

# Universal fluctuations of global quantities in some correlated systems near criticality

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- Some exact results on 2D-XY model fluctuations
- Another example: 2D Ising model

# Outlines

- Gaussian and non gaussian fluctuations for a global order parameter: generalities
- Experimental facts on non-equilibrium systems, turbulence
- Correlated systems: spins systems
- Some exact results on 2D-XY model fluctuations
- Another example: 2D Ising model
- Conclusion and perspectives

# General arguments

## Fluctuations of a global order parameter

Defining  $m = \frac{1}{N} \sum_{\mathbf{r} \text{ or } t} m(\mathbf{r} \text{ or } t)$  ( $m(\mathbf{r} \text{ or } t)$  can be magnetization, local fluid density, dissipated power, etc...).

How  $m$  is fluctuating ?

### Case #1

$m(\mathbf{r})$  large number of independent gaussian variables,

$$\langle m \rangle = \lim_{N \gg 1} \frac{1}{N} \sum_{\mathbf{r}} m(\mathbf{r}), \quad \sigma^2 = \lim_{N \gg 1} \frac{1}{N} \sum_{\mathbf{r}} m(\mathbf{r})^2$$

$$\Rightarrow \text{central limit theorem } P(m) = \frac{1}{\sqrt{2\pi N\sigma}} \exp\left(-\frac{(m - \langle m \rangle)^2}{2N\sigma^2}\right)$$

(Ising model for  $T > T_c$ )

# General arguments

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### Case #2

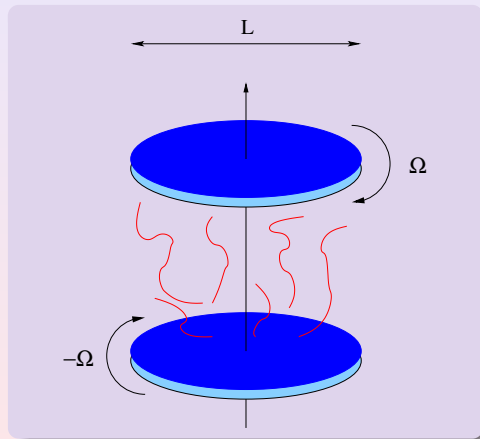
$m(\mathbf{r})$  large number of correlated variables

$$[m(\mathbf{r})m(\mathbf{r}')]_{av} \simeq |\mathbf{r} - \mathbf{r}'|^{-\eta}$$

- Non gaussian fluctuations in general
- Deviation from a gaussian behavior, non-symmetric distribution
- System can not be separated *spatially* or *temporally* in different parts (example: Ising model near critical point)

# Turbulence experiment

[Pinton, Holdsworth, Labbé, Phys. Rev. E, **60** R2452 (1999)]

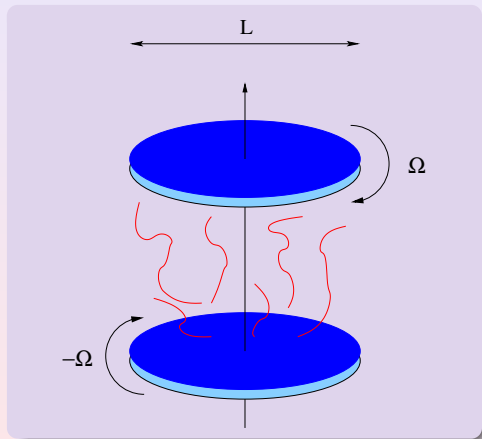


## Principe

- 3D cylinder with counter-rotating disks

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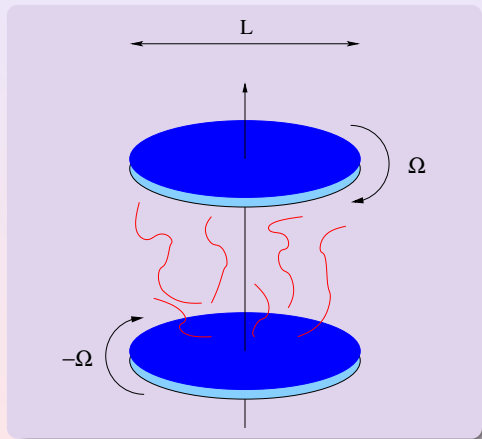


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- 3D cylinder with **counter-rotating disks**
- Constant  $\Omega$

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## Principe

- 3D cylinder with **counter-rotating disks**
- Constant  $\Omega$
- Constant Reynolds number  
 $Re = L^2 \Omega / \nu \Rightarrow$   
injected power  
 $P(t)$  fluctuates

# Fluctuations of total injected power $P(t)$

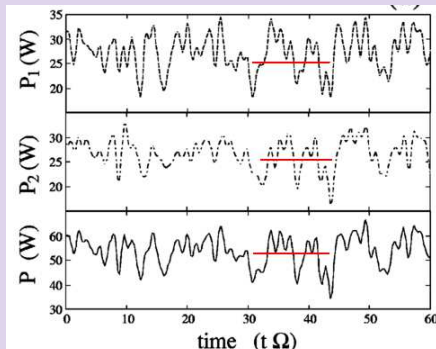
$P_1(t)$  and  $P_2(t)$  time dependent injected power for each disk 1 and 2.

There are correlations between  $P_1$  and  $P_2$ .

Time correlations  $[P_i(t)P_i(t')]$  last over 20 revolutions.

Drops correspond to evolution of coherent large structures.

## Time fluctuations for the 2 disks



# Fluctuations of total injected power $P(t)$

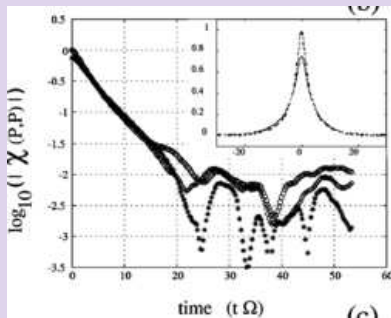
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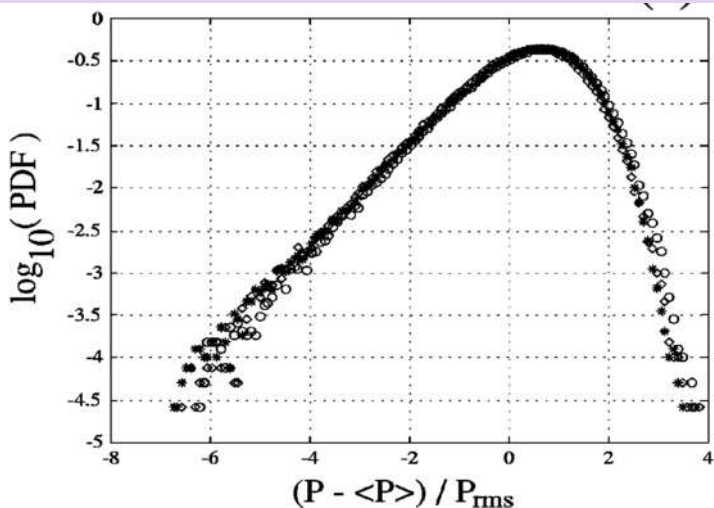
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## Autocorrelation functions $[P_i(t)P_j(t')]$



Probability Density Function (PDF) for  $P_1$ ,  $P_2$ , and the sum  $P = P_1 + P_2$ , normalized by the mean  $\langle P \rangle$  and variance  $P_{rms}$ :  $PDF(P) = \frac{1}{P_{rms}} \Pi((P - \langle P \rangle) / P_{rms})$



Rare events more probable than for a gaussian distribution by a factor  $10^4$  for a deviation  $6P_{rms}$ .

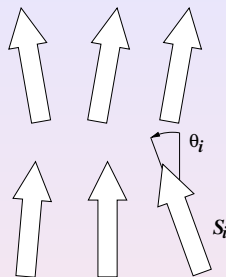
Non gaussian PDF means in general correlated system.

Lack of microscopic theory  $\Rightarrow$  system too complex.

### Questions:

- Are there any simple statistical models with same fluctuation properties ?
- If yes, is it only qualitative or quantitative ?
- Role of universal classes, critical exponents ?

# A simple critical model: 2D-XY



Hamiltonian is angle dependent:

$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

We define a scalar order parameter as the modulus of the sum of the spins:

$$m = \left| \frac{1}{N} \sum_i \mathbf{S}_i \right|$$

This system is critical at low temperature  $T < T_{KT}$ .

Spin wave excitations at low temperature.

$$H \simeq \frac{J}{2} \sum_{\langle i,j \rangle} (\theta_i - \theta_j)^2 \quad (1)$$

The hamiltonian is diagonalisable in Fourier space. The instantaneous magnetization is then:

$$m_{inst} = \frac{1}{N} \sum_i \cos(\theta_i - \bar{\theta})$$

where  $\bar{\theta}$  is the angle of the sum of the spins (average direction).  
For a finite size system:

$$\langle m_{inst} \rangle = (CN)^{-k_B T / 8\pi J} \simeq L^{-\beta/\nu}, \quad \frac{\beta}{\nu} = \frac{k_B T}{4\pi J}$$

We also define the propagator:

$$G(\mathbf{q}) = (4 - 2 \cos q_x - 2 \cos q_y)^{-1}$$

Correlation functions decrease algebraically

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \simeq |r_i - r_j|^{-k_B T / 2\pi J} = |r_i - r_j|^{-\eta}$$

and the variance is proportionnal to  $\langle m_{inst} \rangle$ :

$$\sigma = \sqrt{\langle m_{inst}^2 \rangle - \langle m_{inst} \rangle^2} = \sqrt{\frac{g_2}{2} \frac{k_B T}{J}} \langle m_{inst} \rangle$$

with  $g_2 = \sum_{\mathbf{q} \neq 0} \mathbf{G}(\mathbf{q})^2 / N^2 \simeq 3.87 \times 10^{-3}$ . We have a critical system with long range correlation functions and we would like to study the PDF of the order parameter in the limit  $N \rightarrow \infty$ .

# PDF of the 2D-XY order parameter

The PDF is simply defined by

$$P(m) = \frac{1}{Z} \text{Tr} \delta(m - m_{inst}) \exp(-\beta H)$$

where Tr is the integral over all the angle configurations and Z the partition function. We can write  $P$  as

$$P(m) = \frac{1}{\sigma} \Pi\left(\frac{m - \langle m_{inst} \rangle}{\sigma}\right)$$

and evaluate  $\Pi$ . In the gaussian case (independent gaussian variables),  $\Pi(\theta)$  would simply be  $\exp(-\theta^2/2)/\sqrt{2\pi}$ . 2 ways of computing  $P$  or  $\Pi$ :

- Computing all the moments  $\langle m^p \rangle$ .
- Low temperature expansion which remains however valid thanks to the extension of the critical line.

# Results

We finally find (see [PRE **63** 041106 (2001)] for details):

$$P(m) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\sigma} \exp \left[ ix \frac{m - \langle m_{inst} \rangle}{\sigma} + \sum_{k=2}^{\infty} \frac{g_k}{2k} \left( ix \sqrt{\frac{2}{g_2}} \right)^k \right] \quad (2)$$

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- Non gaussian function ( $\langle m_{inst} \rangle$  non intensive)
- $\Pi$  is function only of  $(m - \langle m_{inst} \rangle)/\sigma$  and does not depend on temperature, nor on  $\beta/\nu \Rightarrow$  **universal function**
- $\Pi$  tends to a finite function in the thermodynamical limit

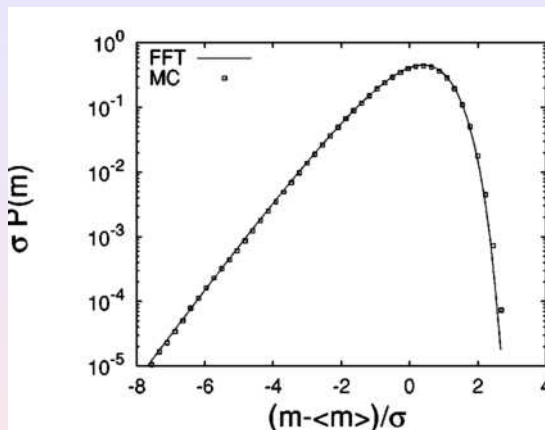
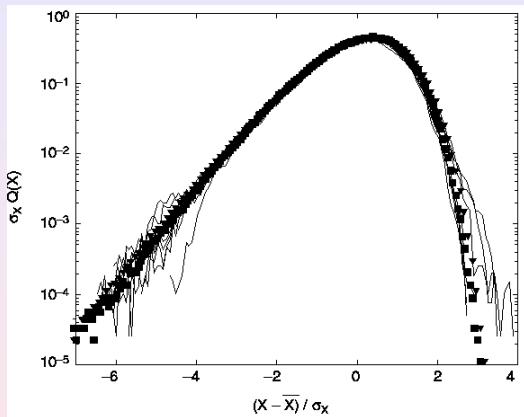


FIG.: Monte Carlo simulation for a system  $N = 32 \times 32$  at  $T=0.1J$  and numerical evaluation of the PDF

Similar curve than the one from the experimental result on turbulence [Bramwell, Holdsworth, Pinton, Nature (1998)]



**FIG.:** PDF of turbulence experiment (for different Reynolds numbers) superposed onto the 2D-XY PDF [ $T/J=0.5$ ,  $N=100$  spins ( $\circ$ ),  $1024$  ( $\star$ ) and  $10\,000$  ( $\triangle$ ), and  $T/J=1$ ,  $N=1024$  ( $\square$ )]

# Properties and asymptotic behavior

We can extract exactly the asymptotic behavior of  $\Pi(\theta)$  for  $\theta \gg 1$  and  $\theta \ll -1$

$\Rightarrow$  saddle points of the function  $\Phi$  such as

$$\Pi(\theta) = \int_{-\infty}^{\infty} \sqrt{\frac{g_2}{2}} \frac{dx}{2\pi} \exp [i\Phi(x)]$$

$$\Phi(x) = ix\theta \sqrt{\frac{g_2}{2}} - i\frac{x}{2} \text{Tr}G/N - \frac{1}{2} \text{Tr} \log (1 - ixG/N)$$

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$$\Pi(\theta) \simeq |\theta| \exp\left(4\pi^2 \sqrt{\frac{g_2}{2}} \theta\right)$$

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- For large and positive deviations, we obtain

$$\Pi(\theta) \simeq \exp \left( -\frac{1}{8\pi} \exp \left[ 8\pi \left( \sqrt{g_2/2} \theta - \text{cst} \right) \right] + 8\pi \sqrt{\frac{g_2}{2}} \theta \right)$$

⇒ probability behaves like a double exponential

The asymptotic form for  $\Pi(\theta)$  is very close to a generalized Gumbel function  $G$

$$G_{\pi/2}(\theta) \simeq \exp \frac{\pi}{2} [b(\theta - \text{cst}) - \exp b(\theta - \text{cst})]$$

with

$$b = 8\pi \sqrt{\frac{g_2}{2}} \simeq 1.105$$

$\Rightarrow$  Gumbel functions are typical of extreme value problems with integer parameter  $G_p$  (PDF of the minimum or maximum of a set of a large number of variables)  $\Rightarrow$  Link with extreme values ?

# Looking for other critical models

We consider the well known 2D Ising model.

System critical with  $T_c = 2.25J$ . Same space dimensionality as 2D-XY model but critical only on one point. Critical point controlled by varying the size  $L$  of the system.

We would like to study the PDF of the order parameter near critical point.

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \quad S_i = \pm 1 \quad (3)$$

Order parameter defined by

$$m_{inst} = \left| \frac{1}{N} \sum_i S_i \right|$$

Numerical results show PDF **similar** to 2D-XY model in a critical region near  $T_c$  (see [PRL **84** 3744 (2000)] and [cond-mat/0405576]):

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- Can these fluctuations be described by the same hamiltonian ?
- How Ising and XY are related at  $T^*(L)$  ?

# Numerical tools

Swendsen-Wang Monte Carlo algorithm for various size  $L$  between 32 and 512.

We define  $T^*(L)$  as the temperature for which the skewness  $\gamma$  of the PDF is equal to that of the 2D-XY model (with error less than 1%):

$$\gamma = \left\langle \left( \frac{m - \langle m_{inst} \rangle}{\sigma} \right)^3 \right\rangle = 0.890$$

$\Rightarrow$  necessary condition but not sufficient.

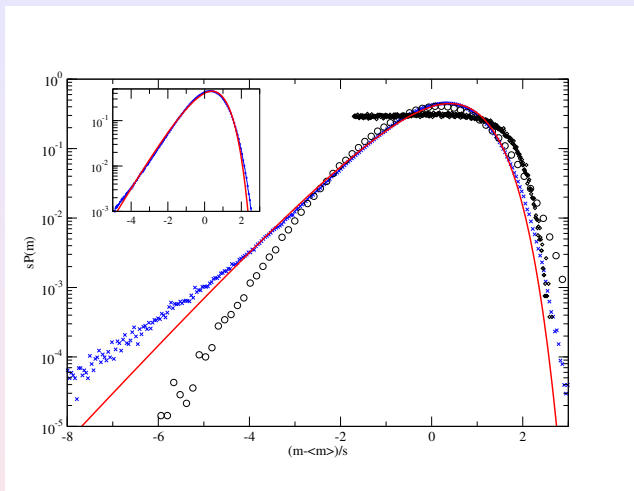


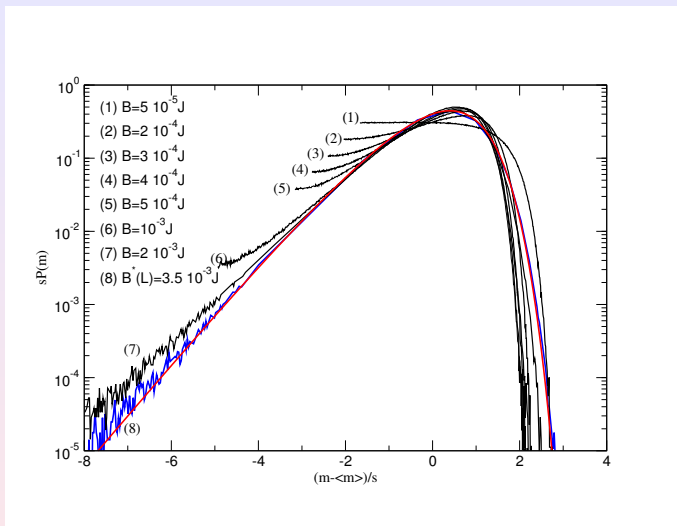
FIG.: PDF for  $T = 2.33J = T_c(L)$  ( $\diamond$ ),  $T^*(L) = 2.11J$  ( $\times$ ) and  $T = 1.54J$  ( $\circ$ ), with  $L = 64$ . The inset is a zoom on  $(m - \langle m_{inst} \rangle)/\sigma \in [-5; 3]$ .

## Comments:

- $T^*(L)$  shifted well below  $T_c$ .  
The PDF is **cutoff** on the left by the constraint  $m = 0$ .
- At low temperature, the PDF looks more like a gaussian with a concave tail for negative deviations.
- At higher temperature, there is no “exponential” tail.  
The system is influenced by the opposite region of the phase diagram where the magnetization changes sign  $\Rightarrow P(m = 0)$  significant
- At  $T^*(L)$ , deviation from the exponential tail for  $m - m_{inst} < -5\sigma \Rightarrow$  PDF cannot be described by the XY-PDF correctly except for statistically low deviations from the mean value.

# Importance of the Phase Space

- Proximity of the 2 phase spaces (magnetization  $\pm$  below  $T_c$ ) near  $T_c \Rightarrow$  the system can tunnel between the 2 regions with  $P(m = 0)$  significant.
- Necessity to introduce a constraint to prevent the system from tunneling from one region (magnetization +) to another one (magnetization -):  
Introducing special boundary conditions (not efficient) or a small external magnetic field (more efficient)



**FIG.:** PDF for a subsystem  $L=32$  inside a system  $L = 128$  at  $T_c(L)$  for various magnetic fields. Plain curve is 2D-XY model

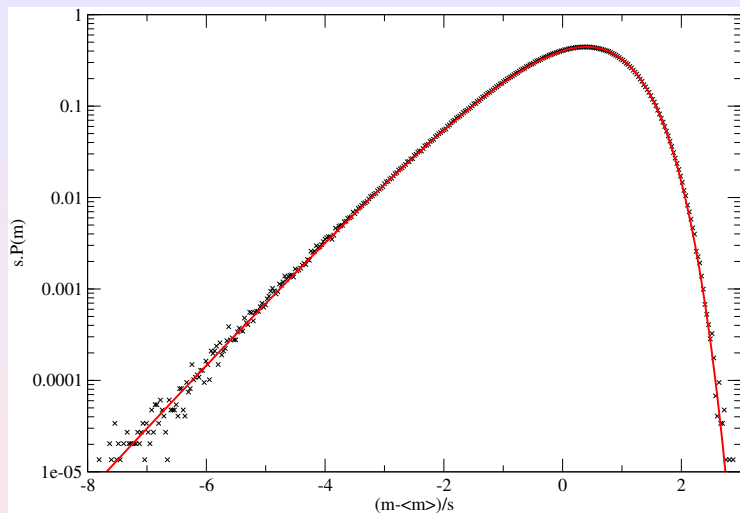


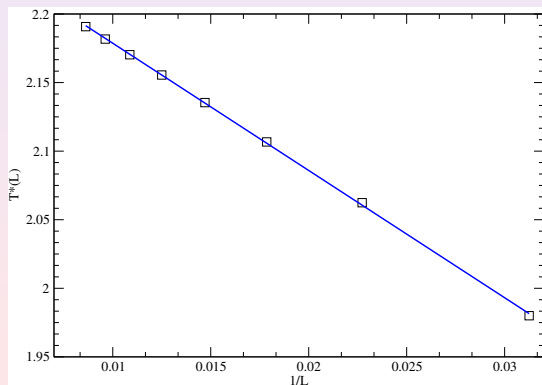
FIG.: Same as before for  $B^*(L = 32) = 0.0035J$ , best fit

- Scaling of the correlation length at  $T^*(L)$  :

$$\xi(L)/L \simeq 0.03$$

ratio quasi-constant for different system sizes.  $\Rightarrow$  small  $\xi(L)$  but diverging with  $L$ , critical behavior.

- $T_c - T^*(L) \simeq 1/L^{1/\nu}$  with  $\nu \simeq 1$ .



# Intermittency and critical fluctuations

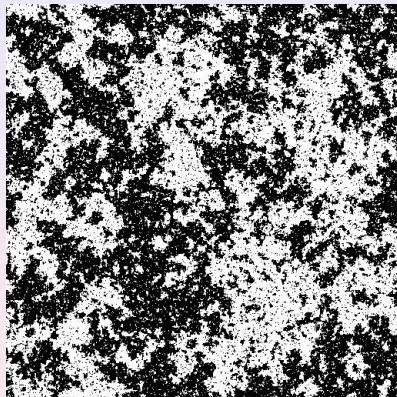


FIG.: Typical configuration at  $T_c(L)$

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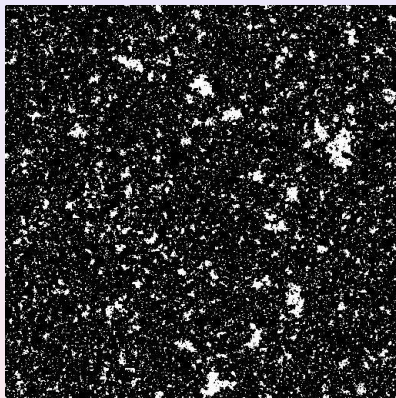


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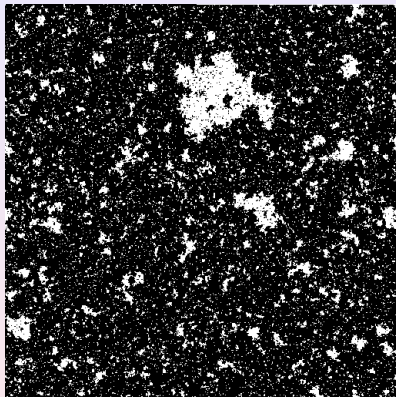


FIG.: Extreme configuration at  $T^*(L)$

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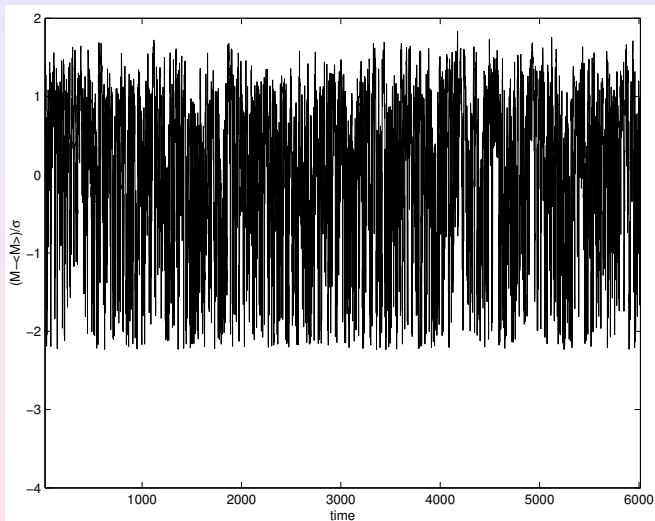


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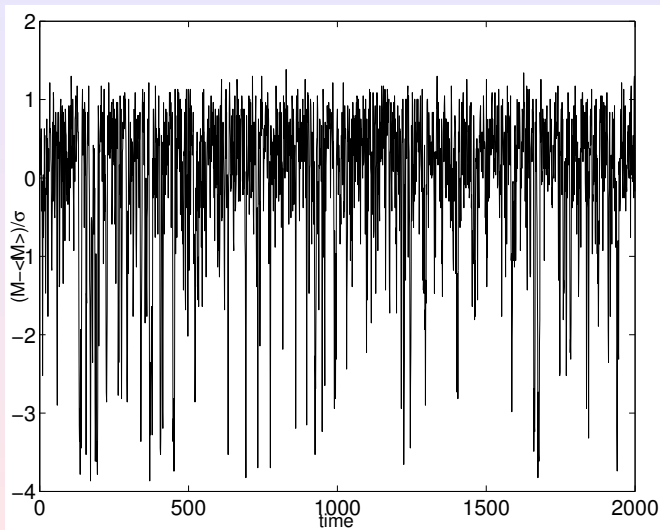


FIG.: Monte Carlo dynamics at  $T^*(L)$

- Magnetization fluctuations at  $T^*(L)$  not too strong on absolute value  $\Rightarrow$  confinement in one region of the phase space
- However fluctuations of  $\theta = (m - \langle m_{inst} \rangle) / \sigma$  stronger at  $T^*(L)$  than at  $T_c(L) \Rightarrow$  fluctuations of  $\theta$  seem unbounded from below rather than constrained
- Rare events occurs more frequently than at  $T_c(L)$  but not enough to modify  $\sigma$

# Some quantitative results

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2D Ising: discrete spin excitations
- How to relate both excitations in the low temperature regime ?
- We can use a Hubbard-Stratonovich transformation: discrete spins become continuous field variables

$$Z_{Ising} = \sum_{S_i} \exp \left( \frac{1}{2} \sum_{i,j} K_{ij} S_i S_j + \beta H \sum_i S_i \right)$$

becomes

$$Z_{\text{Ising}} \propto \int \mathcal{D}\phi \exp \left( -\frac{1}{2} \sum_{i,j} \phi_i K_{ij} \phi_j + \sum_i \log \cosh \left( \sum_j K_{ij} \phi_j + \beta H \right) \right) = \int \mathcal{D}\phi \exp \mathcal{S}[\phi_i]$$

with local magnetization

$$S_i \Rightarrow m_i = \tanh \left( \sum_j K_{ij} \phi_j + \beta H \right)$$

Separation of average value  $\phi = \sum_i \phi_i / N$  and local fluctuations  $\theta_i = \phi_i - \phi$  + expansion for  $\theta_i$  small. We Fourier transform the action and we obtain

$$S = S_0(\phi) - \frac{1}{2} \sum_{\mathbf{q} \neq 0} G_{\mathbf{q}}^{-1}(\phi) \theta_{\mathbf{q}} \theta_{-\mathbf{q}}$$
$$G_{\mathbf{q}}^{-1}(\phi) \propto q^2 + M^2, \quad q \text{ small}$$

The instantaneous magnetization becomes

$$m_{inst} \propto 1 - \frac{1}{N} \sum_{\mathbf{q} \neq 0} s_{\mathbf{q}} \theta_{\mathbf{q}} \theta_{-\mathbf{q}}$$

with  $s_{\mathbf{q}} \sim \cos q_x + \cos q_y$ .

$\Rightarrow$  excitations are like spin wave excitations in 2D-XY.

Expansion not valid near  $T_c(L)$  but at  $T^*(L)$  we expect that this is correct since  $\xi(L)/L$  is small.

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### Perspectives:


- Link with extreme statistics ?[Dahlstedt and Jensen, J Phys A (2001)]

- Fluctuations of global order parameters in correlated systems are not gaussian in general, several systems show similar patterns [PRL **84** 3744 (2000)]: forest fire models, Sneppen model or avalanche models, etc..
- One solvable case at least: 2D-XY model. On the critical line the PDF can be computed exactly and is of universal form
- Numerical study only for 2D Ising model but it shows intermittency like in turbulence experiments: developpement of large coherent structure at  $T^*(L)$  or  $B^*(L)$

### Perspectives:

- Link with extreme statistics ? [Dahlstedt and Jensen, J Phys A (2001)]
- More quantitative study on 2D Ising model  $\Rightarrow$  more precise definition of  $T^*(L)$  or  $B^*(L)$  ? 3D ?

# For further reading

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