Self-duality, helicity and higher-loop QED

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Euler-Heisenberg effective action in a self-dual background is remarkably simple at two-loop

- applications : helicity amplitudes

  β-functions

  non-perturbative effects

[collaboration with: C. Schubert, H. Gies]

hep-th/0111134 PLB
hep-th/0205004 JHEP
hep-th/0205005 JHEP
hep-th/0210240 JHEP

One-Loop Effective Action

$$S[A] \;=\; -\frac{i}{2}\log\det\left(\not{D}^2 + m^2\right)$$

- QED paradigm : Heisenberg and Euler (1936)

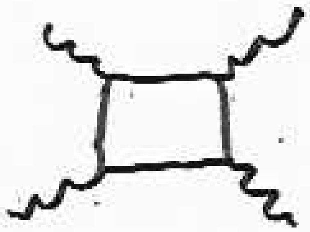
$$\mathcal{L}^{(1)} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}\,e^{-m^2 T}\left[\frac{e^2 ab}{\tanh(eaT)\tan(ebT)} - \frac{1}{T^2} - \frac{e^2}{3}(a^2 - b^2)\right.$$

- constant field strength : $F = \begin{pmatrix} & a & & \\ -a & & & \\ & & & b \\ & & -b & \end{pmatrix}$

$$a\,b = \vec{E}\cdot\vec{B} \qquad , \qquad a^2 - b^2 = \vec{B}^2 - \vec{E}^2$$

Applications : one-loop

- light-light scattering



$$S = \frac{e^4}{360\pi^2 m^4} \int d^4x \left[(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2 \right] + \dots$$

- paradigm of low-energy effective field theory

$$\mathcal{L}^{(1)} = -\hbar c \frac{2}{\pi^2} \left(\frac{mc}{\hbar}\right)^4 \sum_{n=0}^{\infty} \frac{2^{2n} \mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{eB\hbar}{m^2 c^3}\right)^{2n+4}$$

- pair production in constant E field

$$Im\, \mathcal{L}^{(1)} \sim \frac{e^2 E^2}{8\pi^3} \exp\left[-\frac{m^2\pi}{eE}\right]$$

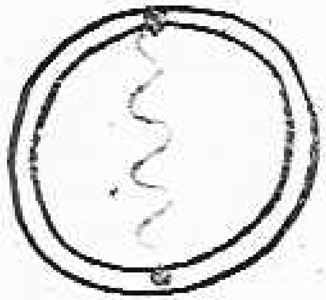
- β-function from strong -field limit

$$\mathcal{L}^{(1)} \sim \left(\frac{e^2}{12\pi^2}\right) \frac{B^2}{2} \ln\left(\frac{eB}{m^2}\right)$$

- beyond QED : QCD, gravity, strings, ...

Two-Loop Effective Action

$$S^{(2)} = -\frac{1}{2}e^2 \int d^4x d^4x' D_{\mu\nu}(x-x') \mathrm{tr}\left[\gamma_\mu G(x,x')\gamma_\nu G(x',x)\right]$$



- $G(x,x')$: propagator in constant background field (Fock, Nambu, Schwinger, ...)

$$G_{\mathrm{scalar}}(x,x') = -i\frac{e^{-i\eta}}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\exp\left[-im^2s - L(s) + \frac{i}{4}z\beta(s)z\right]$$

$$\begin{aligned} z_\mu &\equiv x_\mu - x'_\mu \\ \beta_{\mu\nu} &\equiv \left[eF\coth(eFs)\right]_{\mu\nu} \\ L &\equiv \frac{1}{2}\mathrm{tr}\,\ln\left(\frac{\sinh(eFs)}{eFs}\right) \end{aligned}$$

- V. Ritus (1975) : renormalized expression for $S^{(2)}$ in general constant background field

- new feature at two-loop : mass renormalization

Two-loop Effective Action : Ritus (1975)

$$\mathcal{L}^{(2)}_{\rm unren} = -\frac{\alpha}{32\pi^3}\int_0^\infty dT\int_0^\infty dT' e^{-m^2(T+T')}$$
$$\left\{\frac{e^4a^2b^2\left[4m^2(S\,S' + P\,P')I_0 - I\right]}{\sinh(eaT)\,\sinh(eaT')\,\sin(ebT)\,\sin(ebT')}\right.$$

- where :

$$S \equiv \cosh(eaT)\cos(ebT)$$
$$P \equiv \sinh(eaT)\sin(ebT)$$
$$I_0 \equiv \frac{\log(Q/R)}{(Q-R)}$$
$$I \equiv (U-V)\frac{\log(Q/R)}{(Q-R)^2} - \frac{U/Q - V/R}{Q-R}$$
$$Q \equiv eb\,(\cot(ebT) + \cot(ebT'))$$
$$R \equiv ea\,(\coth(eaT) + \coth(eaT'))$$
$$U \equiv \frac{2e^2a^2\cos(eb(T-T'))}{\sinh(eaT)\,\sinh(eaT')}$$
$$V \equiv \frac{2e^2b^2\cosh(ea(T-T'))}{\sin(ebT)\,\sin(ebT')}$$

- zero-field subtraction; charge and mass renormalization
- very complicated !!! applications difficult

Two-loop : magnetic background

$$\mathcal{L}^{(2)}_{\text{ren}} = \alpha \frac{m^4}{(4\pi)^3}\left(\frac{eB}{m2}\right)^2 \int_0^\infty \frac{ds}{s^3} e^{-m^2 s/(eB)} \int_0^1 du \, \Big[L(s,u) - 2s^2 + \frac{6}{u(1-u)}\left(\frac{s^2}{\sinh^2 s} + s\coth s\right)\Big]$$

$$-12\alpha \frac{m^4}{(4\pi)^3}\frac{eB}{m^2}\int_0^\infty \frac{ds}{s} e^{-m^2 s/(eB)} \left[\coth s - \frac{1}{s} - \frac{s}{3}\right] \times \left[\frac{3}{2} - \gamma - \log\left(\frac{m^2 s}{eB}\right) + \frac{eB}{m^2 s}\right]$$

$$L(s,u) = s\coth s \left[\frac{\log\left(\frac{u(1-u)}{G(u,s)}\right)}{[u(1-u)-G(u,s)]^2} F_1 + \frac{F_2}{G(u,s)[u(1-u)-G(u,s)]} + \frac{F_3}{u(1-u)[u(1-u)-G(u,s)]}\right]$$

$$G(u,s) = \frac{\cosh s - \cosh((1-2u)s)}{2s\sinh s}$$

$$F_1 = 4s(\coth s - \tanh s)G(u,s) - 4u(1-u)$$

$$F_2 = 2(1-2u)\frac{\sinh((1-2u)s)}{\sinh s} + s(8\tanh s - 4\coth s)G(u,s) - 2$$

$$F_3 = 4u(1-u) - 2(1-2u)\frac{\sinh((1-2u)s)}{\sinh s} - 4s\,G(u,s)\tanh s + 2$$

- still very complicated : applications still difficult

Two-loop : self-dual background (GD & C. Schubert 2002)

- self-dual background : $F_{\mu\nu} = \tilde{F}_{\mu\nu}$

$$\mathcal{L}^{(2)}_{\rm spinor} = -\alpha^2 \frac{f^2}{2\pi^2}\left[3\,\xi^2(\kappa) - \xi'(\kappa)\right]$$

$$\mathcal{L}^{(2)}_{\rm scalar} = \alpha^2 \frac{f^2}{4\pi^2}\left[\frac{3}{2}\,\xi^2(\kappa) - \xi'(\kappa)\right]$$

- dimensionless parameter

$$\kappa \equiv \frac{m^2}{2ef} \quad \text{where} \quad \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = f^2$$

- ubiquitous function

$$\xi(\kappa) \equiv = -\kappa\left(\psi(\kappa) - \ln(\kappa) + \frac{1}{2\kappa}\right)$$

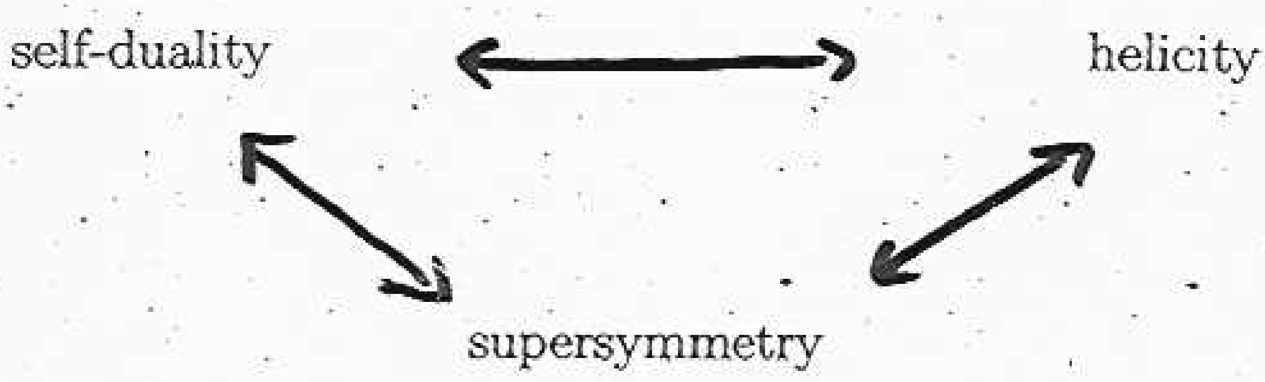
$$= -\kappa \int_0^\infty ds\, e^{-2\kappa s}\left(\coth s - \frac{1}{s}\right)$$

Two-loop : self-dual background

$$\mathcal{L}^{(2)}_{\rm spinor} = -2\alpha \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[3\,\xi^2(\kappa) - \xi'(\kappa)\right]$$

$$\mathcal{L}^{(2)}_{\rm scalar} = \alpha \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\,\xi^2(\kappa) - \xi'(\kappa)\right]$$

- why are the 2-loop answers so simple?
- why are the 2-loop spinor and scalar answers so similar?



Why so simple?

self-duality $\Rightarrow$ propagators simplify dramatically

self-duality $\Rightarrow$ definite **helicity** : $\sigma_{\mu\nu}F_{\mu\nu}\left(\frac{1+\gamma_5}{2}\right)=0$

- (massless) helicity amplitudes in QED, QCD, gravity known to be simple

- generalization to <u>massive</u> case

Why so similar?

self-duality $\Rightarrow$ Dirac operator has QM SUSY

- one-loop : $\mathcal{L}^{(1)}_{\rm spinor}=-2\mathcal{L}^{(1)}_{\rm scalar}+\frac{1}{2}\left(\frac{ef}{2\pi}\right)^2\ln\left(\frac{m^2}{\mu^2}\right)$

- two-loop : effective action not a determinant

- propagators related by helicity projections

$\Rightarrow$ $\mathcal{L}^{(2)}_{\rm spinor}$ and $\mathcal{L}^{(2)}_{\rm scalar}$ have just two terms, with different coefficients

Self-duality

- computational reason for simplicity

$$\text{self-duality} \Rightarrow F_{\mu\nu}F_{\nu\rho} = -f^2\delta_{\mu\rho}$$

- propagators simplify dramatically

$$G_{\text{scalar}}(x,x') = \left(\frac{ef}{4\pi}\right)^2 \int_0^\infty \frac{dt}{\sinh^2(eft)}\, e^{-m^2t-\frac{ef}{4}(x-x')^2\coth(eft)}$$

$$G_{\text{scalar}}(p) = \int_0^\infty \frac{dt}{\cosh^2(eft)}\, e^{-m^2t-\frac{p^2}{ef}\tanh(eft)}$$

- massless case :

$$G_{\text{scalar}}(x,x') = \frac{e^{-\frac{ef}{4}(x-x')^2}}{4\pi^2(x-x')^2}$$

$$G_{\text{scalar}}(p) = \frac{1-e^{-\frac{p^2}{ef}}}{p^2}$$

Self-duality, Helicity and SUSY

- why are spinor and scalar answers so similar?

$$\begin{aligned}
\Delta_B &= -D_\mu D_\mu + m^2 \\
\Delta_F &= -D_\mu D_\mu - \frac{ie}{2}\sigma_{\mu\nu}F_{\mu\nu} + m^2
\end{aligned}$$

- QM SUSY $\Rightarrow$ Δ_B and Δ_F have identical spectra

$$\Rightarrow \det \Delta_F = (\det \Delta_B)^4$$

- one-loop

$$\begin{aligned}
\mathcal{L}_{\text{spinor}} &= \frac{1}{2}\log\det\, \Delta_F \\
&= 2\,\log\det\, \Delta_B \\
&= -2\,\mathcal{L}_{\text{scalar}}
\end{aligned}$$

$$\mathcal{L}^{(1)SD}_{\text{spinor}} = -\frac{e^2 f^2}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(ef)}\left[\coth^2 s - \frac{1}{s^2} - \frac{2}{3}\right]$$

$$\mathcal{L}^{(1)SD}_{\text{scalar}} = \frac{e^2 f^2}{16\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(ef)}\left[\frac{1}{\sinh^2 s} - \frac{1}{s^2} + \frac{1}{3}\right]$$

Self-duality, Helicity and SUSY.

- SUSY of Dirac operator for a self-dual background :

$$i\gamma_\mu D_\mu = \begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix} \quad , \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$Q = i\alpha_\mu D_\mu \quad , \quad \alpha_\mu = (-i\vec{\sigma}, \mathbf{1})$$

$$Q^\dagger = i\bar{\alpha}_\mu D_\mu \quad , \quad \bar{\alpha}_\mu = (i\vec{\sigma}, \mathbf{1})$$

$$\gamma_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \bar{\alpha}_\mu & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

$$\sigma_{\mu\nu} = \frac{1}{4i}(\alpha_\mu\bar{\alpha}_\nu - \alpha_\nu\bar{\alpha}_\mu) \quad ; \quad \bar{\sigma}_{\mu\nu} = \frac{1}{4i}(\bar{\alpha}_\mu\alpha_\nu - \bar{\alpha}_\nu\alpha_\mu)$$

- spectrum : $i\gamma_\mu D_\mu \psi = \lambda\psi$

$$Q\psi_- = \lambda\psi_+ \quad , \quad Q^\dagger\psi_+ = \lambda\psi_-$$

- construct (non-zero-mode) solutions from

$$QQ^\dagger = D_\mu D_\mu + i\bar{\sigma}_{\mu\nu}F_{\mu\nu}$$

- self-duality :

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \quad \Rightarrow \quad \bar{\sigma}_{\mu\nu}F_{\mu\nu} = 0$$

- Δ_B and Δ_F have identical spectra, except for zero-modes, and spinor multiplicity 4

Self-duality, Helicity and SUSY

- two-loop : not a determinant $\Rightarrow \mathcal{L}^{(2)}_{\rm spinor} \neq -2\,\mathcal{L}^{(2)}_{\rm scalar}$

$$\mathcal{L}^{(2)}_{\rm spinor} = -2\alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[3\xi^2(\kappa)-\xi'(\kappa)\right]$$

$$\mathcal{L}^{(2)}_{\rm scalar} = \alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\xi^2(\kappa)-\xi'(\kappa)\right]$$

- spinor and scalar propagators related by SUSY :

$$S = -(\not{D}-m)\,G\left(\frac{1+\gamma_5}{2}\right) - G\not{D}\left(\frac{1-\gamma_5}{2}\right) + \frac{(1+\not{D}G\not{D})}{m}\left(\frac{1-\gamma_5}{2}\right)$$

- $\Rightarrow$ express $\mathcal{L}^{(2)}$ in terms of matrix elements of G

$$= \frac{e^2}{4\pi^2}\int\frac{d^4x'}{(x-x')^2}\left[A\langle x|D_\mu G|x'\rangle\langle x'|D_\mu G|x\rangle + B\langle x|G|x'\rangle\langle x'|D_\mu G D_\mu|x\rangle\right]$$

- spinor : $A=-4, B=8$ scalar : $A=-1, B=-1$

- renormalization (charge and mass) nontrivial
  world-line (GD & C. Schubert, 2002)
  proper-time cut-off (GD, H. Gies & C. Schubert, 2002)

Applications : non-perturbative effects

- pair-production rate

$$Im\,\mathcal{L}^{(1)} = \frac{m^4}{8\pi^3}\left(\frac{eE}{m^2}\right)^2 \sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\frac{m^2\pi k}{eE}\right]$$

- two-loop : Ritus-Lebedev (1984)

$$\mathrm{m}\left(\mathcal{L}^{(1)}+\mathcal{L}^{(2)}\right)=\frac{e^2E^2}{8\pi^3}\sum_{k=1}^{\infty}\left[\frac{1}{k^2}+\alpha\pi\left(-\frac{c_k}{\sqrt{\frac{eE}{m^2}}}+1+O\left(\sqrt{\frac{eE}{m^2}}\right)\right)\right]e^{-\frac{m^2\pi k}{eE}}$$

$$c_1 = 0 \qquad ; \qquad c_k = \frac{1}{2\sqrt{k}}\sum_{l=1}^{k-1}\frac{1}{\sqrt{l(k-l)}} \quad , \quad k \geq 2$$

- numerical Borel analysis (GD & C. Schubert, 2000)

$$\mathrm{Im}\left(\mathcal{L}^{(1)}+\mathcal{L}^{(2)}\right) \sim \frac{e^2E^2}{8\pi^3}\left[1+\alpha\pi\left(1-0.44\sqrt{\frac{eE}{m^2}}+\dots\right)\right]e^{-\frac{m^2\pi}{eE}}$$

Applications : non-perturbative effects

- two-loop self-dual case : $\kappa = \frac{m^2}{2ef}$

  real κ like B field case
  imaginary κ like E field case

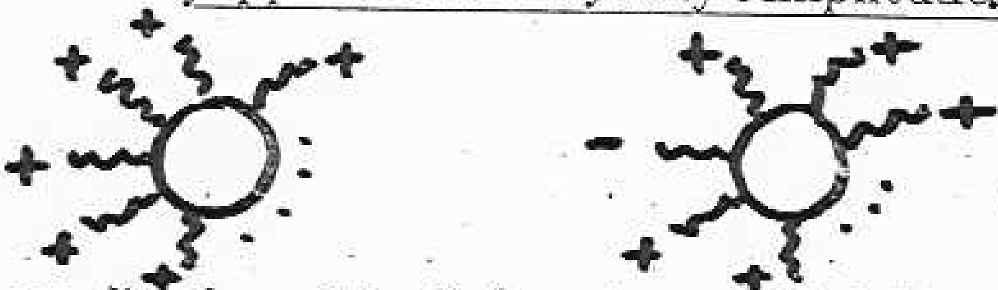
- exact expressions $\Rightarrow$

$$\mathrm{Im}\mathcal{L}^{(1)} = \frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\sum_{k=1}^{\infty}\left(\frac{2\pi\kappa}{k}+\frac{1}{k^2}\right)e^{-2\pi k\kappa}$$

$$\mathrm{Im}\mathcal{L}^{(2)} = \alpha\pi\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\sum_{k=1}^{\infty}\left(2\pi\kappa k-1-3\kappa^2\sum_{l=1}^{\infty}\frac{(-1)^l\mathcal{B}_{2l}}{2l\kappa^{2l}}\right)e^{-2\pi k\kappa}$$

- leading terms differ by factor of $\alpha\pi$
- all orders of Ritus-Lebedev prefactor : asymptotic series
- consistent with Borel analysis

Applications : Helicity Amplitudes



- amplitudes with all (or almost all) helicities alike are known to be especially simple
- tree level :

$$M[p_1+;p_2+;\ldots;p_n+] = 0$$
$$M[p_1-;p_2+;\ldots;p_n+] = 0$$

- one loop : massless QED (Mahlon, 1993)

$$M[p_1+;p_2+;\ldots;p_n+] = 0 \quad , \quad n \neq 4$$
$$M[p_1+;p_2+;p_3+;p_4+] = \frac{ie^4}{2\pi^2}\frac{\langle 12\rangle^*\langle 34\rangle^*}{\langle 12\rangle\langle 34\rangle}$$
$$M[p_1-;p_2+;\ldots;p_n+] = 0 \quad , \quad n \neq 4$$
$$M[p_1-;p_2+;p_3+;p_4+] = \frac{ie^4}{2\pi^2}\frac{\langle 12\rangle\langle 34\rangle^*\langle 24\rangle^*}{\langle 12\rangle^*\langle 34\rangle\langle 24\rangle}$$

- spinor-helicity basis : $\langle ij\rangle \equiv \bar{u}_-(p_i)u_+(p_j)$
- relation to SDYM and integrable models (Nair, Bardeen, Cangemi, ...)
- SUSY Ward identities
- recent results at 2-loop (Bern et al; Glover et al)

Applications : Helicity Amplitudes

- effective action: generating function for helicity amplitudes
- constant self-dual backgrounds
  $\Rightarrow$ low-energy limit of "all +" helicity amplitudes

$$\Gamma^{(1)}[k_1,\epsilon_1^+;k_2,\epsilon_2^+;\ldots;k_N,\epsilon_N^+] = \frac{(2e)^N}{(4\pi)2m^{2N-4}} c^{(1)}_{N/2}\, \chi_N$$

$$\Gamma^{(2)}[k_1,\epsilon_1^+;k_2,\epsilon_2^+;\ldots;k_N,\epsilon_N^+] = \alpha\pi\frac{(2e)^N}{(4\pi)2m^{2N-4}} c^{(2)}_{N/2}\, \chi_N$$

$$\chi_N \equiv \frac{(N/2)!}{2^{N/2}}\left\{[12]^2[34]^2\cdots[(N-1)N]^2 + \text{all perms}\right\}$$

$$c_n^{(1)} = -\frac{\mathcal{B}_{2n}}{2n(2n-2)}$$

$$c_n^{(2)} = \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\mathcal{B}_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\right\}$$

- $[+ + + \ldots +]$ amplitudes for $N \geq 6$ do not vanish in massive QED
- $[- + + \ldots +]$ amplitudes vanish in low-energy limit in massive QED
- generalization to all helicities (hep-th/0301022)

QED β-functions

- scale dependence of running coupling

$$\beta(a) = \frac{d\,a}{d\,\ln\mu^2} \quad , \qquad a \equiv \frac{\alpha}{4\pi}$$

- unquenched :

$$\beta_{MS}(a) = \frac{4}{3}a^2 + 4a^3 - \frac{62}{9}a^4 - \left[\frac{5570}{243} + \frac{832}{9}\zeta(3)\right]a^5 + \ldots$$

- quenched (F_1 function) :

$$\beta_Q(a) = \frac{4}{3}a^2 + 4a^3 - 2a^4 - 46a^5 + \ldots$$

- unsolved problems :

divergence ?

Borel summable? alternating?

zero(es) ?

finite QED? (Johnson, Baker, Willey, 1973)

rationality of quenched coefficients (Kreimer, Broadhurst, 1996 ...)

Applications : QED β-functions

- external field as a probe

- β-function related to strong field limit

$$\mathcal{L}^{(n)} \sim b_n \frac{F^2}{4} \ln\left(\frac{e|F|}{m^2}\right) \quad , \quad n = 1, 2$$

- β-function coefficients related to the b_n

$$\mathcal{L}^{(1)}_{\text{spinor}} \sim \frac{e^2 B^2}{24\pi^2} \ln\left(\frac{eB}{m^2}\right) \quad , \quad \mathcal{L}^{(1)}_{\text{scalar}} \sim \frac{e^2 B^2}{96\pi^2} \ln\left(\frac{eB}{m^2}\right)$$

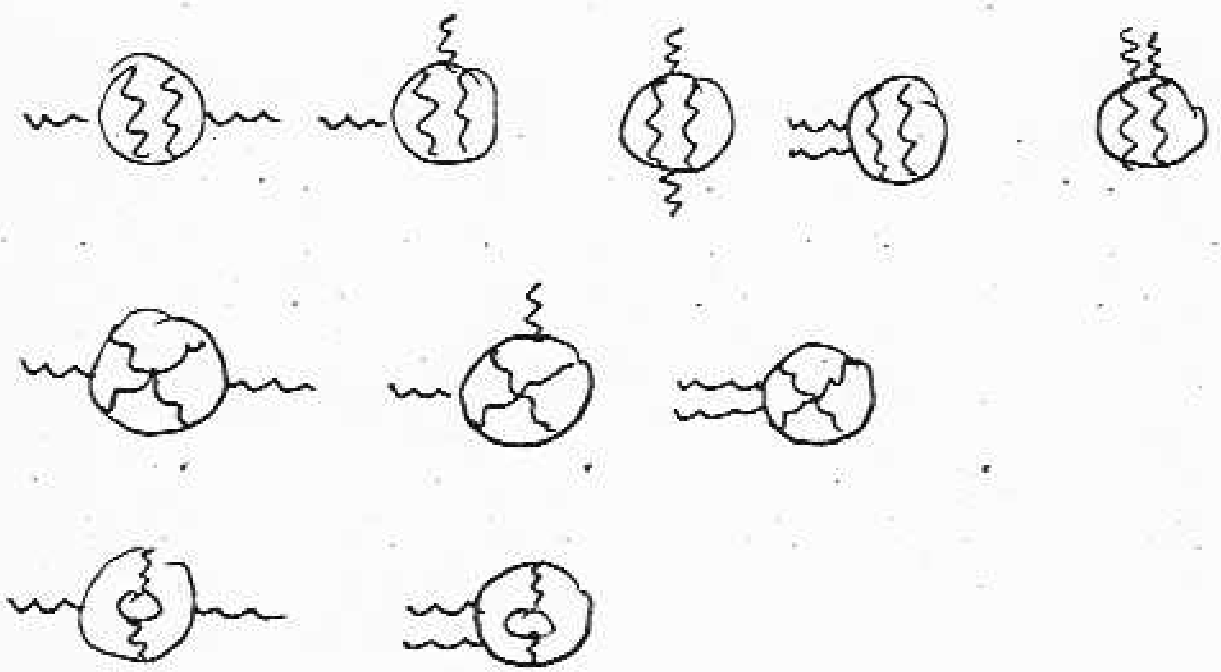
$$\mathcal{L}^{(2)}_{\text{spinor}} \sim \frac{e^4 B^2}{128\pi^4} \ln\left(\frac{eB}{m^2}\right) \quad , \quad \mathcal{L}^{(2)}_{\text{scalar}} \sim \frac{e^4 B^2}{128\pi^4} \ln\left(\frac{eB}{m^2}\right)$$

- compare with β-functions :

$$\beta_{\text{spinor}} = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + \ldots \quad , \quad \beta_{\text{scalar}} = \frac{e^3}{48\pi^2} + \frac{e^5}{64\pi^4} + \ldots$$

- efficient computational approach for higher loops ...

example: 3-loop β-function



background field:

Applications : QED β-functions

- self-dual background appears to be the "simplest"

$$\mathcal{L}^{(1)}_{\rm scalar} \sim \frac{e^2 f^2}{48\pi^2} \ln\left(\frac{ef}{m^2}\right) \quad , \quad \mathcal{L}^{(1)}_{\rm spinor} \sim -\frac{e^2 f^2}{24\pi^2} \ln\left(\frac{ef}{m^2}\right)$$

$$\mathcal{L}^{(2)}_{\rm scalar} \sim \frac{e^4 f^2}{64\pi^4} \ln\left(\frac{ef}{m^2}\right) \quad , \quad \mathcal{L}^{(2)}_{\rm spinor} \sim -\frac{e^4 f^2}{32\pi^4} \ln\left(\frac{ef}{m^2}\right)$$

- compare with β-functions :

$$\beta_{\rm scalar} = \frac{e^3}{48\pi^2} + \frac{e^5}{64\pi^4} + \ldots \quad , \quad \beta_{\rm spinor} = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + \ldots$$

- "works" for scalar QED, but not for spinor QED

- reason : zero modes

- one-loop hint ('t Hooft, 1978):

$$\mathcal{L}^{(1)}_{\rm spinor} = -2\mathcal{L}^{(1)}_{\rm scalar} + \frac{1}{2} N_0 \ln\left(\frac{m^2}{\mu^2}\right)$$

$$N_0 = \left(\frac{ef}{2\pi}\right)^2$$

Applications : QED β-functions

- general scale anomaly argument :

$$\langle \Theta^{\mu}_{\mu} \rangle = \frac{\beta(\bar{e})}{2\bar{e}} \frac{e^2}{\bar{e}^2} (F_{\mu\nu})^2$$

$$\langle \Theta^{\mu\nu} \rangle = -\eta^{\mu\nu} \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}}$$

- these two facts determine form of $\mathcal{L}$ and $\beta(\bar{e})$:

$$\mathcal{L} = -\frac{1}{4} \frac{e^2}{\bar{e}^2(t)} F_{\mu\nu} F^{\mu\nu}$$

$$\beta(\bar{e}(t)) \equiv \frac{d\bar{e}(t)}{dt} \quad , \quad t \equiv \frac{1}{4} \ln\left(\frac{e^2|F^2|}{\mu_0^4}\right)$$

$$t = \int_e^{\bar{e}(t)} \frac{de'}{\beta(e')} \quad , \quad \beta(e) = \beta_1 e^3 + \beta_2 e^5 + \ldots$$

$$\frac{1}{\bar{e}^2(t)} = \frac{1}{e^2} - 2\beta_1 t - 2\beta_2 e^2 t + O(e^4 t^2) \ .$$

$$\mathcal{L} = \frac{1}{16} \left(2\beta_1 e^2 + 2\beta_2 e^4 + \ldots\right) F_{\mu\nu} F^{\mu\nu} \ln\left(\frac{e^2|F^2|}{\mu_0^4}\right)$$

- BUT : assumes existence of well-defined massless limit

Applications : QED β-functions and IR/UV connection

- one-loop scalar QED

$$\mathcal{L}_{sc} = \frac{e^2 f^2}{16\pi^2}\int_{1/\mu^2}^{\infty}\frac{dt}{t}e^{-m^2 t}\left[\frac{1}{\sinh^2(eft)} - \frac{1}{(eft)^2} + \frac{1}{3}\right] - \underbrace{\frac{e^2 f^2}{48\pi^2}\int_{1/\mu^2}^{\infty}\frac{dt}{t}e^{-m^2 t}}_{+\frac{e^2 f^2}{48\pi^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) + \gamma\right]}$$

- massless limit :

$$\mathcal{L}_{sc} \sim \frac{e^2 f^2}{48\pi^2}\ln\left(\frac{ef}{m^2}\right) + \frac{e^2 f^2}{48\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) \sim \frac{e^2 f^2}{48\pi^2}\ln\left(\frac{ef}{\mu^2}\right)$$

- one-loop spinor QED

$$\mathcal{L}_{sp} = -\frac{e^2 f^2}{8\pi^2}\int_{1/\mu^2}^{\infty}\frac{dt}{t}e^{-m^2 t}\left[\coth^2(eft) - \frac{1}{(eft)^2} - \frac{2}{3}\right] - \underbrace{\frac{e^2 f^2}{12\pi^2}\int_{1/\mu^2}^{\infty}\frac{dt}{t}e^{-m^2 t}}_{+\frac{e^2 f^2}{12\pi^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) + \gamma\right]}$$

- massless limit :

$$\mathcal{L}_{sp} \sim -\frac{e^2 f^2}{24\pi^2}\ln\left(\frac{ef}{m^2}\right) + \frac{e^2 f^2}{12\pi^2}\ln\left(\frac{m^2}{\mu^2}\right)$$
$$\sim -2\mathcal{L}_{sc} + \frac{1}{2}\left(\frac{ef}{2\pi}\right)^2\ln\left(\frac{m^2}{\mu^2}\right)$$

- IR divergence of $\mathcal{L}_{unren}$ causes mis-match

Applications : Bernoulli Identities

- Euler-Ramanujan :

$$\sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{B}_{2k}\mathcal{B}_{2n-2k} = -(2n+1)\mathcal{B}_{2n}$$

- Miki's identity (1978) : "exotic"

$$\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)} = \sum_{k=1}^{n-1}\binom{2n}{2k}\frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)(2n-2k)} + \frac{\mathcal{B}_{2n}}{n}\left(\psi(2n+1)+\gamma\right)$$

- 2-loop effective actions give simple proof :

$$\frac{\xi(\kappa)}{\kappa} = -\sum_{n=1}^{\infty}\frac{\mathcal{B}_{2n}}{2n}\frac{1}{\kappa^{2n}}$$

$$= \int_0^\infty ds\, e^{-2\kappa s}\left(\coth s - \frac{1}{s}\right)$$

- many further identities ...

Applications : QED β-functions and IR/UV connection

- two-loop $\mathcal{L}^{(2)\mathrm{unren}}_{\mathrm{spinor}}$ is IR finite !

- zero-mode influence looks like a one-loop effect

- but, enters through mass renormalization

$$\mathcal{L}^{(1)}(ef, m_R^2) = \mathcal{L}^{(1)}(ef, m^2) + \delta m^2 \frac{\partial \mathcal{L}^{(1)}(ef, m^2)}{\partial m^2} + \dots$$

- two-loop effective Lagrangians :

$$\mathcal{L}^{(2)} = \frac{\alpha^2 f^2}{(4\pi)^2} \Bigg[2(A+B)\xi' - 2(A+2B)\xi^2 - 4(A+2B)\kappa\xi \ln\left(\frac{\mu^2}{m^2}\right) + 2(A+B)\ln\left(\frac{\mu^2}{m^2}\right) \Bigg]$$

- mass renormalization term :

$$-4\kappa\xi = 8\pi^2 \left[\left\{ \begin{matrix} 1 \\ -2 \end{matrix} \right\} \frac{m^2}{(ef)^2} \frac{\partial}{\partial m^2} \mathcal{L}^{(1)\,\mathrm{ren}}_{\left\{ \substack{sp \\ sc} \right\}} - \frac{1}{8\pi^2} \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} \right]$$

- charge renormalization term :

$$\frac{e^4 f^2}{(4\pi)^4} \left\{ \begin{matrix} A \\ 2(A+B) \end{matrix} \right\} \ln\left(\frac{\mu^2}{m^2}\right) = -\frac{e^4 f^2}{64\pi^4} \ln\left(\frac{\mu^2}{m^2}\right)$$

sp : A = -4, B = 8
sc : A = -1 = B

Conclusions

- self-dual background an efficient probe
- physically : self-duality $\leftrightarrow$ helicity $\leftrightarrow$ SUSY
- renormalized effective actions have simple form
- applications :

  higher-loop effective actions

  higher-loop nonperturbative contributions

  higher-loop β-functions

  higher-loop helicity amplitudes

- recursive structure ? $\mathcal{L}^{(3)} \sim \xi^4 + \ldots$
- exponentiation of leading prefactors ?
- non-Abelian instanton backgrounds ?