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KIAS

Spacetime Noncommutative Quantum Field Theory

- Question of Unitarity.

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1. Introduction

   Noncommutative QFT

2. Unitarity problem in ST NC QFT

3. Toy model
   - Nonrelativistic nonlinear Schrödinger model in 1+1 D

4. Outlook

hepth/0205193 (v3)

Noncommutativity

1. QM - uncertainty

$$[x, p] = i\hbar \qquad \Delta x \, \Delta p \geq \hbar/2$$

2. space noncommutativity

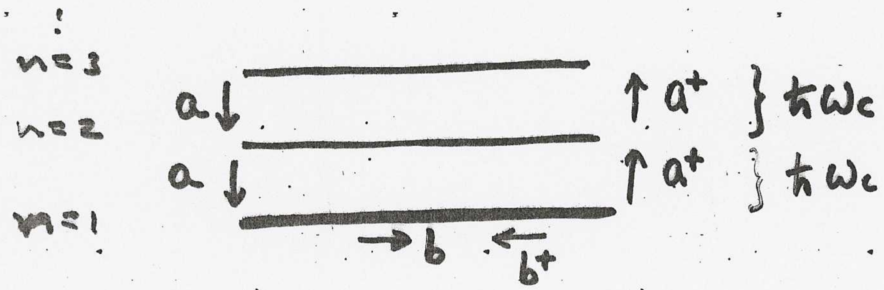
$$[x, y] = i\theta \qquad \Delta x \, \Delta y \geq \theta/2$$

- candidate of UV regularization

  Heisenberg. Snyder (1947)

- Lowest Landau Level

  Strong B-field



$$\omega_c = eB/mc \qquad \text{cyclotron frequency}$$

$$\ell^2 = \hbar c/eB \qquad \text{magnetic length}$$

2D oscillator

$$a = \sqrt{2}(\bar{\partial} + z/4) \qquad a^+ = \sqrt{2}(-\partial + \bar{z}/4)$$

$$b = \sqrt{2}(\partial + \bar{z}/4) \qquad b^+ = \sqrt{2}(-\bar{\partial} + z/4)$$

- $[a, a^+] = 1$. $\quad [b, b^+] = 1$, $\quad [a, b] = [a^+, b] = 0$ ...

$$z = \sqrt{2}(a + b^+), \qquad \bar{z} = \sqrt{2}(a^+ + b)$$

- $[z, \bar{z}] = 0$

Lowest Landau Level projection

$$a \sim 0 \sim a^+$$

$$z \sim \sqrt{2}\, b^+ \qquad \bar{z} = \sqrt{2}\, b$$

- $[z, \bar{z}] = -2 \neq 0$

$\Rightarrow$ One may choose z and $\bar{z}$ (or x and y) as operator b^+ and b.

Or choose z and $\bar{z}$ as commuting one and instead introduce $*$-product.

$*$-product

Weyl's idea (1936)

$$\langle \xi | \hat{O}_f | \xi \rangle = f(\xi, \bar{\xi})$$

$$\langle \xi | \hat{O}_f \hat{O}_g | \xi \rangle = \langle \xi | \hat{O}_{f*g} | \xi \rangle$$

$$b | \xi \rangle = \bar{\xi} | \xi \rangle \qquad \langle \xi | b^{+} = \langle \xi | \xi$$

$$\hat{O}_f (b, b^{+}) = \int \frac{d^2 p}{(2\pi)^2} \tilde{f}(p, \bar{p}) :e^{-i(\bar{p} b^{+} + p b)}:$$

$$f(\xi, \bar{\xi}) = \langle \xi | \hat{O}_f | \xi \rangle$$

$$= \int \frac{d^2 p}{(2\pi)^2} \tilde{f}(p, \bar{p}) \, e^{-i(\bar{p}\xi + p\bar{\xi})}$$

$$f * g \, (\xi, \bar{\xi}) = e^{\partial_{\bar{\xi}} \partial_{\xi'}} f(\xi \bar{\xi}) \, g(\xi' \bar{\xi}') \Big|_{\xi = \xi'}$$

$$\langle \xi | \hat{O}_f \cdot \hat{O}_g | \xi \rangle = \langle \xi | \hat{O}_{f*g} | \xi \rangle$$

$$= f * g \, (\xi, \bar{\xi})$$

- $\xi = \frac{z}{\sqrt{2}}$, $\bar{\xi} = \frac{\bar{z}}{\sqrt{2}}$ commuting variable
- function product is modified

Moyal product (1949)

$$\langle \xi | \hat{O}_f | \xi \rangle = f(\xi, \bar{\xi})$$

$$\langle \xi | O_f \, O_g | \xi \rangle = \langle \xi | O_{f*g} | \xi \rangle$$

$$O_f(b\, b^+) = \int \frac{d^2p}{(2\pi)^2} \, \tilde{f}(p, \bar{p}) \, e^{-i(\bar{p} b^+ + p b)}$$

$$f*g(\xi\bar{\xi}) = e^{(\partial_{\bar{\xi}} \partial_{\xi'} - \partial_\xi \partial_{\bar{\xi}'})} f(\xi\bar{\xi})\, g(\xi', \bar{\xi}')\Big|_{\xi=\xi'}$$

(* product depends on the ordering of the operator)

$$[x^i, x^j] = i\,\theta^{ij}$$

$$\Delta x^i \, \Delta x^j \geq |\theta^{ij}|/2$$

$$\phi_1 * \phi_2(\vec{x}) = e^{\frac{i}{2}\theta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}} \phi_1(\vec{x})\, \phi_2(\vec{y})\Big|_{\vec{y}=\vec{x}}$$

Quantum Field Theory

- $\psi(x) \xrightarrow[\text{NC}]{} \psi(x) \longrightarrow \psi(x)$

  NC — Quantum Field operator

- Moyal product

$$\int d^n x \; \psi(x) * \psi(x) = \int d^n x \; \psi(x)^2$$

Kinetic term is the usual action

- free Quantum Field operator exists in the far past.

$$[\psi_{in}(x), \psi_{in}(0)] = i\Delta(x)$$

Space-time NC QFT

$$[t, x] \neq 0 \quad \Rightarrow \quad \theta^{01} = \theta \varepsilon^{01} \neq 0$$

$$L_I(t) = -\frac{g}{p!} \int d^{D-1}x \; \frac{1}{2} \left(\phi * \phi * \cdots * \phi + h.c. \right)$$

- Serious doubt against Space time NCQFT

unitarity problem

2. Unitarity problem raised in spacetime noncommutative theory

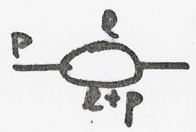
spacetime NC $\boxed{[t, x] = i\theta}$

Gomis & Mehen
NPB 591 (2000) 265

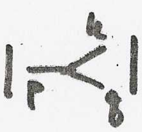
unitarity $\quad S \equiv 1 + iM$

$$SS^\dagger = 1 \quad \Rightarrow \quad 2\,\mathrm{Im}\,M = MM^\dagger$$

$2\,\mathrm{Im}\Big[$  $\Big] \overset{?}{=} \Big|$  $\Big|^2$

- $iM =$ 

$= \frac{g^2}{2}\int \frac{d^D l}{(2\pi)^D} \frac{1+\cos(p\wedge l)}{2} \frac{1}{l^2-m^2+i\varepsilon} \frac{1}{(l+p)^2-m^2+i\varepsilon}$

- $MM^\dagger = \Big|$ 

$\Big|^2 = \frac{g^2}{2}\frac{1}{(2\pi)^{D-2}}\iint \frac{d^{D-1}k}{2k^0}\frac{d^{D-1}q}{2q^0}\,\delta^D(p-k-q)\,\frac{1+\cos p\wedge k}{2}$

- $p^2 > 0$, $(p\theta)^2 = (\theta_{01})^2(p_0^2 - p_1^2) + (\theta_{23})^2(p_2^2 + p_3^2) + \cdots > 0$

$$iM = MM^\dagger$$

- $p\circ p = (p\theta)^2 < 0 \qquad \theta_{01} \neq 0$ (spacetime NC)

$$iM \neq MM^\dagger = 0$$

$$iM = \frac{\lambda^2}{2}\int\frac{d^D l}{(2\pi)^D}\,\frac{1+\cos p\wedge l}{2}\,\frac{1}{l^2-m^2+i\varepsilon}\,\frac{1}{(l+p)^2-m^2+i\varepsilon}$$

$$M = \frac{\lambda^2}{8}\int\frac{d^D l_E}{(2\pi)^D}\int_0^1 dx\int_0^\infty d\alpha\,\alpha\left[e^{-K(\alpha,\,x,\,l_E\,;\,P_E)}\right]$$

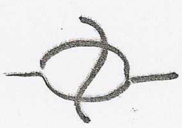
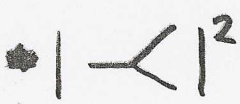
$$K(\alpha, x, l_E | P_E)$$
$$= \alpha\left(l_E^2 + x(1-x)P_E^2 + m^2 - i\varepsilon\right) - i\,l_E\wedge P_E \quad + \text{c.c.}$$

$$\boxed{\begin{array}{l} l^0 = i\,l_E^0, \qquad p^0 = i\,p_E^0 \\ \theta^{0i}\to -i\theta^{0i} \quad ; \quad l\wedge p \to l_E\wedge P_E \end{array}}$$

$$M = \frac{\lambda^2}{4}\,\frac{1}{(4\pi)^{D/2}}\int_0^1 dx\int_0^\infty d\alpha\;\alpha^{1-D/2}\,e^{\left[-\alpha\left(x(1-x)P_E^2+m^2-i\varepsilon\right)-\frac{P_E\circ P_E}{4\alpha}\right]}$$

$$P_E\circ P_E = (\wedge P_E)^2 = \begin{cases} (\theta^{12})^2(P_{E1}^2+P_{E2}^2)+\cdots & \text{space-space} \\ (\theta^{01})^2\underbrace{(-P_{E1}^2-P_{E0}^2)}_{P_0^2-P_1^2}+\cdots & \text{space-time} \end{cases}$$

$\mathrm{Im}\, M \neq 0$ but $MM^\dagger = 0$ for $P_0^2 < P_1^2$

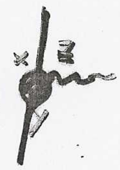
 $\neq$ 

Is the unitary really violated?

OR Is the defect of the formalism?

Bahns et. al PLB $\underline{533}$ (2002) 178

- Non local theory may have unitary problem

$f(x,y,z)$ 

Immamura, Sunakawa, Utiyama
Prog. Theor. Physics $\underline{11}$ (1954) 291

Marnelius, PRD $\underline{10}$ (1974) 3411

$$[\phi_{in}(x), \phi_{in}(0)] \neq [\phi_{out}(x), \phi_{out}(0)]$$

- Space-time NC
  - ~ non-local
  - ~ interaction picture cannot be used
    - time slice is not allowed
    - Evolution operator $U(t)$ is inconsistent

$$\phi_I(t) = U^\dagger(t)\, \phi_{in}(t)\, U(t)$$

$$\phi_I(t) \equiv U^{\dagger}(t)\ \phi_{in}(t)\ U(t)$$

$$\dot{\phi}_{in}(t) \equiv -i\left[L_0(\phi_{in}),\ \phi_{in}(t)\right]$$

$$\dot{\phi}_I(t) = -i\left[L(\phi_I),\ \phi_I(t)\right]$$

$$= \dot{U}^{+}\phi_{in}U + U^{+}\dot{\phi}_{in}U + U^{+}\phi_{in}U$$

$$\Rightarrow \quad \dot{U} = i\left\{\underbrace{U\,L(\phi_I)\,U^{+}}_{\tilde{L}(\phi_{in})\quad ?} - L_0(\phi_{in})\right\}U$$

$\boxed{\text{How to approach this problem?}}$

Use Heisenberg picture

Equation of motion

$$(\Box + m^2)\,\phi(x) = \frac{\delta}{\delta\phi(x)} \int dt\, L_I(t) \equiv \xi(\phi(x))$$

Solution

$$\phi(x) = \phi_{in}(x) + \int \Delta_{ret}(x-y)\,\xi(\phi(y))$$

$$= \phi_{out}(x) + \int \Delta_{ad}(x-y)\,\xi(\phi(y))$$

$$\Delta_{ret}(x) = -\theta(x^0)\,\Delta(x)$$

$$\Delta_{ad}(x) = \theta(-x^0)\,\Delta(x)$$

$$[\phi_{in}(x), \phi_{in}(0)] = i\Delta(x)$$

$$\Rightarrow \quad \phi_{out}(x) = \phi_{in}(x) - \int \Delta(x-y)\,\xi(\phi(y))$$

- $\boxed{\text{Need to check if } [\phi_{out}(x), \phi_{out}(0)] = i\Delta(x).}$

Use perturbation in coupling.

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$$

$$\phi_0 = \phi_{in}$$

$$\phi_1 = -\frac{g}{(p-1)!}\int \Delta_{ret}(x-y)\,(\underbrace{\phi_0 * \cdots * \phi_0}_{p-1}(y))$$

$$\phi_2 = -\frac{g}{(p-1)!}\int \Delta_{ret}(x-y)\Big(\phi_1 * \underbrace{\phi_0 * \cdots \phi_0}_{p-2} + \phi_0 * \phi_1 * \underbrace{\phi_0 \cdots \phi_0}_{p-3} + \cdots + \underbrace{\phi_0 * \cdots \phi_0}_{p-2} * \phi_1\Big)(y)$$

$$\vdots$$

$\Rightarrow$ $[\phi_{out}(x), \phi_{out}(0)] = i\Delta(x)$

Same commutation relation is confirmed.

$\Rightarrow$ $\phi_{out}(x) = S^{-1}\,\phi_{in}(x)\,S$

The task is to find S with $S^\dagger S = 1$.

$$\phi_{out}(x) = \phi_{in}(x) - \int d^p y \, \Delta(x-y) \, \xi(\phi(y))$$

$$= S^{-1} \phi_{in}(x) S$$

$$S = 1 + i \int_{-\infty}^{\infty} dt \; \mathcal{F}(V(\phi_{in}))$$

$$+ i^2 \iint_{-\infty}^{\infty} dt_1 dt_2 \; \mathcal{F}_{12}\left(\Theta_{12} V(\phi_{in}(t_1)) V(\phi_{in}(t_2)) \right)$$

$$+ \cdots$$

$$V(\phi_{in}) = -\frac{g}{p!} \int d^{D-1}x \, [\phi_{in}(x)]^p$$

$$\Theta_{12\cdots n} = \Theta(t_1 - t_2) \cdots \Theta(t_{n-1} - t_n)$$

$$\mathcal{F}_{1\dots n}(V(t_1) \cdots V(t_n)) = L_I(t_1) \cdots L_I(t_n)$$

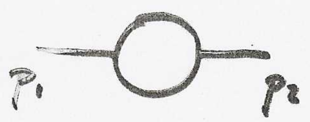
$$\mathcal{F}_{xy}\left(\Theta(x^0 - y^0) \phi^p(x) \phi^p(y) \right)$$

$$= \mathcal{F}_x \mathcal{F}_y \left(\underline{\Theta(x^0 - y^0)} \, \phi(x_1) \cdots \phi(x_p) \, \phi(y_1) \cdots \phi(y_p) \right)$$

$$\mathcal{F}_x = \cos \frac{1}{2} \left[\partial_{x_1} \wedge (\partial_{x_2} + \cdots + \partial_{x_p}) + \partial_{x_2} \wedge (\partial_{x_3} + \cdots + \partial_{x_p}) + \cdots \partial_{x_{p-1}} \wedge \partial_{x_p} \right]$$

$$\Theta(x^0 - y^0) \, \Delta(x_i - y_j) \longrightarrow \Theta(x_i^0 - y_j^0) \, \Delta(x_i - y_i)$$

ϕ^3 - theory



$$= \langle p_1 | S_2 | p_2 \rangle_c$$

$$= -\frac{1}{2} \iint d^D x \, d^D y \, \langle p_1 | T_*\big(V(\phi_{in}(t_1)) \, V(\phi_{in}(t_2)) \big) | p_2 \rangle_c$$

$$= -\left(\frac{g}{3!}\right)^2 \iint d^D x \, d^D y \, \langle p_1 | \mathcal{T}_{xy}\Big(\theta(x^0 - y^0) \, \phi_0^3(x) \, \phi_0^3(y) \Big) | p_2 \rangle_c$$

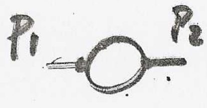
Evaluation

① $$\theta(t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \, \frac{e^{-i\omega t}}{\omega + i\varepsilon}$$

$$\Delta_+(x) = \langle 0 | \phi_{in}(x) \, \phi_{in}(0) | 0 \rangle$$

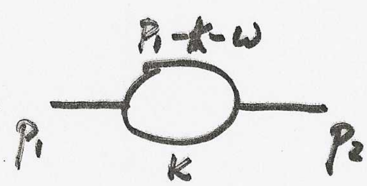
$$= \int \frac{d^D k}{(2\pi)^D} \, e^{-ikx} \, \underbrace{\tilde{\Delta}_+(k)}_{2\pi \delta(k^2 - m^2)\, \theta(k^0)}$$

$$\langle p | \phi_{in} | 0 \rangle = N e^{ipx}$$



$$= \frac{g^2}{2} (2\pi)^D \delta^D(p_1 - p_2) \iint \frac{d^D k \, d^D \ell \, d\omega}{(2\pi)^{2D} \, 2\pi i \, (\omega + i\varepsilon)}$$

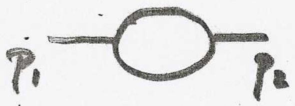
$$(2\pi)^D \delta^D(p_1 - k - \ell - \omega) \, |N|^2 \, \tilde{\Delta}_+(k) \, \tilde{\Delta}_+(\ell) \, \cos^2\!\left(\frac{p_1 \wedge \ell}{2}\right)$$

$=$ 

Evaluation 2

$$\Theta(x^0)\,\Delta_+(x) = \int \frac{d^Dk}{(2\pi)^D}\, e^{-ikx}\, \underbrace{\tilde{\Delta}_R(k)}_{\frac{i}{2\omega_k}\frac{1}{(k_0-\omega_k+i\varepsilon)}}$$

$$\omega_k \equiv \sqrt{m^2+\vec{k}^2}$$

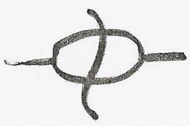


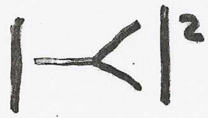
$$= \frac{g^2}{2}(2\pi)^D\,\delta^D(p_1-p_2)\int\frac{d^Dk}{(2\pi)^D}\,|N|^2$$

$$\tilde{\Delta}_R(k)\;\tilde{\Delta}_+(p-k)\;\cos^2\!\left(\frac{p_1\wedge k}{2}\right)$$

Unitarity check

$$\langle p_1 | S_2 + S_2^\dagger | p_2 \rangle_c + \langle p_1 | S_1 S_1^\dagger | p_2 \rangle_c = 0 \quad ?$$





$$\langle p_1 | S_2 + S_2^\dagger | p_2 \rangle = -(2\pi)^D \delta^D(p_1 - p_2)\, F_+(p_1)$$

$$F_+(p_1) = g^2 \int \frac{d^Dk}{(2\pi)^D} |N|^2 \tilde{\Delta}_+(k)\, \tilde{\Delta}_+(p_1 - k) \cos^2\left(\frac{p_1 \wedge k}{2}\right)$$

$$\cdot \quad \frac{1}{\omega + i\varepsilon} = P\left(\frac{1}{\omega}\right) - i\pi \delta(\omega)$$

$$\langle p_1 | S_1 S_1^\dagger | p_2 \rangle_c = \frac{g^2}{2} \iiint \frac{d^Dx\, d^Dy\, d^Dk\, d^Dl}{(2\pi)^{2D}} |N|^2 \tilde{\Delta}_+(k)\, \tilde{\Delta}_+(l)$$

$$e^{ix(p_1 - k - l) - iy(p_2 - k - l)} \cos^2\left(\frac{p_1 \wedge k}{2}\right) + p_1 \leftrightarrow p_2$$

$$= (2\pi)^D \delta^D(p_1 - p_2)\, F_+(p_1)$$

unitarity is OK.

What happened to the ordinary perturbation formalism?

Feynman Propagator

$$i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_-(x)$$

$$-\left(\Delta_F(x)\right)^2 = \theta(x^0)\left(\Delta_+(x)\right)^2 + \theta(-x^0)\left(\Delta_-(x)\right)^2$$

T^*-product 

$$-\Delta_F(x_1-y_1)\ \Delta_F(x_2-y_2)$$
$$\neq \theta(x_1^0-y_1^0)\,\Delta_+(x_1-y_1)\,\Delta_+(x_2-y_2)$$
$$+\,\theta(-x_1^0+y_1^0)\,\Delta_-(x_1-y_1)\,\Delta_-(x_2-y_2)$$

cross terms contributes after T^*-product is evaluated: Source of the trouble.

Toy Model Calculation

Non-relativistic nonlinear Schrödinger Model in 1+1 D.

$$L_I(t) = -\frac{v}{4}\int d\vec{x}\ \psi^+ * \psi^+ * \psi * \psi\ (t, \vec{x})$$

$$[\psi_{in}(\vec{x}, t),\ \psi^+_{in}(\vec{y}, t)] = \delta(\vec{x}-\vec{y})$$

$$\psi_{in}(x) = \int \frac{d^2k}{(2\pi)^2}\ \tilde{D}_+(k)\ a(\vec{k})\ e^{-ik\cdot x}$$

$$\psi^+_{in}(x) = \int \frac{d^2k}{(2\pi)^2}\ \tilde{D}_+(k)\ a^+(k)\ e^{ik\cdot x}$$

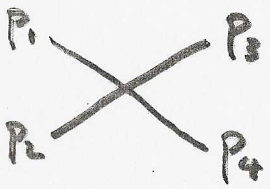
$$[a(\vec{k}),\ a(\vec{l})] = 2\pi\,\delta(\vec{k}-\vec{l})$$

$$\tilde{D}_+(k) = 2\pi\,\delta(k^0 - \vec{k}^2/2)$$

$$D_+(x) = \langle 0|\ \psi_{in}(x)\ \psi^+_{in}(0)\,|0\rangle = \int \frac{d^2p}{(2\pi)^2}\ e^{-ip\cdot x}\ \tilde{D}_+(p)$$

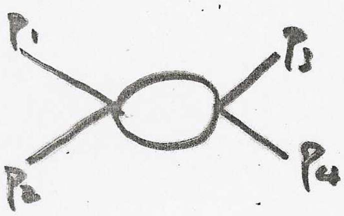
$$D_R(x) = \theta(x^0)\, D_+(x) = \int \frac{d^2p}{(2\pi)^2}\ e^{-ip\cdot x}\ \underbrace{\tilde{D}_R(p)}_{\dfrac{i}{p^0 - \frac{\vec{p}^2}{2} + i\epsilon}}$$

Four point Vertex



$$= -iv\,(2\pi)^2\,\delta^2(p_1+p_2-p_3-p_4)$$

$$\times \cos\left(\frac{p_1\wedge p_2}{2}\right)\,\cos\left(\frac{p_3\wedge p_4}{2}\right)$$



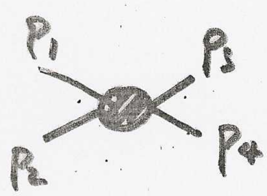
$$= -\frac{v^2}{2}(2\pi)^2\,\delta^2(p_1+p_2-p_3-p_4)\;\xi(p_1\,p_2)$$

$$\times \cos\left(\frac{p_1\wedge p_2}{2}\right)\,\cos\left(\frac{p_3\wedge p_4}{2}\right)$$

$$\xi(p_1\,p_2) = \int\frac{d^2 l}{(2\pi)^2}\,\tilde{D}_R(l)\,\tilde{D}_+(p-l)\,\cos^2\left(\frac{l\wedge p}{2}\right)$$

$$\Rightarrow \frac{1}{|\vec{p}_1-\vec{p}_2|}\cos\left(\frac{\theta|\vec{p}_1|\,|\vec{p}_2|\,|\vec{p}_1-\vec{p}_2|}{4}\right)\,e^{\frac{i\theta|\vec{p}_1||\vec{p}_2||\vec{p}_1-\vec{p}_2|}{4}}$$

when p_1, p_2 on-shell.



$$= (2\pi)^2\,\delta^2(p_1+p_2-p_3-p_4)\,\cos\frac{p_1\wedge p_2}{2}\,\cos\frac{p_3\wedge p_4}{2}$$

$$\times\;\frac{-iv}{1+\frac{iv}{2}\xi(p_1,p_2)}$$

<u>On-shell S-matrix</u>

$$\langle p_3\,p_4|\,S\,|p_1\,p_2\rangle_c = \Big(\delta(\vec{p}_1-\vec{p}_3)\,\delta(\vec{p}_2-\vec{p}_4)+\delta(\vec{p}_1-\vec{p}_4)\,\delta(\vec{p}_2-\vec{p}_4)\Big)\,S_{(2,2)}$$

$$S_{(2,2)} = 1+\left(\frac{\xi(p_1\,p_2)+\xi^*(p_1\,p_2)}{2}\right)\frac{-iv}{1+i\frac{v}{2}\xi(p_1 p_2)} = \frac{1-\frac{iv}{2}\xi^*(p_1\,p_2)}{1+\frac{iv}{2}\xi(p_1\,p_2)}$$

Outlook

1 Space-time NC QFT can be unitary.

But Interaction picture is hard to realize during the <u>finite time</u>.

Also plagued with causality problem

What will be the right quantity to ask?

2. Is the ST NC QFT realized in nature?