EXACT SOLUTION OF THE ASYMMETRIC EXCLUSION PROBLEM WITH PARTICLES OF ARBITRARY SIZES: A MATRIX PRODUCT ANSATZ

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Bethe ansatz

coordinate　　　algebraic　　　functional

Fundamental tool for the integrability of Quantum chains

$$H = \sum_{i=1}^{L} h_{i,i+1} \Rightarrow \begin{cases} \text{EXACT INTEGRABLE} \\ \infty \ \# \ \text{Bethe states} \end{cases}$$

Matrix Product Ansatz (MPA)

- Non-integrable chains but a single state is exactly known (the ground state)
  - VBS Hamiltonian (Affleck, Kennedy, Lieb, Tasaki '87)
  - stochastic models with open boundaries (only stationary states) Derrida, Evans, Hakim, Pasquier '93
- Dynamical Product Ansatz Schultz, Stinchcombe, Popkov '95, 2002
  - Give also non-stationary states (not only the ground-state of the related Hamiltonian)

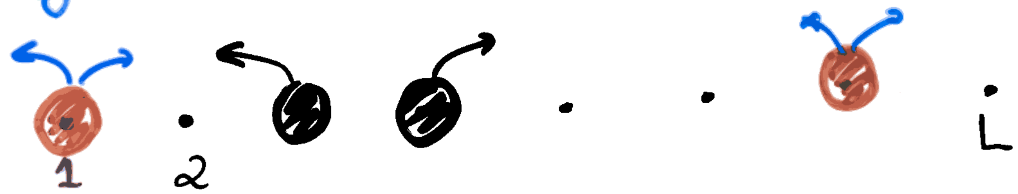
Reason: The associated quantum chain is *exactly* *integrable*

Question

Is it possible to formulate a Matrix Product Ansatz as powerfull as the Bethe ansatz?

Answer

Asymmetric diffusion of particles

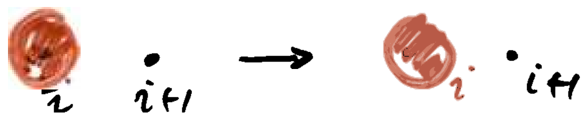


$\underset{i}{\bullet} \equiv 1_i$ $\qquad$ $\underset{i}{\cdot} \equiv 0_i$

Configuration : $\{\beta\} = \{\beta_1, \ldots, \beta_L\} = \{\cdot\cdot\bullet\bullet\cdot\cdot\bullet\cdot\}$

Dynamics → Transition rates

Diffusion to the right }



$\underbrace{W(01\to10)}_{\epsilon_R}$

Diffusion to the left }

$\underbrace{W(10\to01)}_{\epsilon_L}$

Question .

$P(\{\beta\},t) \equiv$ Probability of a given configuration $\{\beta\}$ happen at time t.

Master Equation

$$\frac{\partial}{\partial t} P(\{\beta\},t) = \sum_{\{\beta'\}\neq\{\beta\}} W(\{\beta'\}\to\{\beta\})\, P(\{\beta'\},t)$$

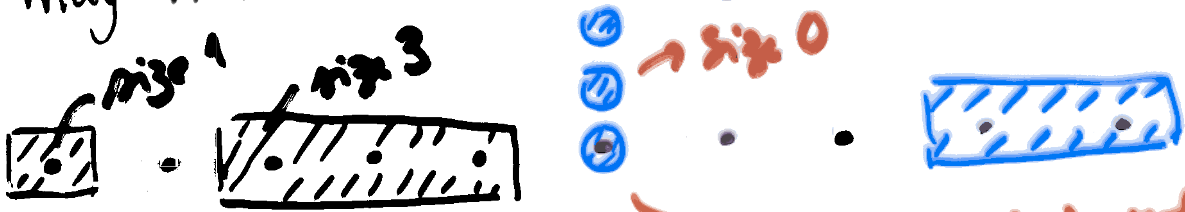
$$\sum_{\{\beta'\}\neq\{\beta\}} W(\{\beta\}\to\{\beta'\})\, P(\{\beta\},t)$$

$$\boxed{\frac{\partial P(\{\beta\},t)}{\partial t} = -\hat{H}\, P(\{\beta\},t)}$$

"Quantum" chain governing time fluctuations

Generalizations of the asymmetric exclusion problem

I Instead of having size 1, each individual particle may have a distinct size $s = 0, 1, 2, 3,$



No excluded volume (keep the ordering)

Site occupations

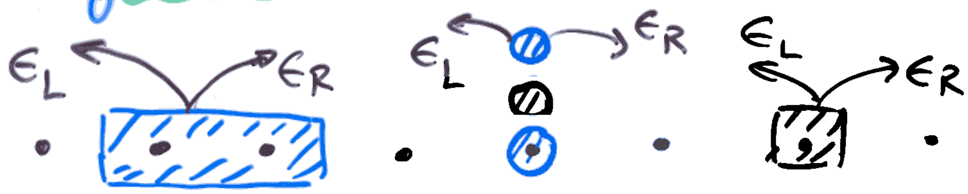
$\beta_i = 0$ site vacant or occupied by a monomer of the molecule on its leftmost.

> 0 site occupied by a molecule with size β_i

< 0 site occupied by $n = |\beta_i|$ molecules of size zero

$$\{\beta\} = \{\beta_1, \cdots, \beta_{10}\} = \{1, 0, 3, 0, 0, -3, 0, 0, 2, 0\}$$

Dynamical Processes



$$\frac{\partial}{\partial t} P(\{\beta\},t) = -H\, P(\{\beta\},t)$$

$$H = -D\,\mathcal{P}\sum_{i=1}^{L}\left(H_i^{>} + H_i^{<}\right)\mathcal{P}$$

$$H_i^{>} = \sum_{\beta=1}^{\infty}\left[\epsilon_+\left(1-E_i^{\beta,0}E_{i+1}^{0,\beta}\right)\mathcal{P}E_i^{0,\beta}E_{i+1}^{\beta,0} + \epsilon_-\left(1-E_i^{0,\beta}E_{i+1}^{\beta,0}\right)\mathcal{P}E_i^{\beta,0}E_{i+1}^{0,\beta}\right]$$

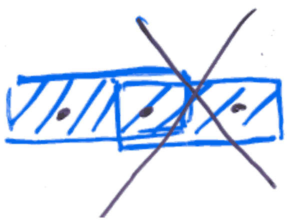
$$H_i^{<} = \sum_{\beta=-\infty}^{-1}\ \sum_{\gamma=-\infty}^{0}\left[\epsilon_+\left(E_i^{\beta+1,\beta}E_{i+1}^{\gamma-1,\gamma} - E_i^{\beta,\beta}E_{i+1}^{\gamma,\gamma}\right) + \epsilon_-\left(E_i^{\gamma-1,\gamma}E_{i+1}^{\beta+1,\beta} - E_i^{\gamma,\gamma}E_{i+1}^{\beta,\beta}\right)\right]$$

$$D = \epsilon_R + \epsilon_L\,, \qquad \epsilon_+ = \frac{\epsilon_R}{\epsilon_R+\epsilon_L}\,, \qquad \epsilon_- = \frac{\epsilon_L}{\epsilon_R+\epsilon_L}$$

$$\left(E^{\alpha,\beta}\right)_{i,j} = \delta_{\alpha,i}\,\delta_{\beta,j} \quad (\alpha,\beta,i,j \in \mathbb{Z})$$

If $s_1 = s_2 = \dots = s_n = s \geq 1$

$$H_{\{s_1=\dots=s_n=s\}} = -\mathcal{P}_s\left(\sum_{i=1}^{L}\left(\epsilon_+\sigma_i^-\sigma_{i+1}^+ + \epsilon\,\sigma_i^+\sigma_{i+1}^- + \frac{1}{4}(\epsilon_+ + \epsilon_-)\left(\sigma_i^z\sigma_{i+1}^z - 1\right)\right)\right)\mathcal{P}_s$$

$\mathcal{P}_s \equiv$ Projects out  .

THE EXACT SOLUTION (MPA)

n molecules sizes $\{s_1, s_2, \dots, s_n\}$

Since no interchange  Order $\underbrace{s_1, s_2, \cdots, s_n}$ fixed

Periodic Boundaries $\Rightarrow$ momentum P is conserved

$$\left.\begin{array}{l} \{s_1, \dots, s_n\} \\ P = \frac{2\pi}{L} j \quad (j = 0, 1, \dots, L-1) \end{array}\right\} \text{good quantum numbers}$$

A general eigenstate:

$$H\,|\psi_{\{s_1,\dots,s_n\},P}\rangle = \varepsilon_n\,|\psi_{\{s_1,\dots,s_n\},P}\rangle$$

$$T\,|\psi_{\{s_1,\dots,s_n\},P}\rangle = e^{iP}\,|\psi_{\{s_1,\dots,s_n\},P}\rangle$$

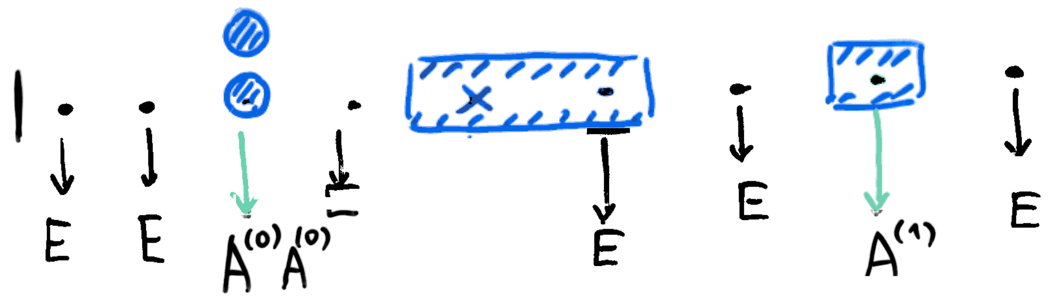
$$P = \frac{2\pi}{L} j \quad (j = 0, \dots, L-1)$$

$$|\psi_{\{s_1,\dots,s_n\},P}\rangle = \sum_{\{c\}}{}' \sum_{\{x\}}{}' \underbrace{f^{s_{c_1},\dots,s_{c_n}}(x_1,\dots,x_n)}_{\text{unknown!}}\,|x_1,\dots,x_n\rangle$$

$$x_{i+1} - x_i \geq s_{c_i}, \quad i = 1, \dots, n-1$$

$$s_{c_1} \leq x_n - x_1 \leq L - s_{c_n}$$

Matrix Product Ansatz



$$f^{s_1,\dots,s_n}(x_1,\dots,x_n)=\mathrm{Tr}\left(E^{x_1-1}A^{(s_1)}E^{x_2-x_1-1}A^{(s_2)}\dots E^{x_n-x_{n-1}-1}A^{(s_n)}E^{L-x_n}\Omega_p\right)$$

$E,\ A^{(s_i)},\ \Omega_p \rightarrow$ matrices forming an associative algebra $\Big\}$ Unknown!

Translation Invariance $\Rightarrow$ $\underbrace{T|\psi_{\{s_1,\dots,s_n\},p}\rangle = e^{ip}|\psi_{\{s_1,\dots,s_n\},p}\rangle}$

$$\Downarrow$$

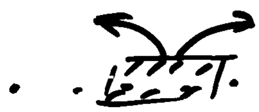
$$f^{s_1,\dots,s_n}(x_1+m,\dots,x_n+m)=e^{-ipm}f^{s_1,\dots,s_n}(x_1,\dots,x_n)$$

$$\left\{ E\Omega_p = e^{-ip}\Omega_p E\,,\qquad A^{(s_i)}\Omega_p = e^{-ip}\Omega_p A^{(s_i)}\right.$$

n = 1 particle (size s)

$$H_{,p}|\psi_{1,p}\rangle = \mathcal{E}_{1,p}|\psi_{1,p}\rangle\,,\qquad |\psi_{1,p}\rangle = \sum_{1\le x\le L} f^{s}(x)\,|x\rangle$$

$$\mathcal{E}_{1,p}\,\mathrm{Tr}[E^{x-1}A^{(s)}E^{L-x}\Omega_p] = -\epsilon_+\mathrm{Tr}[E^{x-2}A^{(s)}E^{L-x+1}\Omega_p]$$



$$-\epsilon_-\mathrm{Tr}[E^{x}A^{(s)}E^{L-x-1}\Omega_p] + \mathrm{Tr}[E^{x-1}A^{(s)}E^{L-x}\Omega_p$$

The cyclic property of the trace $\Longrightarrow$ $\mathcal{E}_{1,p} = -(\epsilon_+ e^{-ip} + \epsilon_- e^{ip} - 1)$

$$AE = e^{-ip} E A$$

It is convenient to define $A = A_k E^{2-s}$

$$E A_k = e^{ik} A_k E$$
$$A_k \Omega_p = \Omega_p A_k$$

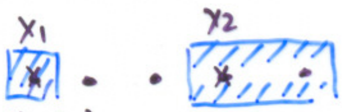
$$\mathcal{E}_{1,p} = \mathcal{E}(k) = -(\epsilon_+ e^{-ik} + \epsilon_- e^{ik} - 1)$$
$$k = p = \frac{2\pi}{L} j \quad (j = 0, 1, \dots, L-1)$$

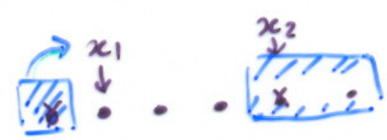
n = 2 particles

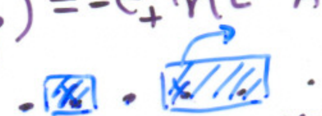
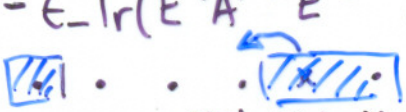
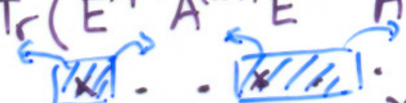
$$H |\psi_{2,p}^{\{s_1,s_2\}}\rangle = \mathcal{E}_2^{\{s_1,s_2\}} |\psi_{2,p}^{\{s_1,s_2\}}\rangle$$

- two types of relations $\begin{cases} x_2 > x_1 + s_1 \\ x_2 = x_1 + s_1 \end{cases}$

- $x_2 > x_1 + s_1$





$$\mathcal{E}_{2,p}^{(s_1,s_2)} \mathrm{Tr}(E^{x_1-1} A^{(s_1)} E^{x_2-x_1-1} A^{(s_2)} E^{L-x_2} \Omega_p) = -\epsilon_+ \mathrm{Tr}(E^{x_1-2} A^{(s_1)} E^{x_2-x_1} A^{(s_2)} E^{L-x_2} \Omega_p)$$
$$- \epsilon_- \mathrm{Tr}(E^{x_1} A^{(s_1)} E^{x_2-x_1-2} A^{(s_2)} E^{L-x_2} \Omega_p) - \epsilon_+ \mathrm{Tr}(E^{x_1-1} A^{(s_1)} E^{x_2-x_1} A^{(s_2)} E^{L-x_2-1} \Omega_p)$$
$$\underbrace{- \epsilon_- \mathrm{Tr}(E^{x_1-1} A^{(s_1)} E^{x_2-x_1-2} A^{(s_2)} E^{L-x_2-1} \Omega_p) + 2 \mathrm{Tr}(E^{x_1-1} A^{(s_1)} E^{x_2-x_1-1} A^{(s_2)} E^{L-x_2} \Omega_p)}$$

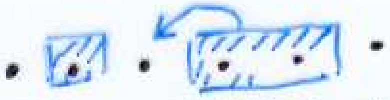
Solution: $\left\{ A^{(s_j)} = \left(A_{k_1}^{(s_j)} + A_{k_2}^{(s_j)}\right) E^{(2-s_j)}, \quad j = 1, 2 \right.$

Solution: $A^{(\Delta_j)} = \left(A^{(\Delta_j)}_{k_1} + A^{(\Delta_j)}_{k_2}\right)$ $\quad j=1,2$, $\quad E A^{(\Delta_\ell)}_{k_j} = A^{(\Delta_\ell)}_{k_j} E$

$$A^{(\Delta_\ell)}_{k_j}\Omega_p = e^{iP(1-\Delta_\ell)}\,\Omega_p A^{(\Delta_\ell)}_{k_j}, \qquad j=1,2\,;\ \ell=1,2$$

$$P = k_1 + k_2 \qquad \epsilon^{(\Delta_1,\Delta_2)}_{2,P} = \sum_{j=1}^{2}\varepsilon(k) = -\sum_{j=1}^{2}\left(\epsilon_+ e^{-ik_j} + \epsilon_- e^{ik_j} - 1\right)$$

• Second Type $\quad x_2 = x_1 + \Delta_1$



$$\underbrace{\begin{aligned}&\mathcal{E}^{(\Delta_1,\Delta_2)}_{2,P}\,\mathrm{Tr}\left(E^{x_1-1}A^{(\Delta_1)}E^{\Delta_1-1}A^{(\Delta_2)}E^{L-x_1-\Delta_1}\Omega_p\right) = -\epsilon_+\,\mathrm{Tr}\left(E^{x_1-2}A^{(\Delta_1)}E^{\Delta_1}A^{(\Delta_2)}E^{L-x_1-\Delta_1}\Omega_p\right)\\ &-\epsilon_-\,\mathrm{Tr}\left(E^{x_1-1}A^{(\Delta_1)}E^{\Delta_1}A^{(\Delta_2)}E^{L-x_1-\Delta_1-1}\Omega_p\right) + \mathrm{Tr}\left(E^{x_1-1}A^{(\Delta_1)}E^{\Delta_1-1}A^{(\Delta_2)}E^{L-x_1-\Delta_1}\Omega_p\right)\end{aligned}}$$

Solution

$$A^{(\Delta_\ell)}_{k_j}A^{(\Delta_r)}_{k_m} = S(k_j,k_m)\,A^{(\Delta_\ell)}_{k_m}A^{(\Delta_r)}_{k_j}, \quad (j\neq m,\ \ell,r=1,2);$$

$$A^{(\Delta_\ell)}_{k_j}A^{(\Delta_r)}_{k_j} = 0, \qquad S(k_j,k_m) = -\frac{\epsilon_+ + \epsilon_- e^{i(k_j+k_m)} - e^{ik_j}}{\epsilon_+ + \epsilon_- e^{i(k_j+k_m)} - e^{ik_m}}$$

$k\ \ k_2 \to$ Unknown

Cyclic Property of trace $\Rightarrow$ $\mathrm{Tr}\left(A^{(\Delta_1)}_{k_\ell}A^{(\Delta_2)}_{k_j}E^{L-\Delta_1-\Delta_2+2}\Omega_p\right) =$

$$e^{-ik_jL}\,e^{ik_j(\Delta_1+\Delta_2-2)}\,e^{-iP(\Delta_2-1)}\,S(k_j,k_\ell)\cdot\mathrm{Tr}\left(A^{(\Delta_2)}_{k_\ell}A^{(\Delta_1)}_{k_j}E^{L-\Delta_1-\Delta_2+2}\Omega_p\right)$$

$$= \underbrace{\left[e^{-ik_jL}\,e^{ik_j(\Delta_1+\Delta_2-2)}\,S(k_j,k_\ell)\right]^2 e^{-iP(\Delta_1+\Delta_2-2)}}_{1}\cdot\mathrm{Tr}\left(A^{(\Delta_1)}_{k_\ell}A^{(\Delta_2)}_{k_j}E^{L-\Delta_1-\Delta_2+2}\Omega_p\right)$$

$$e^{ik_jL} = e^{i\frac{2\pi}{2}m}\left(\frac{e^{ik_j}}{e^{ik_\ell}}\right)^{\bar{s}-1} S(k_j,k_\ell)\,; \quad \underbrace{j\neq \ell = 1,2\,;\ s_1\neq s_2 \quad m=0,1}_{}$$

Due to the distinguibility of particles

General n

$$A^{(s_j)} = \sum_{\ell=1}^{n} A_{k_\ell}^{(s_j)} E^{2-s_j}, \quad A_{k_j}^{(s_\ell)}\Omega_p = e^{iP(1-s_\ell)}\Omega_p A_{k_j}^{(s_\ell)}$$

$$E\, A_{k_\ell}^{(s_j)} = e^{ik_\ell} A_{k_\ell}^{(s_j)} E$$

$$A_{k_j}^{(s_t)} A_{k_\ell}^{(s_u)} = S(k_j,k_\ell) A_{k_\ell}^{(s_t)} A_{k_j}^{(s_u)} \quad (j\neq \ell)$$

$$A_{k_j}^{(s_t)} A_{k_j}^{(s_u)} = 0 \qquad (j,\ell,t,u = 1,\dots,n)$$

$$e^{ik_jL} = e^{i\frac{2\pi}{n}m}\prod_{\ell=1}^{n} S(k_j,k_\ell)\left(\frac{e^{ik_j}}{e^{ik_\ell}}\right)^{\bar{s}-1}, \quad m=0,1,\dots,n-1, \quad j=1,\dots,n$$

$$\bar{s} = \frac{1}{n}\sum_{j=1}^{n} s_j$$

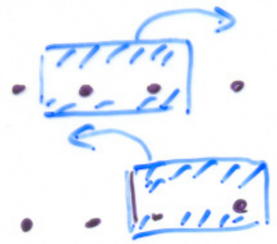
$$\epsilon_n = \sum_{j=1}^{n}\epsilon(k_j) = -\sum_{j=1}^{n}\left(\epsilon_+ e^{-ik_j} + \epsilon_- e^{ik_j} - 1\right)$$

(II) Generalization of the asymmetric diffusion problem

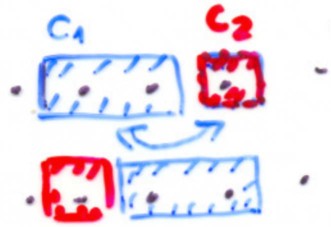
- Particles belonging to N distinct classes with hierarchical order
- Each particle has a size $s_i = 0, 1, 2, \cdots$ and belong to a given clan $c_i = 1, 2, \cdots, N$.

$\boxed{\text{Stochastical Processes}}$

Difusion $\left\{ \begin{array}{l} \text{rate } \epsilon_R \\ \text{rate } \epsilon_L \end{array} \right\}$ As before!



Interchange $\left\{ \begin{array}{l} \text{rate } \epsilon_R \text{ if } c_1 < c_2 \\ \phantom{\text{rate }} \epsilon_L \text{ if } c_1 > c_2 \end{array} \right.$



Good quantum numbers $\{c_1, s_1; \cdots, c_n s_n\}$ $\Big\}$ do not keep order!

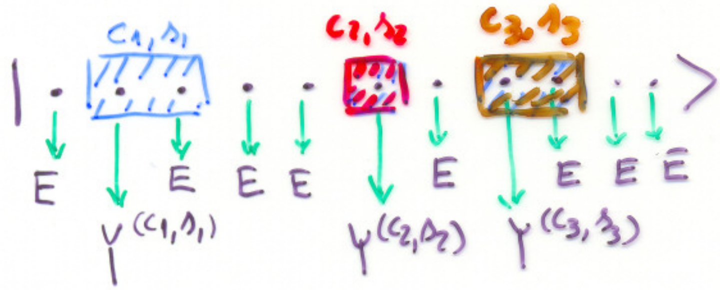
$c_j = 1, \cdots, N \quad , s_j = 0, 1, 2, \cdots$

- The Hamiltonian for $s_1 = s_2 = \cdots = s_n = 1$ is related to the $SU_q(N)$ Perk-Schultz model $q = \sqrt{\epsilon_R / \epsilon_L}$

We want to find the eigenfunctions

$$|c_1, s_1; \cdots, c_n, s_n\rangle = \sum_{\{c, s\}} \sum_{\{x\}} f^{(c_1, s_1; \cdots, c_n, s_n)}_{(x_1, \cdots, x_n)} |x_1, x_2, \cdots, x_n\rangle$$

Matrix Product Ansatz



$$f^{c_1 s_1, \dots, c_n s_n}_{(x_1,\dots,x_n)} = \mathrm{Tr}\left[E^{x_1-1} Y^{(c_1,s_1)} E^{x_2-x_1-1} Y^{(c_2,s_2)} \cdots E^{x_n-x_{n-1}-1} Y^{(c_n,s_n)} E^{L-x_n} \Omega_p\right]$$

Periodic Boundaries $\Big) \Longrightarrow$ Momentum Conservation $\Big) \Longrightarrow$ $P = \frac{2\pi}{L} j \;, \; (j = 0, 1, \dots, L-1)$

$$E\,\Omega_p = e^{-iP}\,\Omega_p\, E \quad , \quad Y^{(c,s)}\Omega_p = e^{-iP}\,\Omega_p\, Y^{(c,s)}$$

$$H\,|c_1 s_1, \dots, c_n s_n\rangle = \mathcal{E}_n\,|c_1 s_1; \dots, c_n s_n\rangle$$

- Relations where all particles at no "colliding" positions

$$Y^{(c,s)} = \sum_{j=1}^{n} Y^{(c,s)}_{k_j} E^{2-s} \;, \quad E\,Y^{(c,s)}_{k_j} = e^{ik_j}\, Y^{(c,s)}_{k_j} E$$

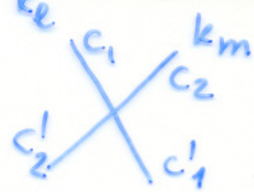
$$Y^{(c,s)}_{k_j}\Omega_p = e^{iP(1-s)}\,\Omega_p\, Y^{(c,s)}_{k_j} \; ; \; (j = 1, \dots, n; \; c = 1, 2, \dots, N; \; s = 0, 1, 2, \dots)$$

- Relations where two particles at "colliding" positions

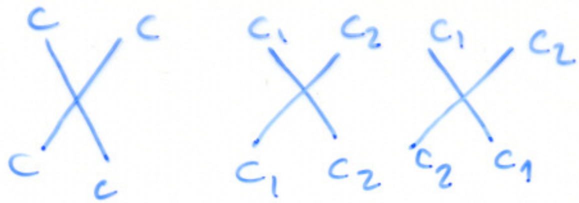
$$Y^{(c_1,s_1)}_{k_\ell}\, Y^{(c_2,s_2)}_{k_m} = \sum_{c_1', c_2' = 1}^{N} S^{c_1 c_2}_{c_1' c_2'}(k_\ell, k_m)\, Y^{(c_2',s_1)}_{k_m}\, Y^{(c_1',s_2)}_{k_\ell} \quad (k_\ell \neq k_m)$$

$$Y^{(c_1,s_1)}_{k_\ell}\, Y^{(c_2,s_2)}_{k_\ell} = 0 \qquad (\ell, m = 1, \dots, n \; ; \; c_1, c_2 = 1, \dots, N)$$

Structure Constants (like a S-matrix)

$S^{c_1 c_2}_{c'_1 c'_2}(k_\ell, k_m) \equiv$ 

$(2N^2 - N)$ non-zero values -1



$N = 1 \quad S^{11}_{11}(k_1, k_2) = S(k_1, k_2)$

$N = 2 \rightarrow$ 6 possible values

But the associativity of the algebra

$$\cdots Y^{(\alpha,\Delta_1)}_{k_1} Y^{(\beta,\Delta_2)}_{k_2} Y^{(\gamma,\Delta_3)}_{k_3} \cdots \begin{array}{l} \nearrow \widehat{\alpha\beta}\gamma \to \beta\widehat{\alpha\gamma} \to \widehat{\beta\gamma}\alpha \searrow \\ \searrow \alpha\widehat{\beta\gamma} \to \widehat{\alpha\gamma}\beta \to \gamma\widehat{\alpha\beta} \nearrow \end{array} \cdots Y^{(\gamma,\Delta_1)}_{k_3} Y^{(\beta,\Delta_2)}_{k_2} Y^{(\alpha,\Delta_3)}_{k_1} \cdots$$

$$\sum_{\gamma,\gamma',\gamma''=1}^{N} S^{\alpha\alpha'}_{\gamma\gamma'}(k_1,k_2)\, S^{\gamma\alpha''}_{\beta\gamma''}(k_1,k_3)\, S^{\gamma'\gamma''}_{\beta'\beta''}(k_2,k_3) =$$

$$\sum_{\gamma,\gamma',\gamma''=1}^{N} S^{\alpha'\alpha''}_{\gamma'\gamma''}(k_2,k_3)\, S^{\alpha\gamma''}_{\gamma\beta''}(k_1,k_3)\, S^{\gamma\gamma'}_{\beta\beta'}(k_1,k_2)$$

Yang-Baxter Equation

Necessary and sufficient condition for the associativity

satisfyed for arbitrary N !

The constants k_j $(j=1,\dots,n)$ are fixed by the cyclic property of the trace

$$\mathrm{Tr}\left[Y_{k_1}^{c(c_1,s_1)}\,Y_{k_2}^{c(c_2,s_2)}\cdots Y_{k_n}^{c(c_n,s_n)}\,E^{L-\sum_{l=1}^{n}(s_l-1)}\,\Omega_p\right]=$$

$$e^{ink_jL}\,e^{i\sum_{l=1}^{n}(s_l-1)[P-nk_j]}\,\times$$

$$\sum_{c'_1,\dots,c'_n}\underbrace{\langle c_1,\dots,c_n|\tau^n|c'_1,\dots,c'_n\rangle}\,\mathrm{Tr}\left[Y_{k_1}^{(c'_1,s_1)}\cdots Y_{k_n}^{c'_n,s_n}\,E^{L-\sum_{j=1}^{n}(s_j-1)}\,\Omega_p\right].$$

$\tau \equiv$ Transfer matrix of a $(2N^2-N)$-inhomogeneous vertex model with Boltzmann Weights $S_{c'_1c'_2}^{c_1c_2}$ and Periodic boundary conditions

$$\tau|\psi\rangle=\underbrace{\Lambda(k_j,\{k_l\})}|\psi\rangle$$

evaluated by the coordinate or algebraic Bethe ansatz

$$e^{ik_j[L+n-\sum_{j=1}^{n}s_j]}=e^{i\frac{2\pi}{n}r}\,e^{iP(\bar{s}-1)}\,\Lambda(k_j,\{k_l\})$$

CONCLUSIONS

1) It is possible to introduce a Matrix Product ansatz for the stochastic models. exactly integrable

2) The MPA provide an unified prescription enabling the inclusion of hard-core interactions

(Produce solutions more easily than the standard Bethe ansatz)

3) The MPA can be applied to quantum chains related or not to stochastic models

Spin-1 models) Fateev-Zamolodchikov, Izergin Korepin

Spin-$\frac{3}{2}$ models) t-J model, Hubbard model, Perk-Schultz model, etc

etc.