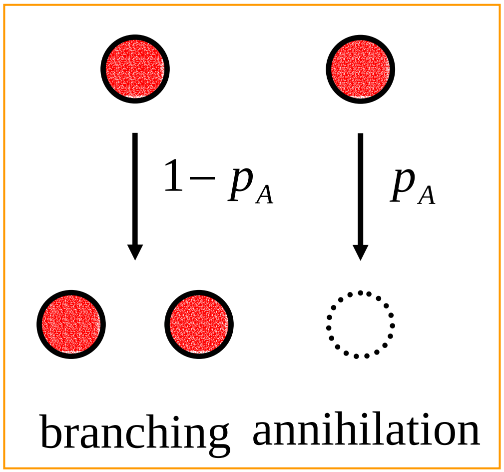
- Simulation results in 1D
  - A : DP critical behavior
  - B, C, … : dressed
  - negative coupling is irrelevant.

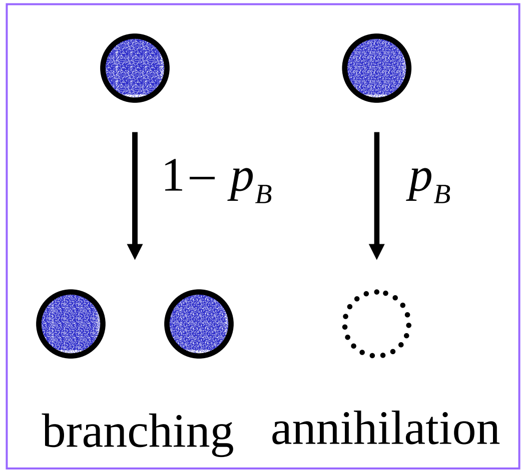
- Unidirectionally Coupled DP



Coupled Directed Percolation Systems (2 species)

- Two species particle system (A and B)



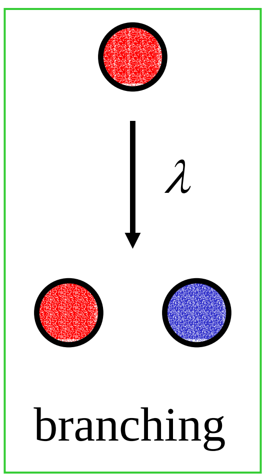


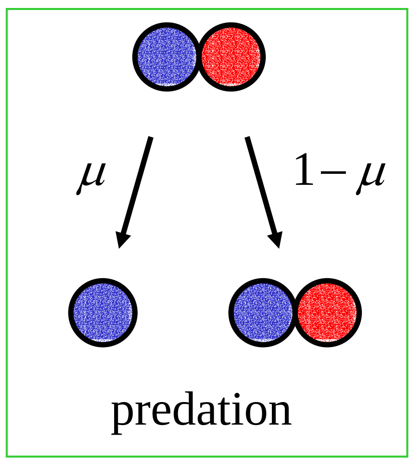
$$A \xrightarrow{p_A} \varnothing$$

$$A \xrightarrow{(1-p_A)(1-\lambda)} AA$$

$$B \xrightarrow{p_B} \varnothing$$

$$B \xrightarrow{(1-p_B)} BB$$



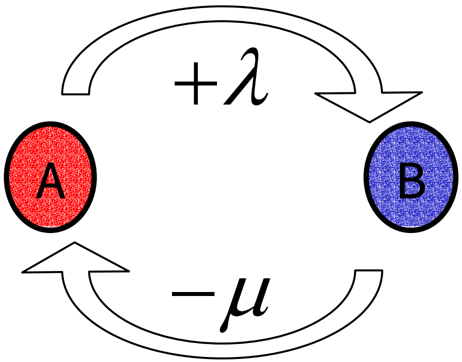


$$A \xrightarrow{(1-p_A)\lambda} AB$$

$$AB \xrightarrow{\mu} B$$

Coupled Directed Percolation Systems

- Asymmetric coupling



- $\lambda \neq 0, \ \mu = 0$: UCDP

  A species belongs to DP. B species is dressed.

- Criticality for $\lambda \neq 0, \ \mu \neq 0$

- Epidemic spreading

A : virus	$\Leftrightarrow$	B : vaccine
A : antigen	$\Leftrightarrow$	B : antibody

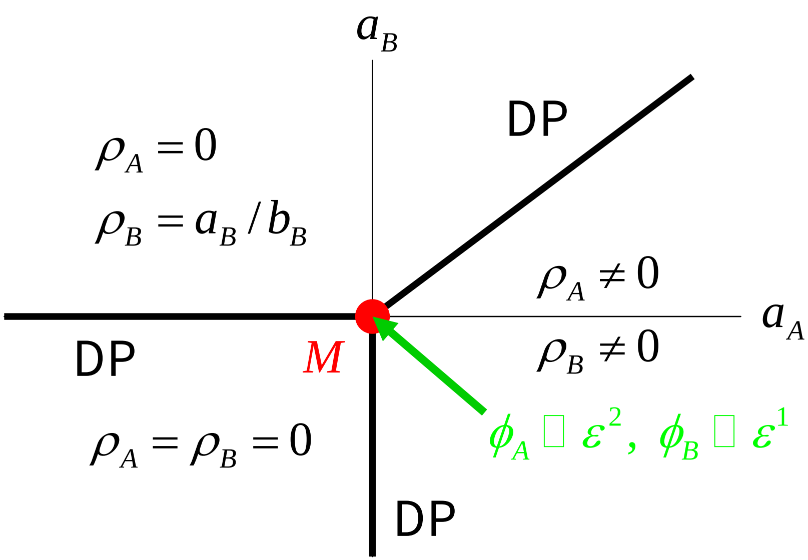
Epidemic spreading in the presence of reactive immunization systems

Mean Field Theory

- Single site mean field equation

$$\dot{\rho}_A = a_A\rho_A - b_A\rho_A{}^2 - \mu\rho_A\rho_B$$
$$\dot{\rho}_B = a_B\rho_B - b_B\rho_B{}^2 + \lambda\rho_A$$

- Phase diagram



- Multicritical behavior at M

- Critical exponent

$$\beta_A = 2, \ \nu_{xA} = 1/2, \ \nu_{tA} = 1$$
$$\beta_B = 1, \ \nu_{xB} = 1/2, \ \nu_{tB} = 1 \qquad \in \quad \text{DP}$$

- Density decay

(i) $2b_B > \mu$

$$\rho_A \sim t^{-2} \ \ and \ \ \rho_B \sim t^{-1}$$

(ii) $2b_B = \mu$

$$\rho_A \sim \frac{1}{2t^2 \ln t} \quad and \quad \rho_B \sim \frac{1}{t}\left(1 + \frac{1}{2\ln t}\right)$$

(iii) $2b_B < \mu$

$$\rho_A \sim t^{-\mu/b_B} \quad and \quad \rho_B \sim t^{-1}$$

continuously varying

A : dressed, B : DP ⟷ UCDP

Numerical Simulations

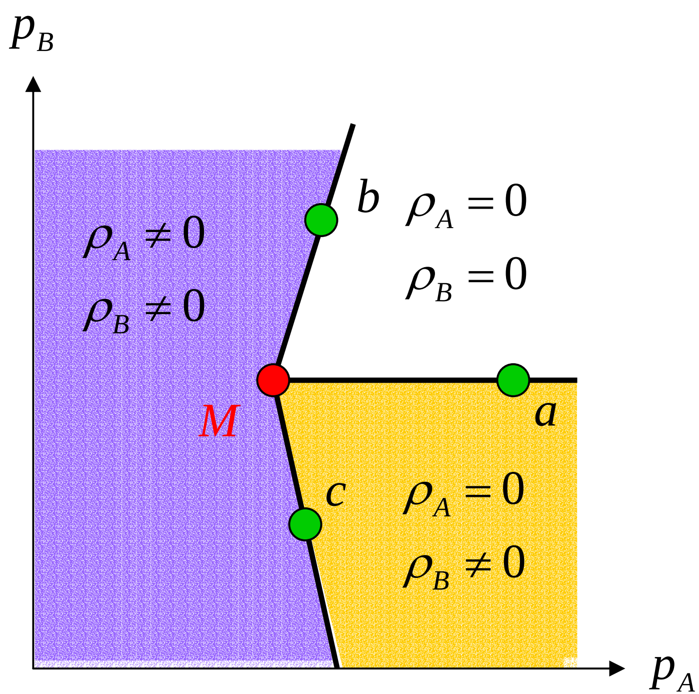
- 1D lattices
  - relevance/irrelevance of the negative coupling μ

- Watts-Stogatz Small-World Networks
  - validity of the mean field theory
  - irrelevance/relevance of the negative coupling μ
  - finite-size-scaling in infinite dimension network
  - upper critical dimension

- Scale-Free Networks
  - absence/presence of epidemic phase transition

CDP in 1D

Phase Diagram



$\lambda = 0.2\ ,\ \ \mu = 0.2$

- Scaling properties
  - a (0.3, 0.23267)

    $\rho_A \sim \exp(-t/\tau)\ ,\ \ \rho_B \sim t^{-0.159}$
  - b (0.1561, 0.5)

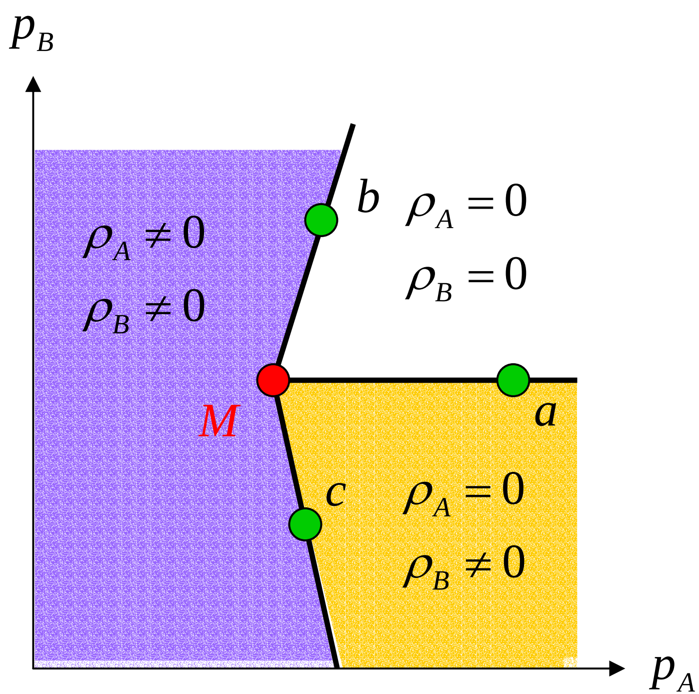
    $\rho_A \sim \rho_B \sim t^{-0.159}$
  - c (0.1472, 0.15)

    $\rho_A \sim t^{-0.159},\ \ \rho_B \sim const. + t^{-0.159}$

➔ DP universality class

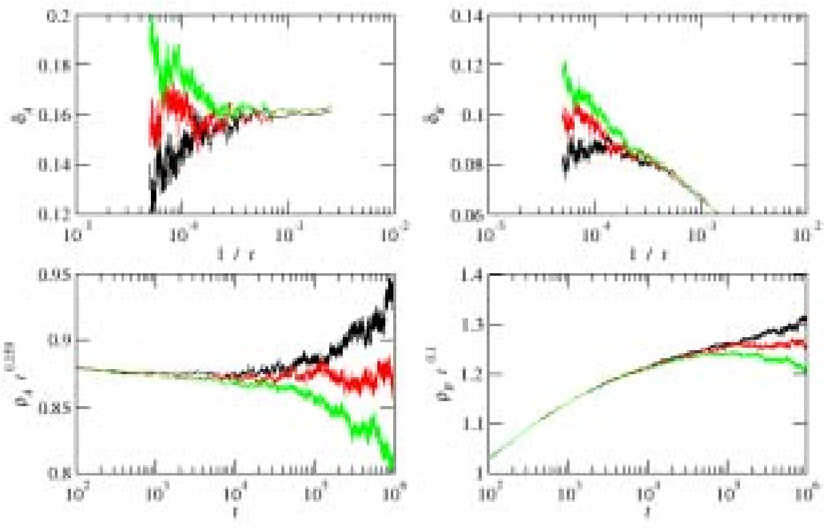
CDP in 1D

Phase Diagram



$\lambda = 0.2\,,\ \mu = 0.2$

- Scaling at multicritical point *M*



$\rho_A \sim t^{-0.16}\,,\ \ \rho_B \sim t^{-0.10} \quad \in\ \text{UCDP}$

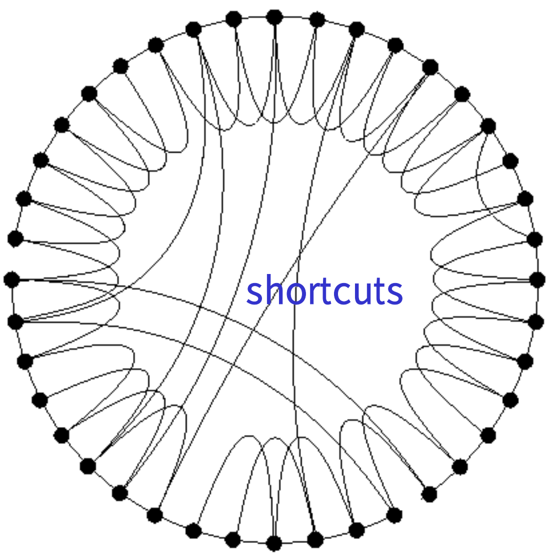
as Alon's model belongs to UCDP

Negative coupling is irrelevant in 1D lattice

CDP in Infinite Dimensional Lattice

- Small World Network : infinite dimensional lattice

[Watts and Strogatz Nature `98]



small–world network
(Watts–Strogatz)

1. 1D ring with bonds up to Kth n.n. sites

2. Rewire each bond with probability P_{rew} randomly ← shortcuts

Regular lattice for $p_{rew} = 0$

$\Updownarrow$

Small-World network for $0 < p_{rew} < 1$

$\Updownarrow$

Random network for $p_{rew} = 1$

Mean field critical behaviors in Ising, XY, Synchronization, etc

[Kim, Hong, Choi]

Finite-Size-Scaling in Infinite Dim. Systems

- FSS in finite dim. systems
  - Correlation length

$$\xi \sim \varepsilon^{-\nu}$$

  - Order parameter

$$\rho(\varepsilon, L) = \varepsilon^{\beta}\ \Psi(L/\xi)$$

L : linear system size

- FSS in infinite dim. systems
  - Correlation length? ➔ Correlation volume!

$$\xi_V \sim \varepsilon^{-\bar{\nu}}$$

  - Order parameter

$$\rho(\varepsilon, L) = \varepsilon^{\beta_{MF}}\ \Psi(N/\xi_V)$$

N : total number of sites

- Conjecture by Botet, Jullien, and Pfeuty [PRL `82]

$$\bar{\nu} = \nu \times d_c$$

d_c : upper critical dimension

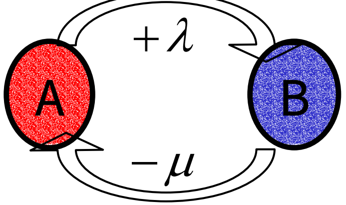
upper critical dimension

- Extension to Non-eq. systems

$$\beta^A \to \beta^A_{MF} = 2 \qquad \nu^A_t \to \nu^A_{t,MF} = 1 \qquad \nu^A_x \to \bar{\nu}^A_{x,MF} = d^A_c/2 \qquad z^A \to \bar{z}^A_{MF} = 2/d^A_c$$

$$\beta^B \to \beta^B_{MF} = 1 \qquad \nu^B_t \to \nu^B_{t,MF} = 1 \qquad \nu^B_x \to \nu^B_{x,MF} = d^B_c/2 \qquad z^B \to \bar{z}^B_{MF} = 2/d^B_c$$

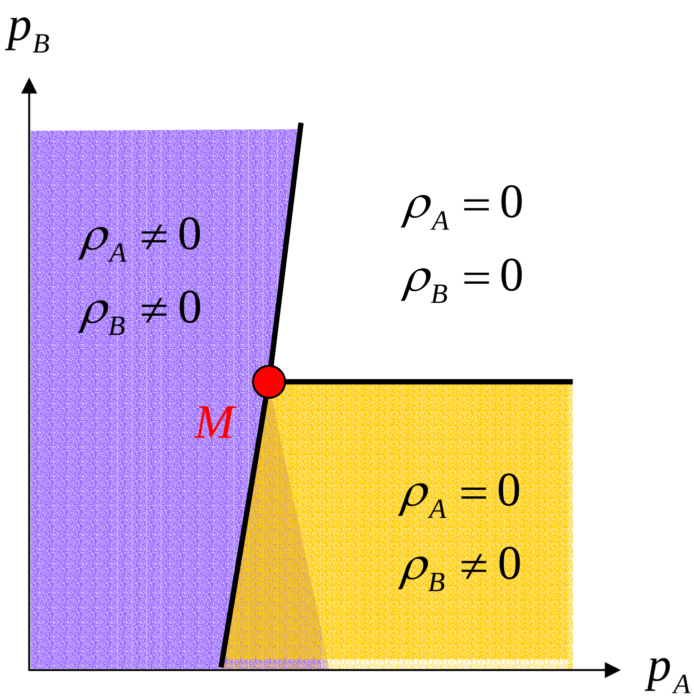
CDP on SW Net. with μ=0



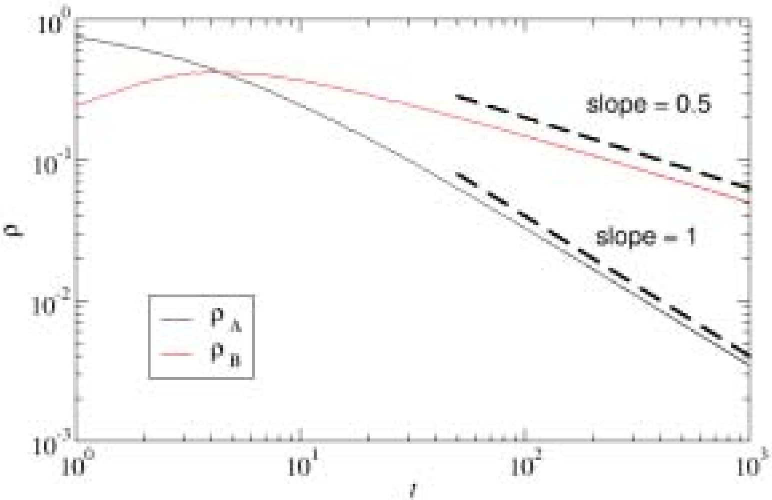
- SW network with K = 10 and P_{rew} = 0.5

- Unidirectionally coupled limit

$$\lambda = 0.5, \;\; \mu = 0.0$$



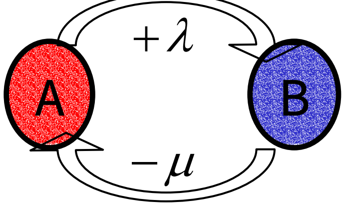
- Density decay at *M* (0.3202,0.48505)
  N = 2000000



$$\beta^A_{MF} / \nu^A_{t,MF} = 1 \quad , \quad \beta^B_{MF} / \nu^B_{t,MF} = 1/2$$

: consistent with the MF exponent

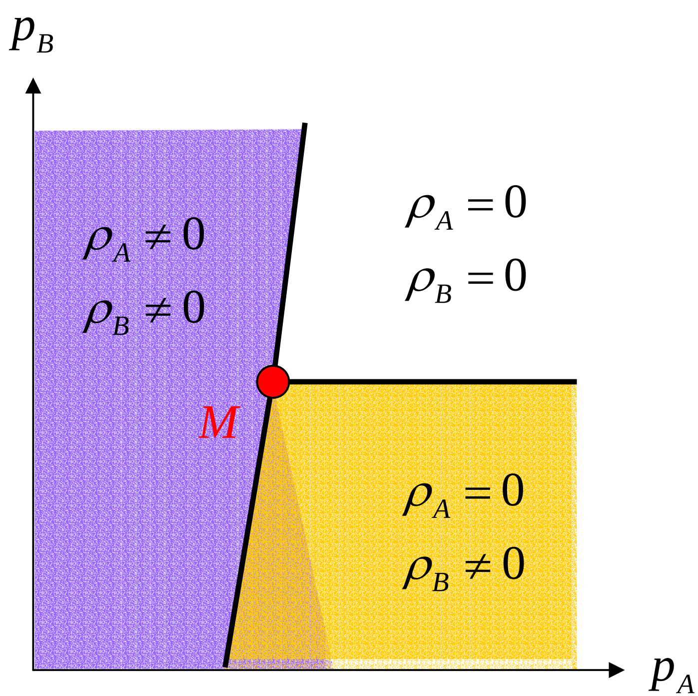
CDP on SW Net. with μ=0



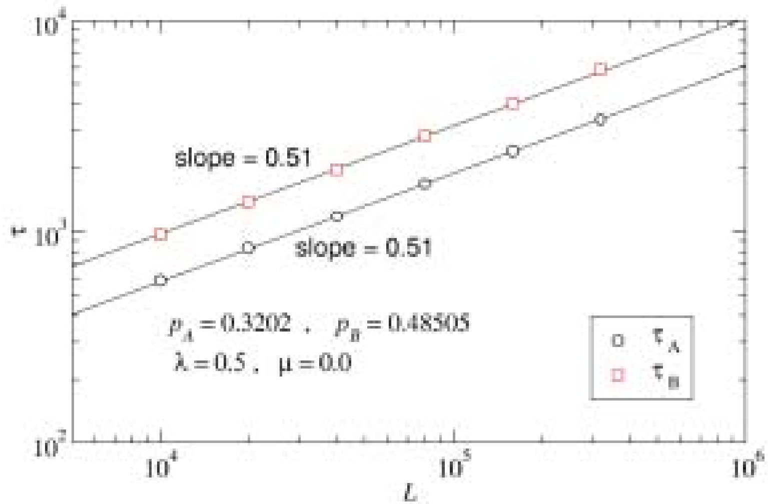
- SW network with K = 10 and P_{rew} = 0.5

- <u>Unidirectionally coupled limit</u>

$$\lambda = 0.5,\ \ \mu = 0.0$$



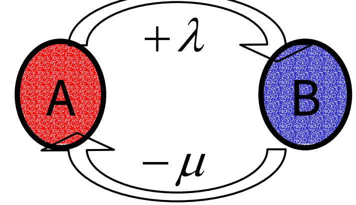
- Scaling of correlation time at *M*



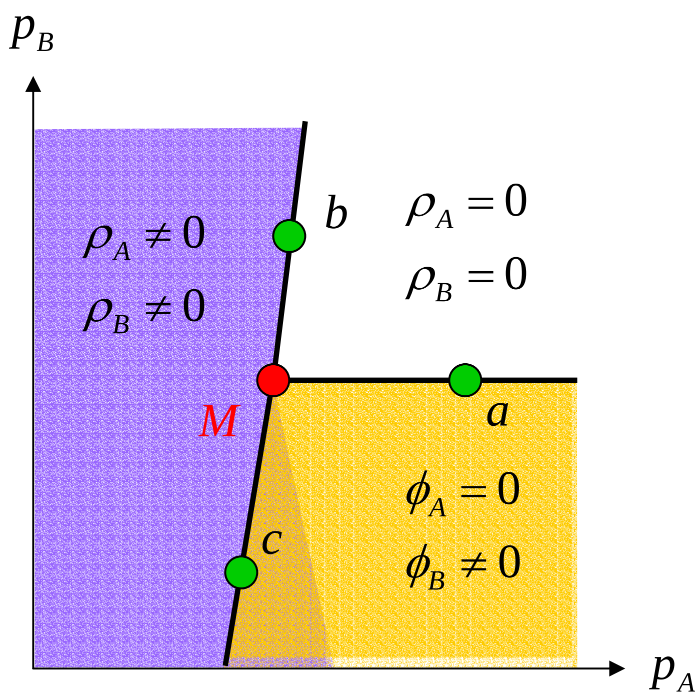
$$\bar{z}^A \approx 0.5 \quad , \quad \bar{z}^B \approx 0.5$$

$$\bar{z} = z_{MF} / d_c \quad \Rightarrow \quad d_c^A = d_c^B = 4$$

CDP on SW Net. with Nonzero μ



- SW network with K = 10 and P_{rew} = 0.5



$\lambda = 0.1\,,\ \mu = 1.0$

- Scaling properties
  - a (0.8, 0.48505)

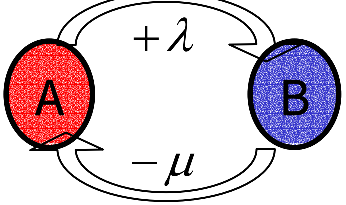
    $\rho_A \sim \exp(-t/\tau)\,,\ \rho_B \sim t^{-1.0}$
  - b (0.4562, 0.8)

    $\rho_A \sim \rho_B \sim t^{-1.0}$
  - c (0.2635, 0.4)

    $\rho_A \sim t^{-1.0},\ \rho_B \sim const.$

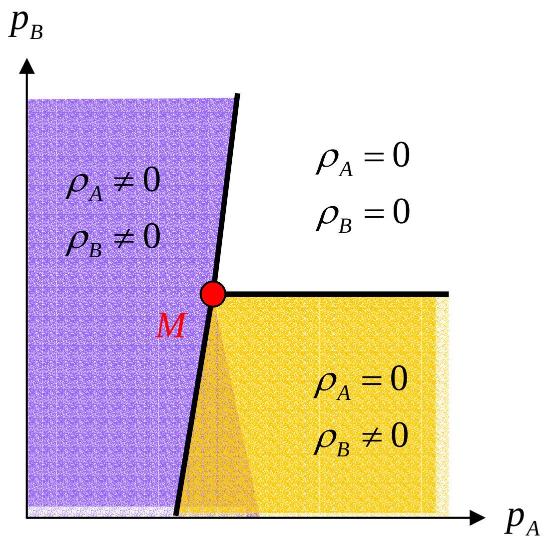
➔ MF DP universality class

CDP on SW Net. : Multicriticality

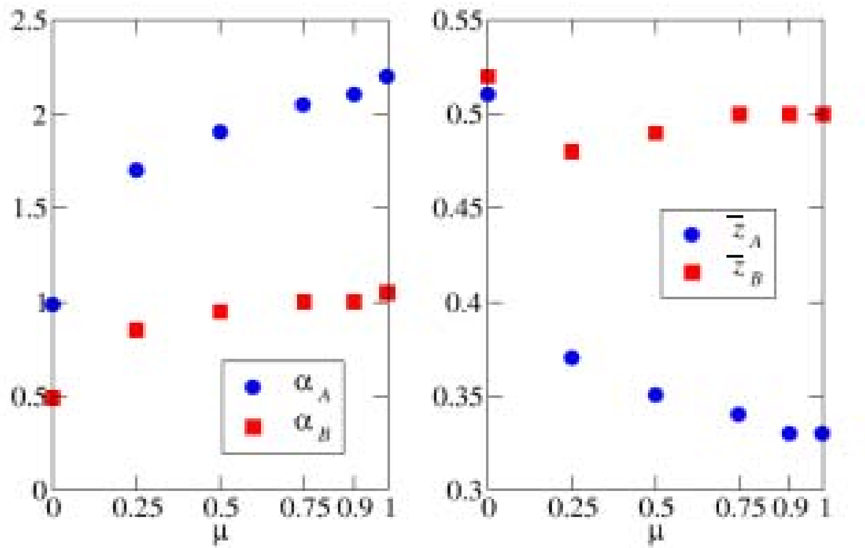


- SW network with K = 10 and P_{rew} = 0.5

$\lambda = 0.5$: fixed, $M = M(\mu)$



simulation results



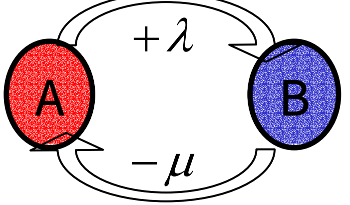
– critical density decay $(N = 2000000)$

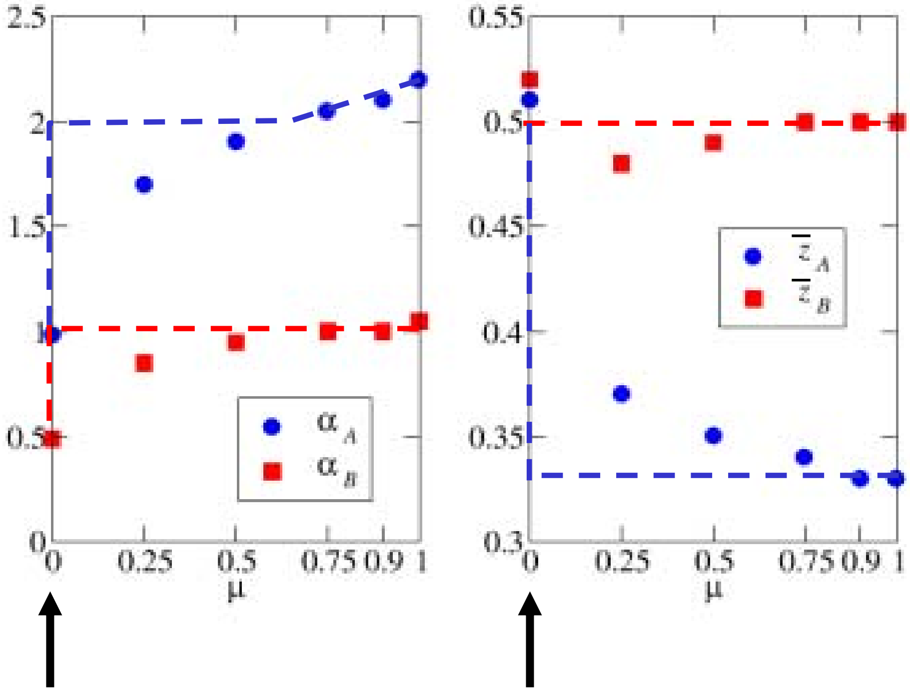
$\rho_A \sim t^{-\alpha_A}$ and $\rho_B \sim t^{-\alpha_B}$

– relaxation time $(N \leq 320000)$

$\tau_A \sim N^{\bar{z}_A}$ and $\tau_B \sim N^{\bar{z}_B}$

MF vs SW





UCDP

UCDP

$\alpha_A = 1\,,\ \alpha_B = 1/2\,,\ \bar{z}_{A,B} = 1/2$

Negative coupling is relevant

- Mean Field Theory

  $\alpha_A \geq 2$ (large μ), $\alpha_A = 2$ (small μ)

  $\alpha_B = 1$

  $z_A = z_B = 2$

- Simulations on SW

  $\alpha_A\,,\ \alpha_B$

  $\bar{z}_A \neq \bar{z}_B\ \ (\bar{z}_A \approx 1/3,\ \bar{z}_B \approx 1/2)$

$$\bar{z} = z_{MF} / d_c$$

$$\Downarrow$$

$$d_c^A (\approx 6) \neq d_c^B (\approx 4)$$

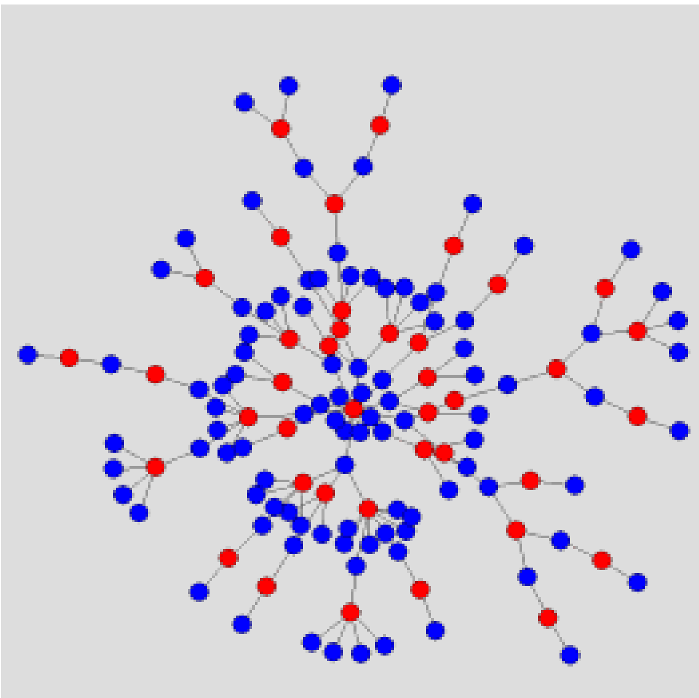
$cf\,)$ power counting

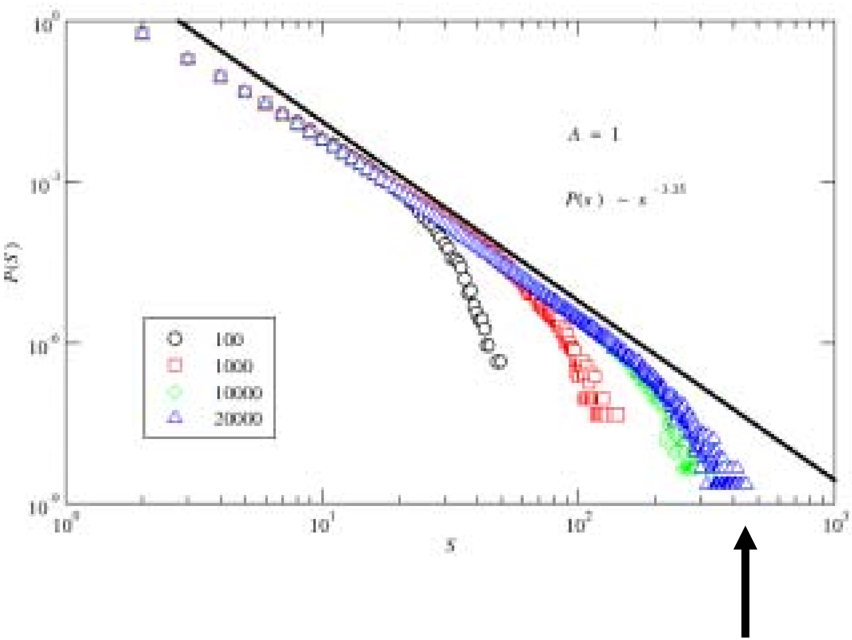
$$\begin{cases} \dot{\rho}_A = a_A \rho_A - b_A {\rho_A}^2 - \mu \rho_A \rho_B + \eta_A \\ \dot{\rho}_B = a_B \rho_B - b_B {\rho_B}^2 + \lambda \rho_A + \eta_B \end{cases}$$

CDP on Scale-Free Network

- Virus spreading through the Internet (A : virus, B : vaccine)
- Proper geometry : Scale-free network with power-law degree distribution

$$P(k) \sim k^{-\gamma}$$

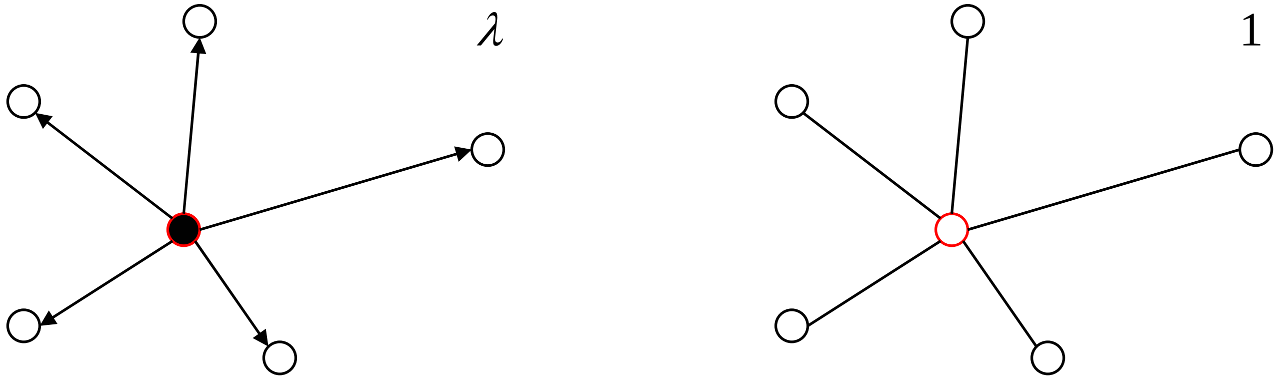




hub

Epidemic Spreading Model on SF Network

- Contact Process on SF Network [Pastor-Satorras and Vespignani, PRL `01]



infecting all neighbors   spontaneous healing

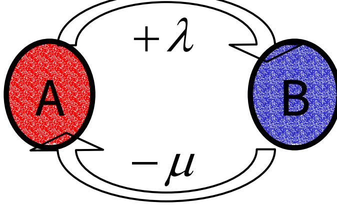
- Crucial role of the hub : $\lambda_c \propto \frac{1}{\langle k^2 \rangle} \rightarrow 0$ for $\gamma \leq 3$
- Viruses are always there for any small value of λ

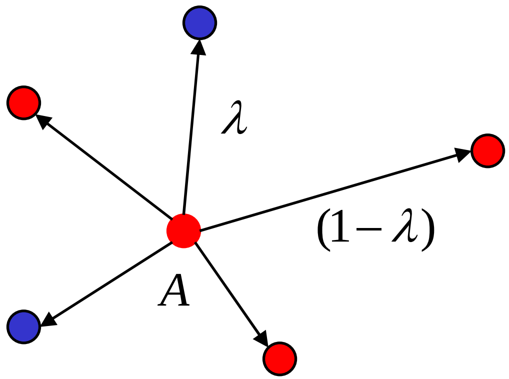
- How to control the epidemics on SF net.
  ← introducing reactive vaccine

Constraint :
"Never flood the network with viruses or vaccines"

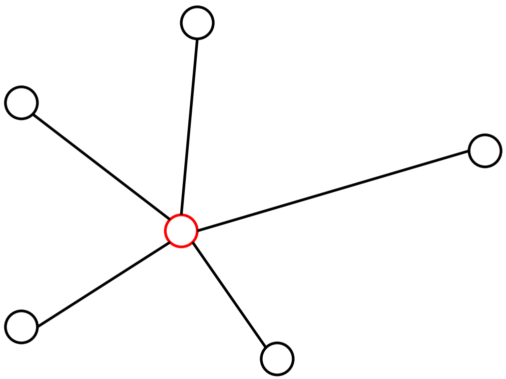
→ CDP

CDP on SF Network

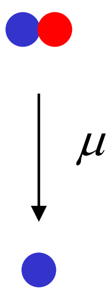




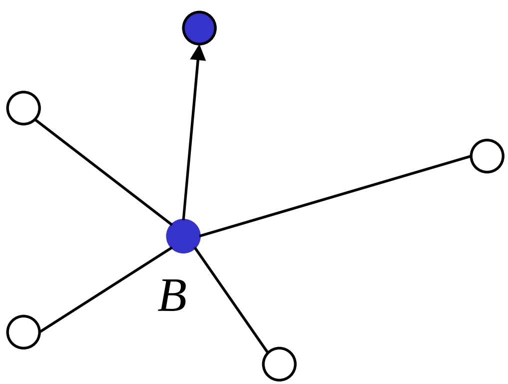
infecting or activating vaccine
at "all neighbors" with prob. $1-p_A$



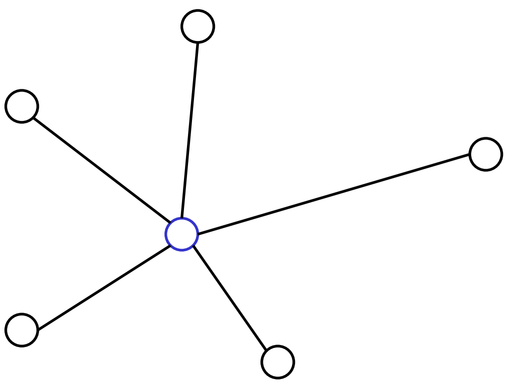
spontaneous annihilation
with prob. p_A



predation

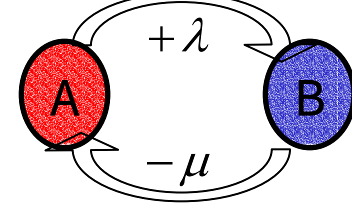


branching "one offspring"
with prob. $1-p_B$

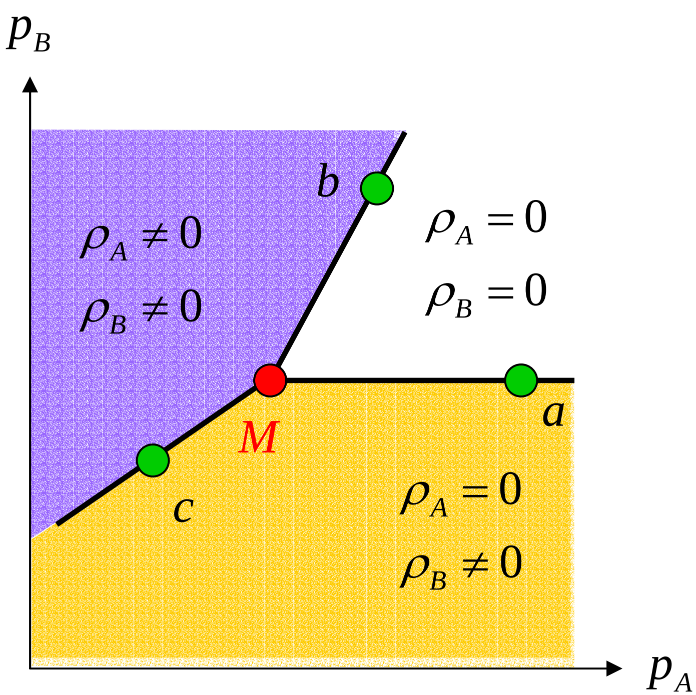


spontaneous annihilation
with prob. p_B

Simulations on BA Network (γ=3)



- Phase Diagram



$\lambda = 0.5,\ \mu = 1.0$

- phase boundary
  - a (1,0.4660)
  - b (0.90,0.52)
  - c (0.75,0.40)

- Multicritical point at *M* (0.87,0.466)

$$\rho_A \sim t^{-6.5},\ \rho_B \sim t^{-1.2}$$

$$\bar{z}_A \approx 0.12,\ \bar{z}_B \approx 0.44$$

- Virus-free and Vaccine-free region

↓

Efficient control of Epidemics

Summary

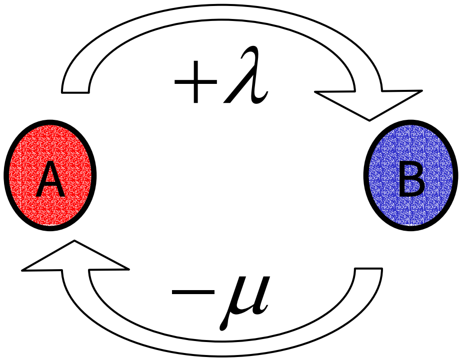
- Non-equilibrium phase transitions in Coupled Directed Percolation

$$A \xrightarrow{p_A} \varnothing \qquad A \xrightarrow{(1-p_A)(1-\lambda)} AA$$

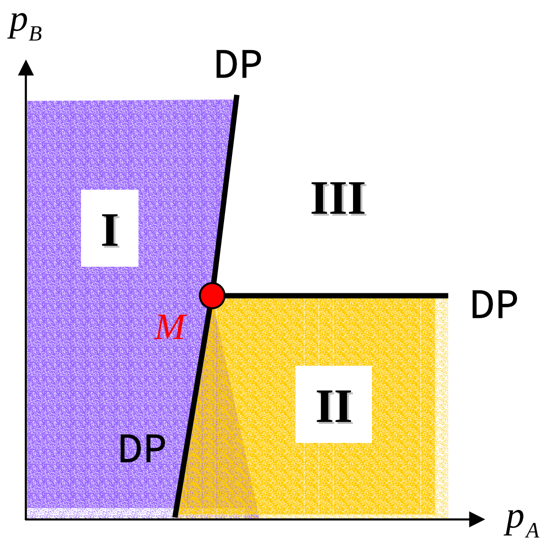
$$B \xrightarrow{p_B} \varnothing \qquad B \xrightarrow{(1-p_B)} BB$$

$$A \xrightarrow{(1-p_A)\lambda} AB$$

$$AB \xrightarrow{\mu} B$$

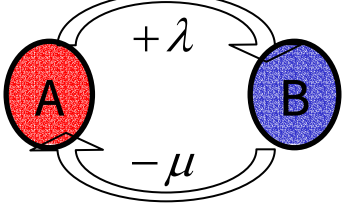


- Phase diagram (1D lattice, SW, <u>SF</u>)



- Region I $(\rho_A \neq 0, \rho_B \neq 0)$
  Viruses and vaccines are battling.
- Region II $(\rho_A = 0, \rho_B \neq 0)$
  Vaccines prevail over viruses.
- Region III $(\rho_A = 0, \rho_B = 0)$
  Healthy state

Summary



- Multicriticality at *M*
  - 1D lattice : Negative feedback is irrelevant ➔ UCDP
    - ✓ *A* : DP ➔ *B* : dressed
  - SW network : Mean field theory (negative feedback is relevant) works.
    - ✓ *A* : dressed 🡐 *B* : MF DP
    - ✓ $\overline{z}_A \approx 1/3 \Leftrightarrow \overline{z}_B \approx 1/2$

      If $\overline{z} = z_{MF} / d_c$, $d_c^A = 6$ and $d_c^B = 4$.

- Further questions
  - Existence of d* : μ is $\left\{\begin{matrix}\text{relevant} \\ \text{irrelevant}\end{matrix}\right\}$ for $\left\{\begin{matrix} d > d^* \\ d < d^* \end{matrix}\right\}$
  - Validity of $\overline{z} = z_{MF} / d_c$
  - Multispecies CDP (interesting MF theory)