STATPHYS22 SATELLITE MEETING 29 JUNE - 2 JULY 2004

Collective phase synchronization of spatially extended oscillators with random frequencies

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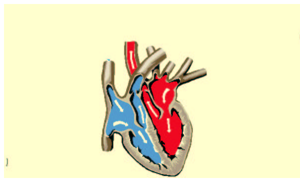
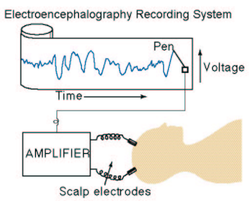
- Fireflies blinking in unison

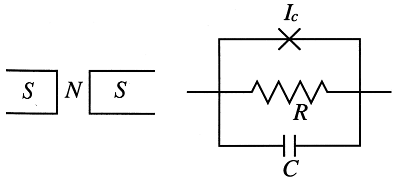
- Pacemaker cells in the heart : ECG



- Neural networks : EEG

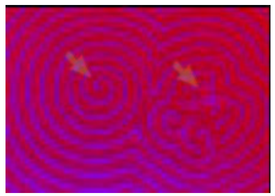


- Josephson junction arrays



- Clapping in the theater

- Chemical reactions : BZ



▽ Locally coupled oscillators

$$\frac{d\phi_i}{dt} = \omega_i - K \sum_{j \in \Lambda_i} \sin(\phi_i - \phi_j) \qquad i = 1, \cdots, N$$

ϕ_i : phase of the ith oscillator

K : coupling strength

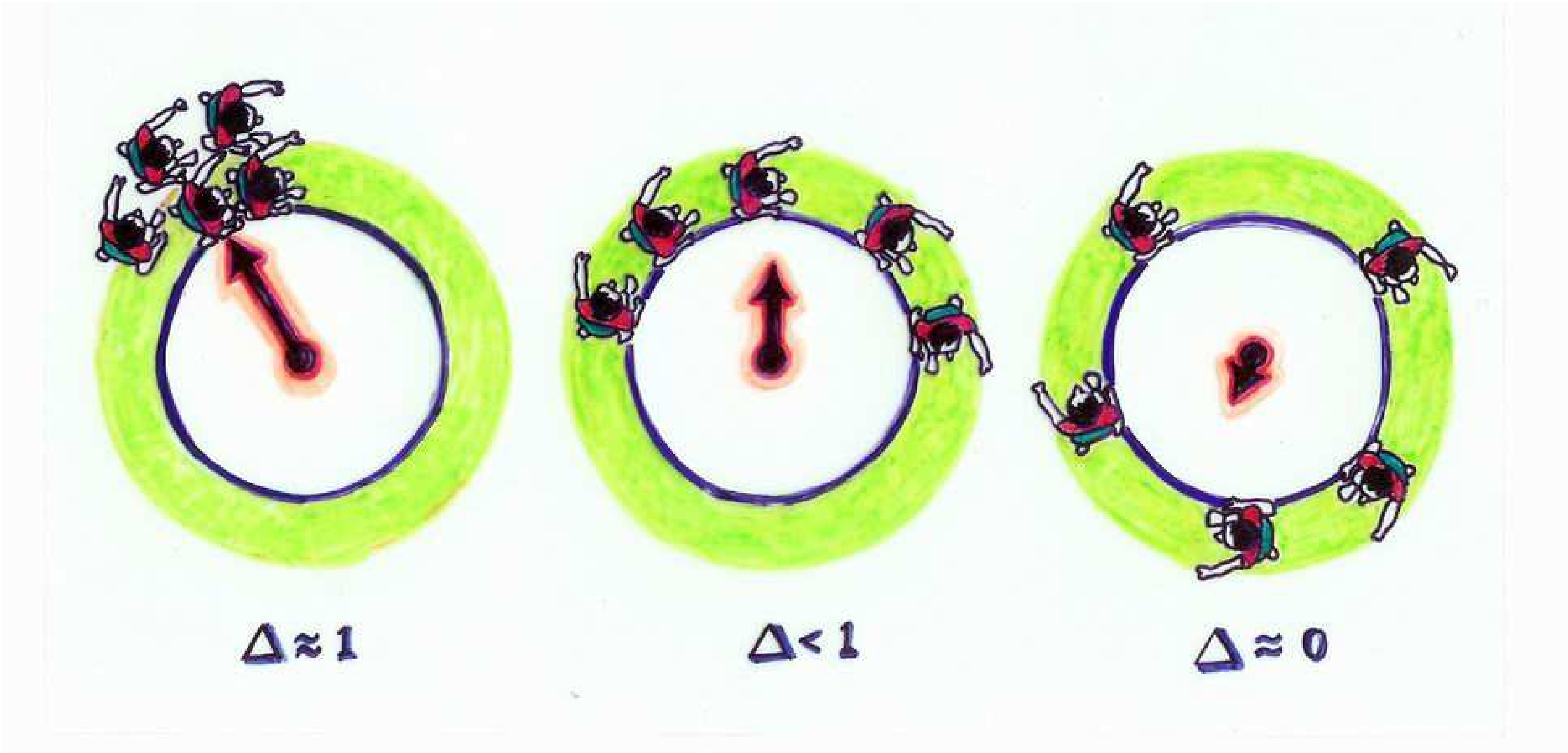
Λ_i : nearest neighbor set of $\mathbf{i}$th oscillator

ω_i : intrinsic frequency of the $\mathbf{i}$th oscillator : $\mathbf{g}(\omega)$

$$\langle \omega_i \omega_j \rangle = 2\sigma\delta_{ij} \qquad \langle \omega_i \rangle = 0$$

- **Order parameter**

$$\Delta \equiv \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j} \right| \right\rangle$$



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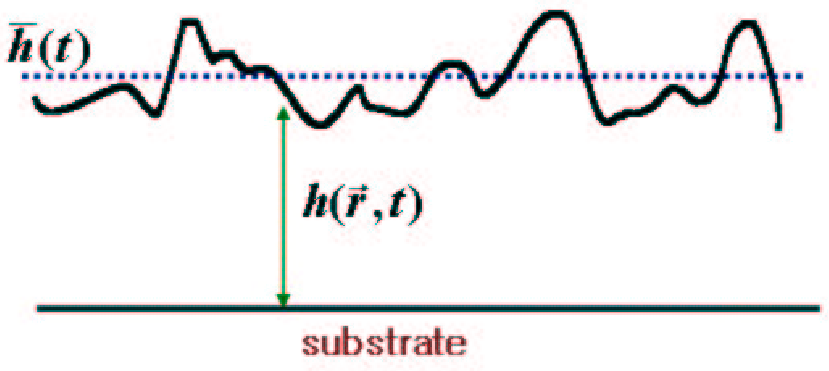
▽ Continuum limit

$$\frac{\partial}{\partial t}\phi(\mathbf{r},t) = \omega(\mathbf{r}) + K\nabla^2\phi + \mathcal{O}[\nabla^4\phi, (\nabla\phi)^2\nabla^2\phi]$$

$$\langle\omega(\mathbf{r})\rangle = 0, \quad \langle\omega(\mathbf{r})\omega(\mathbf{r}')\rangle = 2\sigma\delta(\mathbf{r}-\mathbf{r}')$$
$$0 \le \phi \le 2\pi \quad \longrightarrow \quad (-\infty, \infty)$$

phase $\phi(\mathbf{r},t) \longleftrightarrow$ height $h(\mathbf{r},t)$:



Edwards-Wilkinson type equation with **columnar disorder** !

- **Mean-square surface fluctuation width**

$$W^2 \equiv \frac{1}{L^d}\left\langle \int^L d^d\mathbf{r}\left[\phi(\mathbf{r},t) - \bar{\phi}(t)\right]^2\right\rangle$$

(i) In the long time limit ($Kt \gg L^2$)

$$\begin{aligned} W^2 &\sim (2\sigma/K^2)L^{4-d}, & d < 4 \\ &\simeq (\sigma/4\pi^2K^2)\ \mathrm{ln}\, L, & d = 4 \\ &\sim 2\sigma/K^2, & d > 4. \end{aligned}$$

(ii) In the early time limit ($Kt \ll L^2$)

$$\begin{aligned} W^2 &\sim (2\sigma/zK^2)(Kt)^{(4-d)/z}, & d < 4 \\ &\simeq (\sigma/4\pi^2zK^2)\ \mathrm{ln}(Kt), & d = 4 \end{aligned}$$

$W \sim L^\alpha$ where $\alpha = (4-d)/2$, $W \sim L^\beta$ where $\beta = (4-d)/4$

- **Steady-state probability distribution $P[\{\phi\}]$**

$$\frac{\partial}{\partial t}\phi(\mathbf{k},t) = \omega(\mathbf{k}) - Kk^2\phi(\mathbf{k},t)$$

$$P_{\omega_{\mathbf{k}}}(\{\phi_{\mathbf{k}}\}) = \prod_{\mathbf{i}} \delta\left(\phi_{\mathbf{k_i}} - \frac{\omega_{\mathbf{k_i}}}{K\mathbf{k_i}^2}\right)$$

Averaging over random frequencies $\{\omega_{\mathbf{k}}\}$, we find

$$P[\{\phi\}] \sim \int \mathcal{D}\omega_{\mathbf{k}} P_{\omega_{\mathbf{k}}}(\{\phi_{\mathbf{k}}\})$$

$$= \int d\omega_{\mathbf{k_1}} \cdots d\omega_{\mathbf{k_N}} \exp\left[-\sum_{\mathbf{i}} \frac{\omega^2_{\mathbf{k_i}}}{4\sigma}\right] \prod_{\mathbf{i}} \delta\left(\phi_{\mathbf{k_i}} - \frac{\omega_{\mathbf{k_i}}}{K\mathbf{k_i}^2}\right)$$

$$= \exp\left[-\sum_{\mathbf{i}} \frac{K^2}{4\sigma}\mathbf{k_i^4}\phi^2_{\mathbf{k_i}}\right]$$

$$= \exp\left[-\frac{K^2}{4\sigma}\int \left(\nabla^2\phi\right)^2 \mathbf{dx}\right].$$

Gaussian property of $P[\{\phi\}]$ guarantees that $\langle e^{\mathbf{i f}(\phi)}\rangle = e^{-\langle \mathbf{f}^2(\phi)/2\rangle}$ for an arbitrary $f(\phi)$.

$$\Delta \equiv \left\langle e^{i(\phi-\bar{\phi})}\right\rangle$$

$\hookrightarrow$ $$\boxed{\Delta = \exp(-W^2/2)}$$

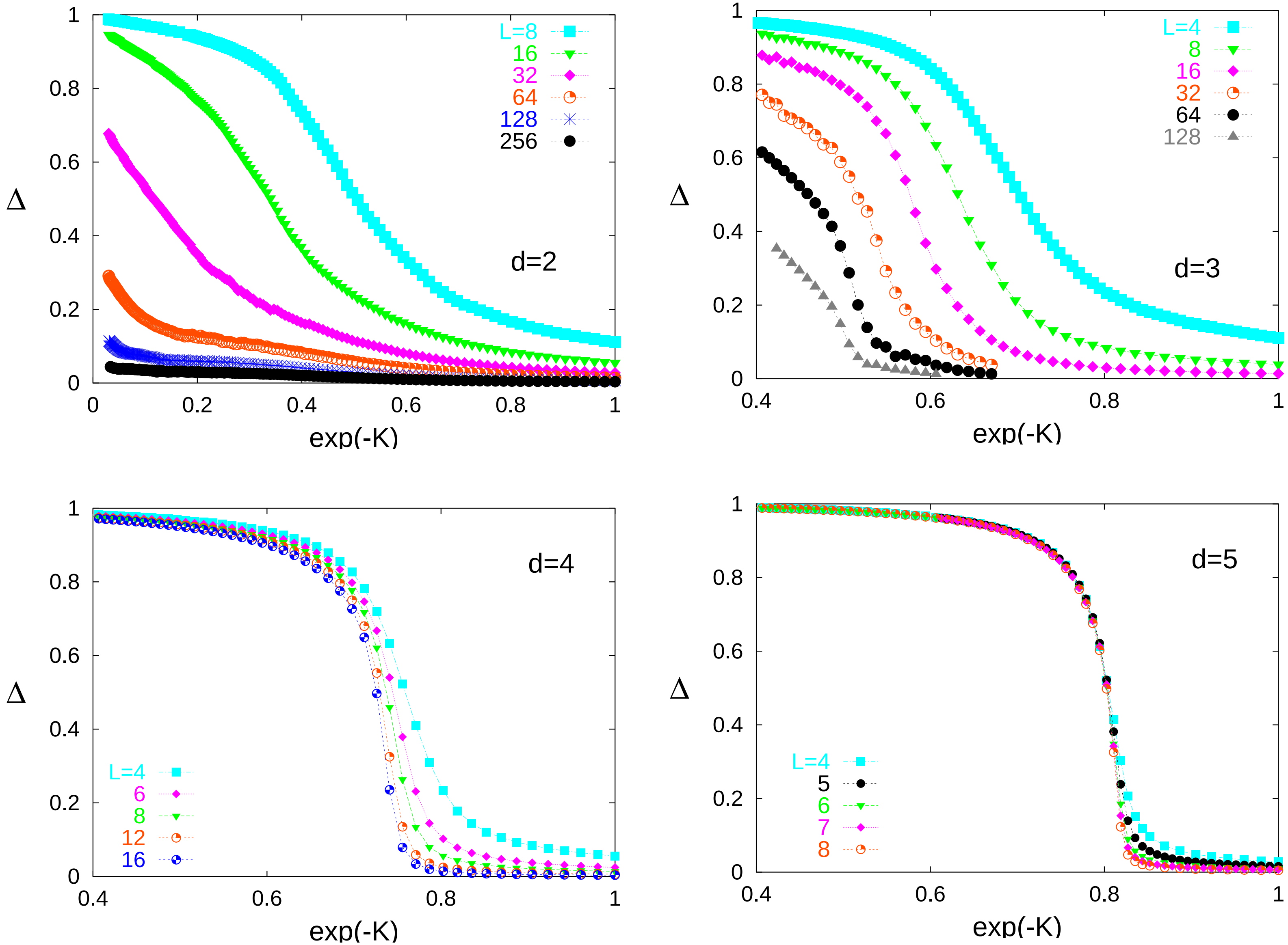
- **For $d \leq 4$,**

  ROUGH $(W^2 \to \infty)$ $\Longleftrightarrow$ DESYNC $(\Delta = 0)$

- **For $d > 4$,**

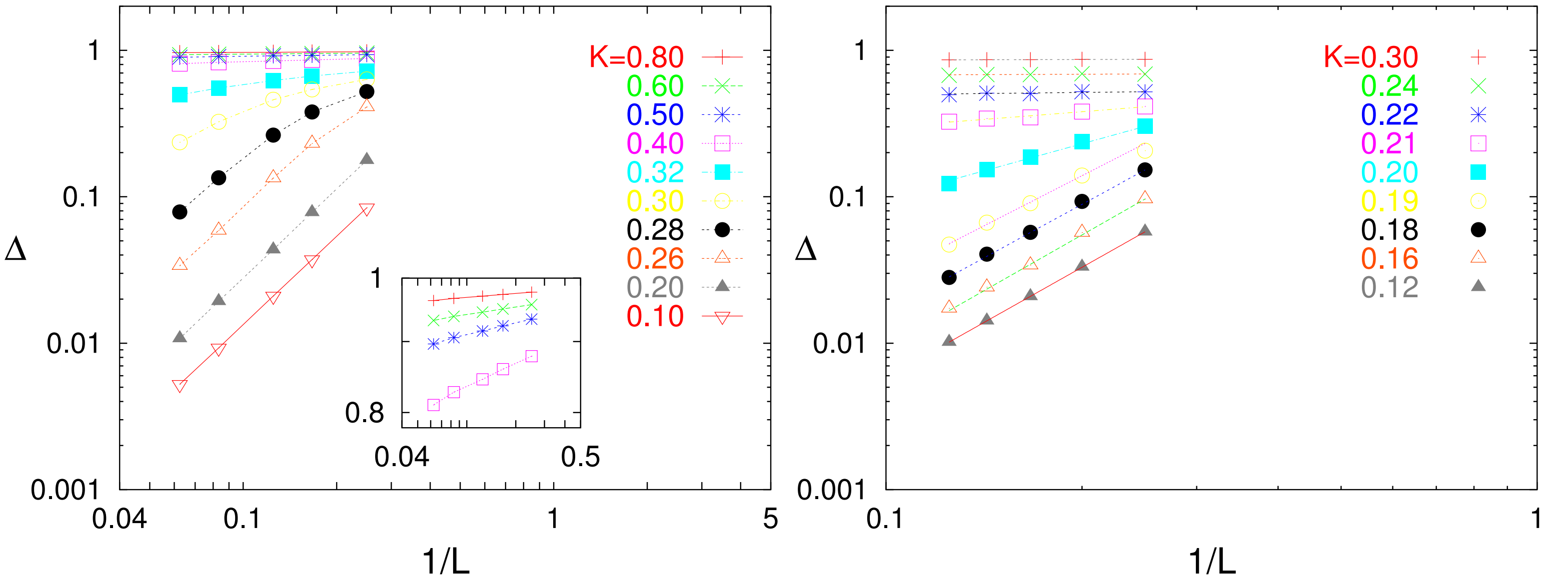
  SMOOTH $(W^2 < \infty)$ $\Longleftrightarrow$ SYNC $(\Delta \neq 0)$

▽ Numerical results



Finite-size analysis

$d = 4$ $\qquad$ $d = 5$

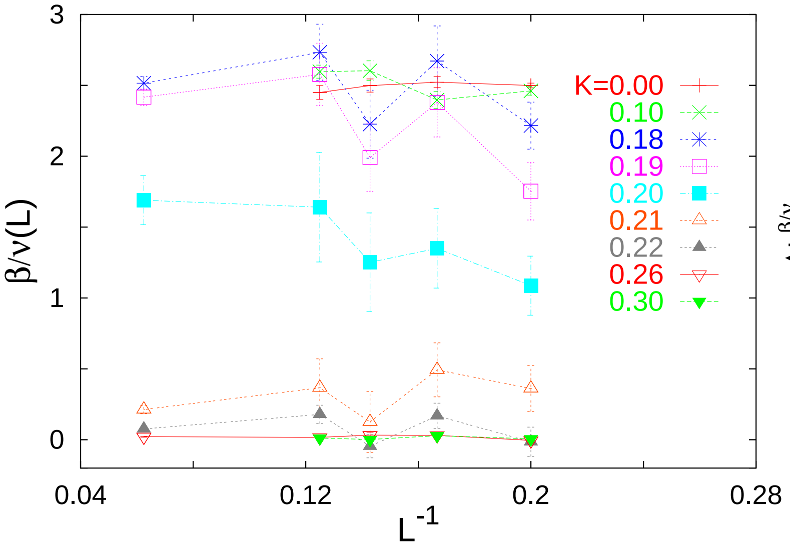


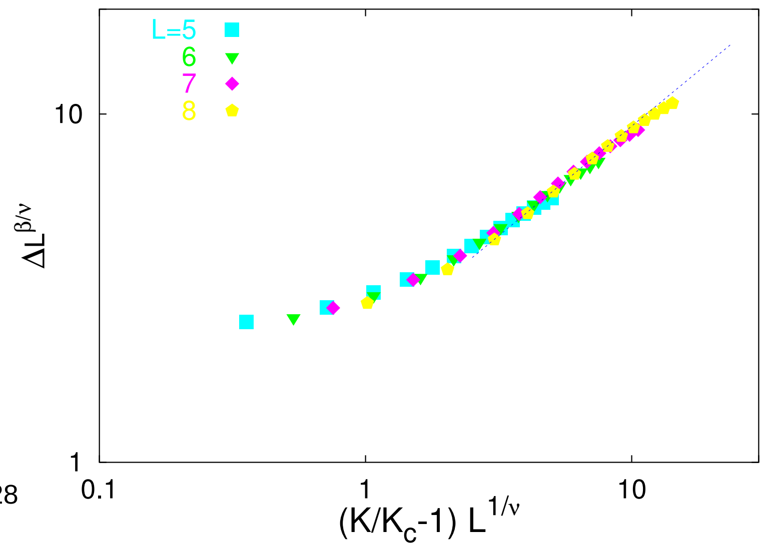
$K_c = 0.200(5)$

Critical behavior for $d = 5$

$$\Delta = L^{-\beta/\nu} f[(K - K_c)L^{1/\nu}], \qquad \Delta(K_c, L) \sim L^{-\beta/\nu}$$

$$\beta/\nu(L) = -\ln[\Delta(L')/\Delta(L)]/\ln(L'/L)$$





$$\beta/\nu = 1.5(3), \quad \nu = 0.45(10), \quad K_c = 0.200(5)$$

Summary

- SYNC-DESYNC of the coupled oscillators — ROUGH-SMOOTHENING of the growing surface

- Order parameter: Δ — Surface fluctuation width: W^2

$$\Delta = \exp(-W^2/2)$$

- Lower critical dimension for the phase synchronization:

  $d_\ell = 4$