Langevin equations with multiplicative noise

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Introduction

In many fields, stochastic (partial) differential equations with multiplicative noise arise

$$\partial_t \rho(\mathbf{r},t) = D\,\nabla^2 \rho + P(\rho,\rho^2,\ldots) + \sigma \rho^{\alpha}\, \eta(\mathbf{r},t)$$

where ρ is a positive density field, $P(\rho,\rho^2,\ldots)$ is a polynomial, η is a delta-correlated (Gaussian) noise.

- $\alpha = 0$: usual, additive noise
- $\alpha = 1$: multiplicative linear noise
- $\alpha = \frac{1}{2}$: multiplicative square-root noise (Poissonian local fluctuations)

Example:consider reaction-diffusion process where A particles diffuse, branch, and spontaneously disappear ($A \to 2A, 2A \to 0$).

One easily arrives, either phenomenologically or rigorously, to the following SDE for the (positive) density of particles:

$$\partial_t \rho(\mathbf{r}, t) = D\,\nabla^2 \rho + a\rho - b\rho^2 + \sigma\sqrt{\rho}\,\eta(\mathbf{r}, t)$$

where a, b terms correspond to branching and annihilation. Note that when $\rho = 0$ everywhere nothing happens: the empty state is *absorbing*.

This is the field theory for the archetypical absorbing phase transition, that of directed percolation. Hereafter called "DP equation".

Part I:

Efficient method for integrating multiplicative-noise SPDEs

Motivation

Why is it desirable to integrate such stochastic PDEs numerically?

- analytical methods (statistical field theory, renormalisation group approaches) are very difficult and sometimes impossible to implement on such Langevin equations, and often limited to the vicinity of some upper critical dimension

- their predictions are often contradicted by numerical results obtained on microscopic models

- ever-increasing computer power makes integration feasible

The example of square-root noise equations

Why it is difficult:

Naïve Euler schemes unavoidably run into severe problems:

For the zero-dimensional problem, we have:

$$\rho(t+\Delta t) = \rho(t) + \Delta t[a\rho(t) - b\rho^2(t)] + \sigma\sqrt{\Delta t \rho(t)}\Delta W_t$$

where $\Delta W_t = N(0,1)$ is the Gaussian increment of the Wiener process.

Ineluctably, $\rho(t+\Delta t) < 0$ for any finite Δt . . .

Other methods were proposed in the past:

- Dickman (1994)

  Discretize *also* the density: $\rho = n.\rho_{\text{min}}$, $n = 0, 1, 2, \ldots$, with $\rho_{\text{min}} \propto \Delta t$ (plus a truncate of the Gaussian noise such that $\rho(t + \Delta t) > 0$)

- Balanced implicit method (Schurz 1996)

  Implicit Euler scheme with suitably chosen weights:

$$\rho(t + \Delta t) = \frac{\rho(t) + \sigma\sqrt{\Delta t \rho(t)}(\Delta W_t + |\Delta W_t|)}{1 + \sigma\sqrt{\Delta t/\rho(t)}|\Delta W_t|}$$

... also requires a cut-off when $\rho(t) < \epsilon$

These schemes converge to a continuous limit when $\Delta t \to 0$ but:

1. they are *not* simulating a continuous field (especially since we will often be interested in the $\rho \to 0$ limit)

2. indeed they show similar transients/crossovers as the underlying microscopic models (see below)

What is the meaning of a (Itô sense) stochastic differential equation of the type:

$$dX_t = f(X_t)dt + \sigma g(X_t)dW_t \qquad ?$$

↪ for infinitesimal dt, the increment dX_t is locally Gaussian,of mean $f(X_t)dt$ and variance $\sigma^2 g^2(X_t)dt$:

$$\text{Prob.}\{X_{t+dt} = x | X_t = x_0\}dx = \frac{\exp\left[-\frac{(x-x_0-f(x_0)dt)^2}{2\sigma^2 g^2(x_0)dt}\right]dx}{\sqrt{2\pi\sigma^2 g^2(x_0)dt}}$$

In simple explicit Euler : "$d \to \Delta$" :

$$\begin{aligned} X_{t+\Delta t} &\equiv N(x_0 + f(x_0)\Delta t, \sigma^2 g^2(x_0)\Delta t) \\ &\equiv x_0 + f(x_0)\Delta t + \sigma g(x_0)\sqrt{\Delta t}N(0,1) \end{aligned}$$

but for a singular noise such as $g(X) = \sqrt{X}$ the conditional pdf is not Gaussian near the absorbing barrier $X = 0$ whatever the finite Δt . . .

Idea #1: Exact integration of stochastic degrees of freedom via the solution of the associated Fokker-Planck equation:

For the zero-dimensional equation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \sigma\sqrt{\rho(t)}\eta(t)\ ,$$

calculate the conditional probability

$$P(\rho, t) = \text{Proba.}\{\rho(t) = \rho | \rho(0) = \rho_0\}$$

which is the solution of the Fokker-Planck equation:

$$\partial_t P(\rho, t) = \frac{\sigma^2}{2}\partial_\rho^2[\rho P(\rho, t)]$$

We find explicitly:

$$P(\rho, t) = \delta(\rho) e^{-\frac{2\rho_0}{\sigma^2 t}} + \frac{2e^{-\frac{2(\rho_0+\rho)}{\sigma^2 t}}}{\sigma^2 t}\sqrt{\frac{\rho_0}{\rho}} I_1\left(\frac{4\sqrt{\rho_0\rho}}{\sigma^2 t}\right)$$

Note that, by construction, $\forall \rho_0, t, \sigma,\ \rho(t) \geq 0$

The SDE with an affine linear part

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \alpha + \beta\rho(t) + \sigma\sqrt{\rho(t)}\eta(t), \quad \alpha \geq 0$$

is also solvable:

$$P(\rho, t) = \lambda e^{-\lambda(\rho_0 e^{\beta t} + \rho)} \left[\frac{\rho}{\rho_0 e^{\beta t}}\right]^{\frac{\mu}{2}} I_\mu\left(2\lambda\sqrt{\rho_0 \rho e^{\beta t}}\right)$$

where

$$\lambda = \frac{2\beta}{\sigma^2(e^{\beta t} - 1)} \text{ and } \mu = -1 + \frac{2\alpha}{\sigma^2}$$

Note that each local equation is not “linear”...

Idea #2: Operator splitting

After discretizing the Laplacian, one first integrates stochastically at each site between t and $t + \Delta t$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \alpha + \beta\rho + \sigma\sqrt{\rho}\eta$$

with

$$\begin{aligned}
\alpha &= \alpha(\mathbf{r}, t) = \frac{D}{(\Delta x)^2} \sum_{v=1}^{2d} \rho(\mathbf{r} + \mathbf{e}_v, t) \\
\beta &= a - \frac{2dD}{(\Delta x)^2}
\end{aligned}$$

This yields ρ^* which is then used as an initial condition for integrating what left of the deterministic part, i.e. for the DP equation

$$\partial_t \rho(\mathbf{r}, t) = -b\rho^2(\mathbf{r}, t)$$

Any usual scheme for ODEs can be used, but for our example, one has exactly:

$$\rho(\mathbf{r}, t + \Delta t) = \frac{\rho^*}{1 + \rho^* b \Delta t}$$

Some remarks:

- the positivity of $\rho(\mathbf{r}, t)$ is ensured $\forall \mathbf{r}, t$ (via that of $\alpha(\mathbf{r}, t)$) if true initially
- the only constraint on the timestep arises from the integration of the deterministic part (here $\Delta t < \tau_D = \frac{(\Delta x)^2}{d.D}$ for the linear stability of the Laplacian)
- there is no delta-peak in the p.d.f. with $\alpha, \beta \neq 0$ but one can show that the distribution of the absorbing time τ_{abs} behaves at the origin like $\tau_{\text{abs}}^{\mu-1}$, $\Rightarrow \langle \tau_{\text{abs}} \rangle < \infty$ for $\mu < 0$, i.e. in the low-density regions.

But how can one generate random numbers according to this awesome-looking Bessel-functions p.d.f.?

Idea #3: Recognize a mixture of standard laws:

Developing the Bessel function in series:

$$P(\rho,t) = \sum_{n=0}^{\infty} \frac{\left(\lambda\rho_0 e^{\beta t}\right)^n e^{-\lambda\rho_0 e^{\beta t}}}{n!} \frac{\lambda e^{-\lambda\rho}(\lambda\rho)^{n+\mu}}{\Gamma(n+\mu+1)}$$

We obtain in distribution:

$$\rho^* \equiv \mathrm{Gamma}[\mu + 1 + \mathrm{Poisson}[\lambda\rho_0 e^{\beta t}]]/\lambda$$

where

$$\begin{aligned} \mathrm{Prob.}\{\mathrm{Poisson}[\lambda\rho_0 e^{\beta t}] &= n\} = \frac{\left(\lambda\rho_0 e^{\beta t}\right)^n e^{-\lambda\rho_0 e^{\beta t}}}{n!} \\ \mathrm{Prob.}\{\mathrm{Gamma}[\omega] &= v\} = \frac{e^{-v} v^{\omega-1}}{\Gamma(\omega)} \end{aligned}$$

↪ “Numerically exact” procedure

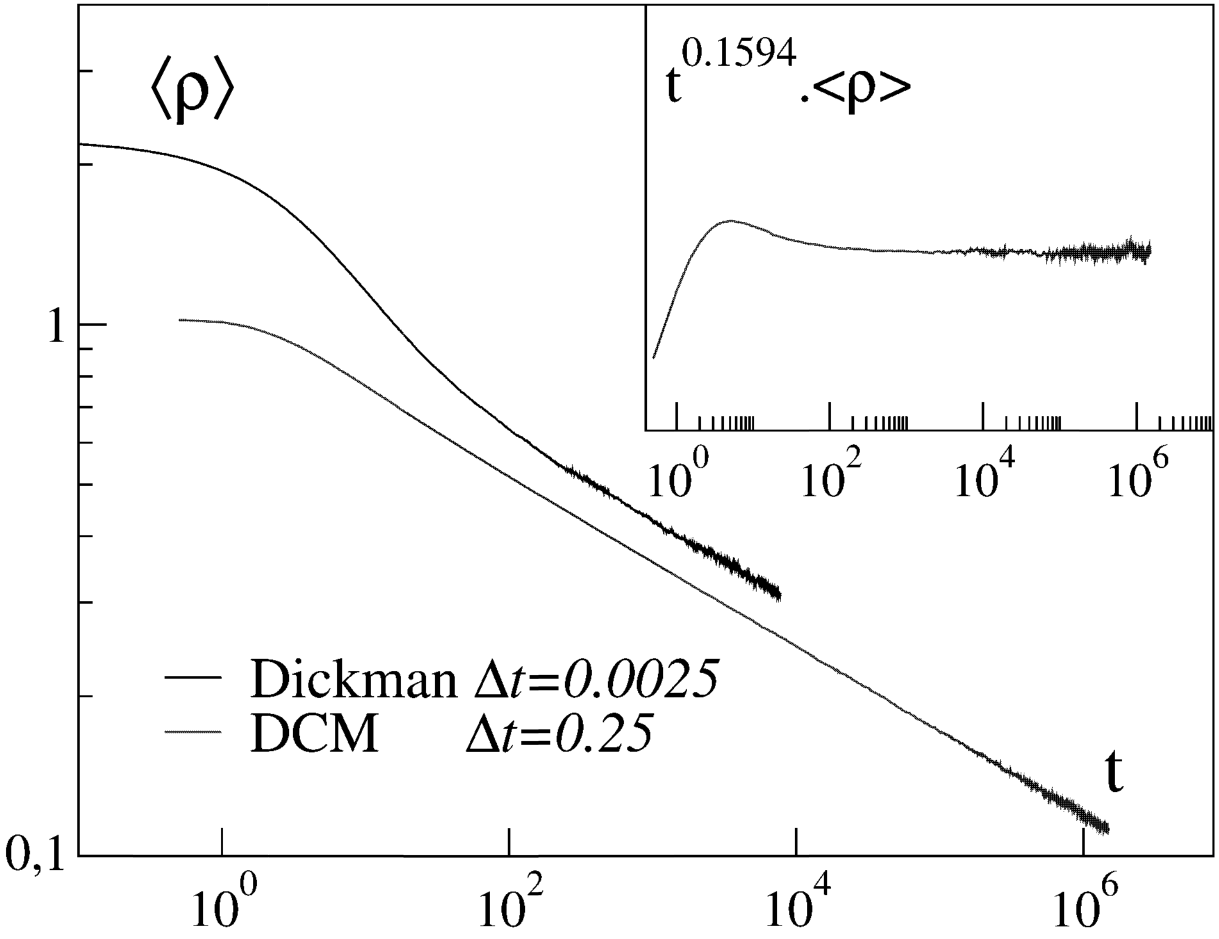
Typical results: one-dimensional DP equation

$\Delta t = 0.25$ possible, with a quality of scaling comparable to that of the best microscopic models; fast convergence of threshold when $\Delta t \to 0$; for 0.25 already within 1% of continuous limit.

Example: decay of activity during critical quench

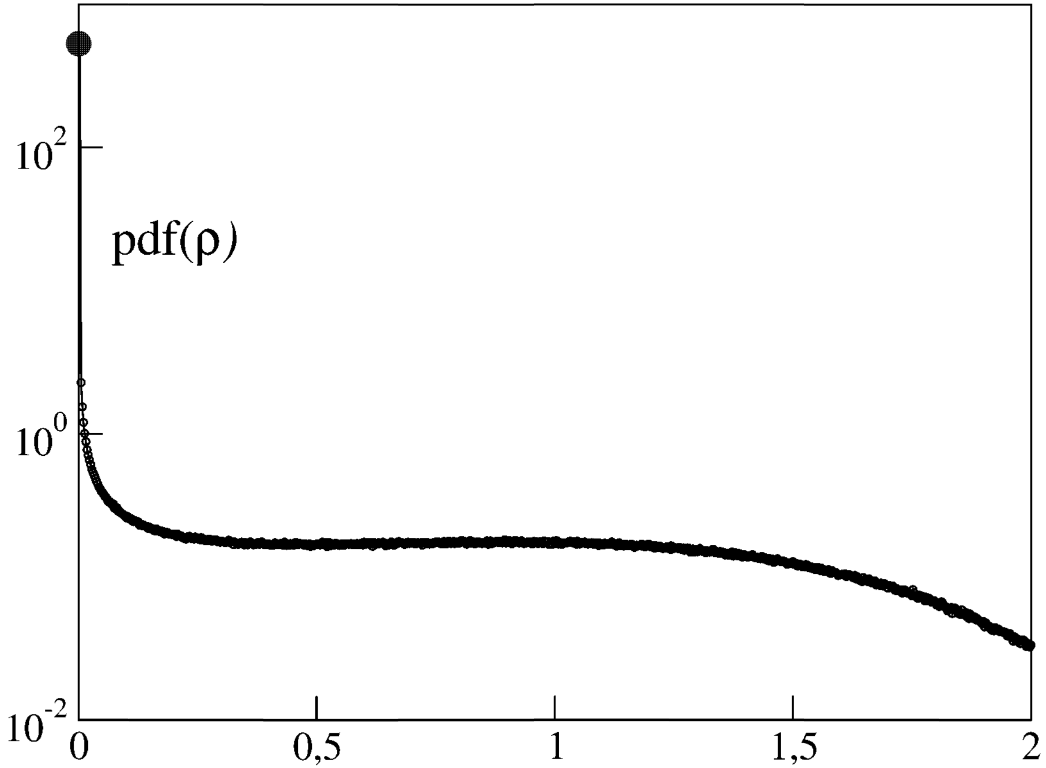
$$\langle \rho(\mathbf{r}, t) \rangle_{\mathbf{r}} \propto t^{-\theta}, \quad \theta = \beta / \nu_{||}$$

with $\theta = 0.1594(3)$ vs. 0.15946 via series expansion for bond DP to order 171 (Jensen)



2^{22} sites, $a = a_c = 1.75623(2)$, $D/(\Delta x)^2 = 0.25$, $\sigma^2 = 2$, $b = 1$.

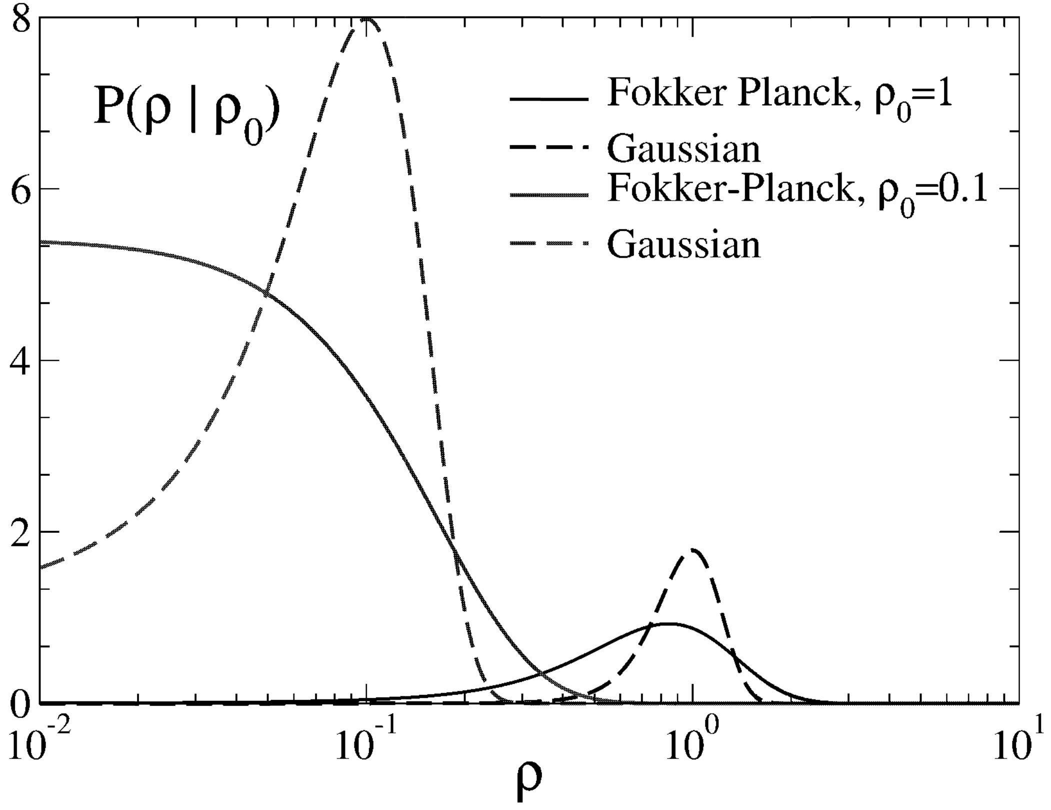
Deep reason for efficiency and quality of results: the scheme accounts well for the non-Gaussian character of the (unconditional) probability distribution of the order parameter



This is very similar to the zero-dimensional result:

$$p(\rho) = \int \mathrm{d}\rho_0 P(\rho, t|\rho_0, t-\Delta t) = \sum_{n=0}^{\infty} \frac{\lambda e^{-\lambda\rho}(\lambda\rho)^{n+\mu}}{\Gamma(n+\mu+1)}$$

Look also at the modification of the local conditional distribution under the presence of the absorbing barrier:



↪ clearly, drawing Gaussian random numbers (Dickman and BIM methods) is *not* a good idea, even with small Δt . . .

The case of linear multiplicative noise

The linear stochastic differential equation (Itô sense)

$$dX_t = (\alpha + \beta X_t)dt + \sigma X_t dW_t$$

can be integrated exactly:

$$\begin{aligned} X_t &= \exp\left[(\beta - \frac{1}{2}\sigma^2)t + \sigma W_t\right] . \left\{X_0 + \alpha \int_0^t \exp\left[(\beta - \frac{1}{2}\sigma^2)s + \sigma W_s\right] ds\right\} \\ &\quad + \int_0^t \exp\left[-(\beta - \frac{1}{2}\sigma^2)s - \sigma W_s\right] ds \end{aligned}$$

Solution of the associated Fokker-Planck equation: explicit form for $\alpha = 0$ (geometric Brownian motion, or Black-Scholes process) but not with $\alpha \neq 0$!:

$$dX_t = \beta X_t dt + \sigma X_t dW_t$$

Switching to Stratonovich, then changing variable $Y_t = \ln X_t$, one obtains

$$dY = \left(\beta - \sigma^2/2\right) dt + \sigma dW$$

which, since it has constant coefficients, can be either interpreted in the Stratonovich or in the Ito sense. This is the Brownian motion with drift, with solution:

$$P(Y_{t+\Delta t} = t | Y_t = y_0) = N(y_0 + (\beta - \sigma^2/2)\Delta t, \sigma^2 \Delta t).$$

The distribution of $X_t = \exp Y_t$ is therefore lognormal, and can be easily sampled:

$$\begin{aligned} P(X_{t+\Delta t} = x | X_t = x_0) &= P(Y_{t+\Delta t} = y = \ln x | Y_t = y_0 = \ln x_0) |dy/dx| \\ &= \exp\left[N(\ln x_0 + (\beta - \sigma^2/2)\Delta t, \sigma^2 \Delta t)\right] \\ &= x_0 \exp\left[(\beta - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} N(0,1)\right] \end{aligned}$$

Note that to lowest order in Δt one recovers Euler's scheme.

Typical results: one-dimensional MN equation

The standard linear noise equation (for related physics, wait for the second talk)

$$\partial_t \rho(\mathbf{r},t) = D\,\nabla^2 \rho + a\rho - b\rho^2 + \sigma\,\rho\,\eta(\mathbf{r},t)$$

is easily integrated in two-steps (operator splitting):

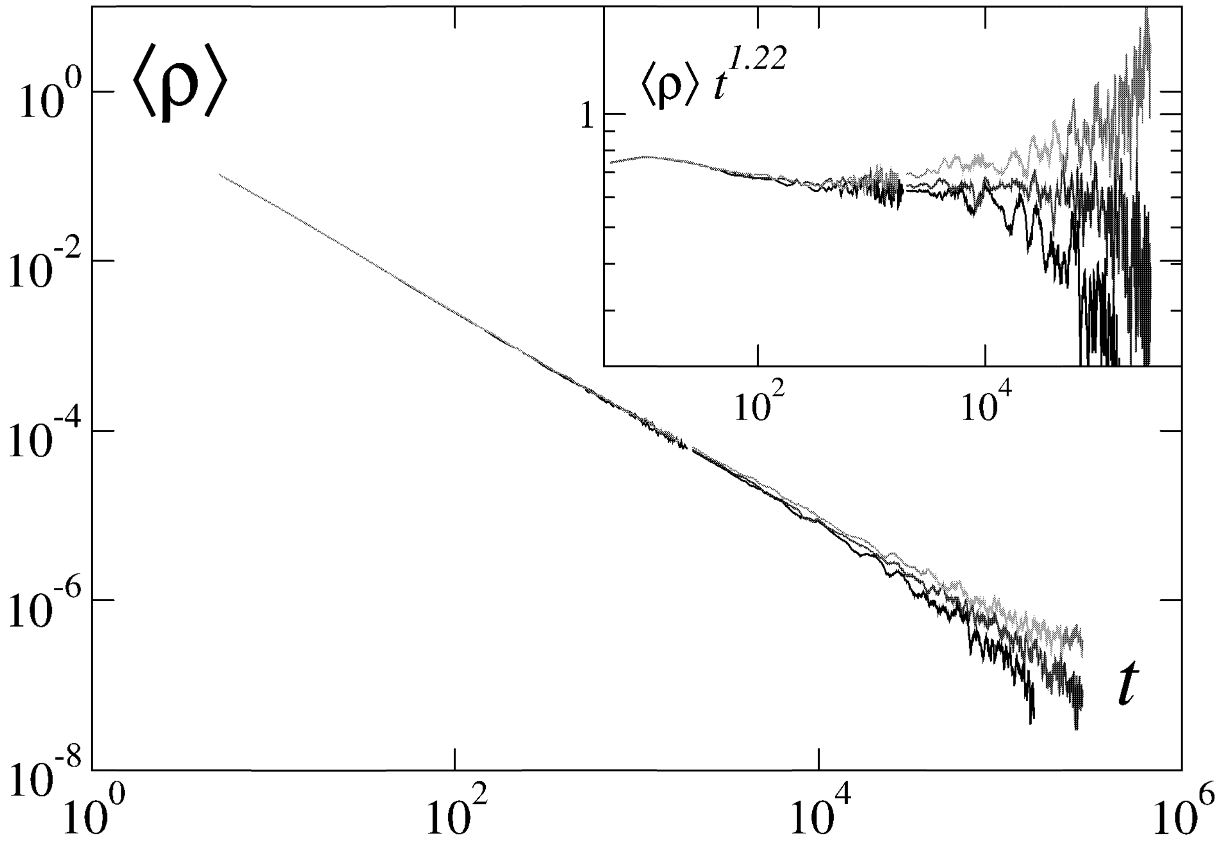
1. (local) stochastic sampling of

$$\partial_t \rho(\mathbf{r},t) = -\left(a + \frac{2dD}{(\Delta x)^2}\right)\rho(\mathbf{r},t) + \sigma.\rho(\mathbf{r},t).\eta$$

2. followed by the deterministic integration of

$$\partial_t \rho(\mathbf{r},t) = \frac{D}{(\Delta x)^2}\left(\sum_{v=1}^{2d} \rho(\mathbf{r}+\mathbf{e}_v,t)\right) - b\rho^2(\mathbf{r},t)$$

Typical results in $d = 1$ (critical decay):



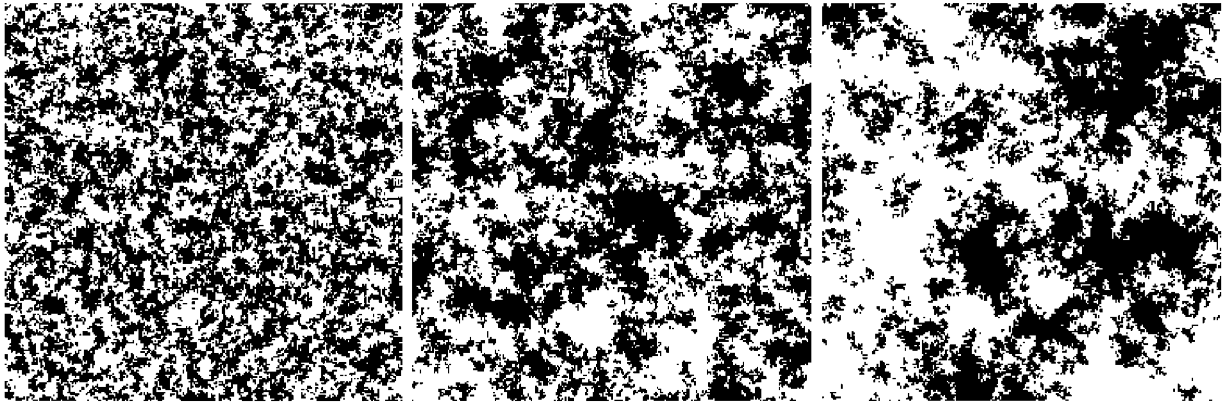
$\Delta t = 0.3$, 2^{22} sites, $a = a_c = 0.1946(1)$, $D/(\Delta x)^2 = 0.25$, $\sigma^2 = 2$, $b = 1$.

Two symmetric absorbing states: the Voter universality class

The Voter universality class: two symmetric absorbing states (only interfacial noise) and direct transition from disordered phase to one of the absorbing states.

Original discrete model: spins $\sigma = \pm 1$ evolving in continuous time: $\sigma(\mathbf{r}, t + \Delta t) = \sigma(\mathbf{r} + \mathbf{e}_v, t)$ with proba. $1/2d$

Critical domain growth without surface tension in $d = 2$:



$$\rho_I = 1 - \langle \sigma(\mathbf{r}, t) \sigma(\mathbf{r} + \mathbf{e}_v, t) \rangle \propto 1/\ln t$$

l theory proposed by Dickman and Tretyakov, for continuous $\rho \in [-1, 1]$ (never studied, nor simulated):

$$\partial_t \rho = D\,\nabla^2 \rho + \sigma \sqrt{1-\rho^2}\,\eta(\mathbf{r}, t)$$

ker-Planck equation for the associated zero-dimensional prob-

$$\partial_t P(\rho, t) = \frac{\sigma^2}{2} \partial_\rho^2 [(1-\rho^2).P(\rho, t)]$$

tion via an expansion in eigenmodes on the basis of Gegen-
r polynomials $G_n(\rho)$ of index 3/2 (orthogonal on $[-1, 1]$ for
weight $(1-\rho^2)$), *with in addition* two Dirac peaks at $\rho = \pm 1$

$$P(\rho, t) = f_c(\rho) + p.\delta(\rho - 1) + q.\delta(\rho + 1)$$