Phases of $\mathcal{N}=1$ Supersymm. SO/Sp Gauge Theories (in 4-d) via Matrix Model

(Dijkgraaf-Vafa)

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C.A.

hep-th/0302150 (JHEP)

1.* Cachazo, Seiberg and Witten
hep-th/0301006 U(N)

2. Cachazo, Intriligator and Vafa
hep-th/0103067 U(N)

3. Cachazo and Vafa
hep-th/0206017 U(N)

⋮

Deform $\mathcal{N}=2$ $U(N)$ YM theory by a tree level superpotential

$$W_{tree}(\Phi) = \sum_{i=1}^{n+1} g_i u_i \,, \quad u_i \equiv \frac{1}{i} \mathrm{Tr}\, \Phi^i$$

Classically, solutions to the F and D terms equations are given by Φ being diagonal with eigenvalues solutions of

$$W_n'(x) = g_{n+1} x^n + \cdots + g_1$$

$$= g_{n+1} \prod_{i=1}^{n} (x - a_i) \underset{\substack{\uparrow \\ \text{F-term}}}{=} 0$$

Different choices of the number N_i of eigenvalues of Φ equal to a_i

$\longrightarrow$

The gauge group $U(N)$ is broken down to $U(N_1) \times \cdots \times U(N_n)$

where $\sum_{i=1}^{n} N_i = N$

In the Coulomb branch, the $\mathcal{N}=2$ theory is described by an $U(1)^N$ effective theory, at low energy.

The SW curve for a pure $U(N)$ gauge theory is given by

No flavor

$$y^2 = P_N^2(x) - 4\Lambda^{2N}$$

where $P_N(x, u_k) = \langle \det(x - \Phi) \rangle$

$$\text{and} \quad u_k = \frac{1}{k} \mathrm{Tr}\, \Phi^k$$

Once W_{tree} is introduced, all points in the Coulomb moduli space are lifted EXCEPT those for which $(N-n)$ mutually local magnetic monopoles become massless

$\rightarrow$ a condensate of monopoles:

Those points are where $\langle u_k \rangle$'s are solution to

$$P_N^2(x) - 4\Lambda^{2N} = F_{2n}(x)\, \underline{\underline{H_{N-n}^2(x)}}$$

arbitrary poly. (simple zeroes)

Only $\underline{\underline{U(1)^n}}$ remains unbroken

Monopole Condensation and Confinement

Breaking $\mathcal{N}=2$ down to $\mathcal{N}=1$, one can add a superpotential

$$W = m \underbrace{\mathrm{Tr}\, \Phi^2}$$

represented by a chiral superfield U where

$$\langle u \rangle = \langle \mathrm{Tr}\, \phi^2 \rangle$$

Near the point at which there are massless monopoles, the monopoles are represented by chiral superfields M and $\widetilde{M}$

$$\hat{W} = \sqrt{2}\, A_D M \widetilde{M} + m U(A_D)$$

dual gauge field

If $m \neq 0$, vacuum states correspond to

$$\begin{cases} \sqrt{2}\, M \widetilde{M} + m \dfrac{du}{da_0} = 0 \\ a_D M = a_D \widetilde{M} = 0 \end{cases}$$

$$M, \widetilde{M} \neq 0 \rightarrow a_D = 0$$

$$M = \widetilde{M} = \sqrt{-mu'(0)/\sqrt{2}}$$

Expanding around this vacuum there is a Mass gap

The gauge field gets a mass

Condensation of monopoles induce

Confinement of electric charge!

For $U(N)$ the Coulomb moduli space has dimension N, parametrized by the roots of $P_N(x)$: $\phi_1, \cdots, \phi_N$

In order to produce $(N-n)$ double roots on the RHS, $(N-n)$ of those roots have to be tuned $\rightarrow$ n parameter family of such factorization

For this subspace $\langle u_k \rangle$'s are functions of n parameters

$\rightarrow$ An effective superpotential

$$W_{eff} = \sum_{k=1}^{n+1} g_k \langle u_k \rangle : W_{low} = W_{low}(g_i, \Lambda)$$

This extremization problem in purely algebraic method

$$\rightarrow \; g_{n+1}^2 F_{2n}(x) = W'(x)^2 + f_{n-1}(x)$$

<u>Low energy dynamics</u> of the $\mathcal{N}=1$ theory from the following problem

Find $P_N(x)$ such that

$$P_N^2(x) - 4\Lambda^{2N} = \frac{1}{g_{n+1}^2}\left(W'(x)^2 + f_{n-1}(x)\right) H_{N-n}^2(x)$$

where $W'(x) = g_{n+1} \prod_{i=1}^{n} (x - a_i)$ is given

and $P_N(x) \longrightarrow \prod_{i=1}^{n} (x-a_i)^{N_i}$ as $\Lambda \to 0$

Strong Gauge Coupling Approach: SO(2N) Case

$$W_{2k+2}(\Phi) = \sum_{r=1}^{k+1} \frac{g_{2r}}{2r} \mathrm{Tr}\,\Phi^{2r} \equiv u_{2r}$$

adjoint scalar chiral superfield

$$\Phi_{2N\times 2N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(i\phi_1, \cdots, i\phi_N)$$

$\pm\phi_I$ $(I=1,2,\cdots,N)$ are the eigenvalues of Φ

The classical vacua can be obtained from

$$W'(\pm\phi_I) = 0$$

The vacua in which the pert. by $W(\Phi)$ remains ONLY $U(1)^n$ gauge group at low energy

$$W_{eff} = \sqrt{2} \sum_{\ell=1}^{N-n} \underbrace{M_\ell(u_{2r})}_{\text{mass}} \underbrace{q_\ell \tilde{q}_\ell}_{\text{monopole fields}} + W(\Phi)$$

near a point with $(N-n)$ massless monopoles

$$M_\ell(\langle u_{2r}\rangle) = 0 \quad \ell = 1, 2, \ldots, N-n \quad \text{and} \quad \sum_{\ell=1}^{N-n} \frac{\partial M_\ell(\langle u_{2r}\rangle)}{\partial u_{2r}} \langle q_\ell \tilde{q}_\ell \rangle + g_{2r} = 0$$

In this vacuum

$$W_{eff} = \sum_{r=1}^{k+1} g_{2r} \langle u_{2r} \rangle$$

Singular Point where $(N-n)$ monopoles are massless: $\mathcal{N}=2$ SW curve degenerates — $y^2, \left.\frac{\partial y^2}{\partial x^2}\right|_{\substack{x=0 \\ x=\pm\rho_i}}$

$$y^2 = P_{2N}^2(x) - 4\Lambda^{4N-4}x^4$$

Brandhuber Landsteiner '95

$$= \underbrace{x^2 H_{2N-2n-2}^2(x)}\, \underbrace{F_{2(2n+1)}(x)}_{\text{genus } 2n}$$

where

$$H_{2N-2n-2}(x) = \prod_{i=1}^{N-n-1}(x^2-\rho_i^2)$$

$$F_{2(2n+1)}(x) = \prod_{i=1}^{2n+1}(x^2-g_i^2)$$

$$P_{2N}(x) = \det(x-\Phi_{cl})$$

$$= \prod_{I=1}^{N}(x^2-\phi_I^2)$$

When the degree $(2k+1)$ of $W'(x)$ is equal to $(2n+1)$,

$$\boxed{F_{2(2n+1)}(x) = \frac{1}{g_{2n+2}^2} W'_{2n+1}(x)^2 + \mathcal{O}(x^{2n})}$$

$2n = 2k$

Edelstein, Oh '01

Tatar, Feng '02

$$2n+2 < 2k+2 < 2N$$

$$W_{eff} = \sum_{r=1}^{k+1} g_{2r} u_{2r}$$

$$+ \sum_{i=0}^{2N-2n-2} \left[L_i \oint \frac{P_{2N}(x) - 2\varepsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)} dx + B_i \oint \frac{P_{2N}(x) - 2\varepsilon_i x^2 \Lambda^{2N-2}}{(x-p_i)^2} dx \right]$$

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Lagrange multipliers

p_i : locations of the double roots of y^2

The contour integration encloses all p_i

$p_0 \equiv 0$

$p_{N-n-1+i} = -p_i$ where $i = 1, 2, \cdots, (N-n-1)$

$$\left(P_{2N}(x) - 2x^2\varepsilon_i\Lambda^{2N-2}\right)\Big|_{x=\rho_i} = 0 \quad \bullet\, L_i$$

$$\frac{\partial}{\partial x}\left(P_{2N}(x) - 2x^2\varepsilon_i\Lambda^{2N-2}\right)\Big|_{x=\rho_i} = 0$$

• The variation of W_{eff} w.r.t. B_i

$$P'_{2N}(x) = P_{2N}(x)\sum_{I=1}^{N}\frac{2x}{x^2-\phi_I^2}$$

$$P_{2N}(x)\underbrace{\left(\mathrm{Tr}\frac{1}{x-\Phi_a} - \frac{2}{x}\right)\Big|_{x=\rho_i}}_{=0} = 0$$

$\Rightarrow$

New term

$$\rho_i = \rho_i(u_{2r})$$

$$P_{2N}(x) = \left[x^{2N}\exp\left(-\sum_{r=1}^{\infty}\frac{u_{2r}}{x^{2r}}\right)\right]_+$$

- Variation of W_{eff} w.r.t. ρ_j

$$0 \;=\; 2B_j \underbrace{\oint \frac{P_{2N}(x) - 2x^2\varepsilon_i \Lambda^{2N-2}}{(x-\rho_i)^3}\, dx}_{\neq\, 0}$$

eq. of motion for B_i

$$\Rightarrow \quad B_i = 0$$

- Variation of W_{eff} w.r.t. u_{2r}

$$0 \;=\; g_{2r} - \sum_{i=0}^{2N-2n-2} \oint \underline{\underline{\left[\frac{P_{2N}}{x^{2r}}\right]_+}} \frac{L_i}{x-\rho_i}\, dx$$

eq. of motion for B_i

$$\frac{Q(x)}{x H_{2N-2n-2}(x)}$$

$$\equiv$$

$$W'(z) = \sum_{r=1}^{k+1} g_{2r} z^{2r-1}$$

$$= \boxed{\sum_{i=0}^{2N-2n-2}} \oint_C \sum_{r=1}^{k+1} \frac{P_{2N}(x)}{x^{2r}} \boxed{\frac{L_i}{(x-\rho_i)}} z^{2r-1}\, dx$$

z is inside the C

$$W'(z) = \oint \sum_{r=1}^{k+1} \frac{z^{2r-1}}{x^{2r}} \frac{Q(x) P_{2N}(x)}{x H_{2N-2n-2}(x)} dx$$

$$= \oint \sum_{r=1}^{\infty} \frac{z^{2r-1}}{x^{2r}} \frac{Q(x) P_{2N}(x)}{x H_{2N-2n-2}(x)} dx$$

$$= \oint \frac{z\, Q_{2k-2n}(x)\, P_{2N}(x)}{x(x^2-z^2) H_{2N-2n-2}(x)} dx$$

$$\left(P_{2N}(x) = x \sqrt{F_{2(2n+1)}(x)}\, H_{2N-2n-2}(x) + \mathcal{O}(x^{-2N+4}) \dashrightarrow \text{does not contribute} \right)$$

$$W'(z) = \oint \frac{z\, y_m(x)}{x^2 - z^2} dx$$

$$y_m^2(x) = F_{2(2n+1)}(x)\, Q^2_{2k-2n}(x)$$

$$= W'^2_{2k+1}(x) + f_{2k}(x)$$

When $2k = 2n$, $Q_0 = g_{2n+2}$

- When the k is arbitrary

$$y_m^2 = F_{2(2n+1)}(x)\, \overline{Q}^2_{2k-2n}(x)$$

$$\equiv F_{2(2n+1)}(x) \left(\underline{R_{2k-2N+2}}(x)\, H_{2N-2n-2}(x) + \underline{R_{2N-2n-2}}(x) \right)^2$$

$$\left\langle \mathrm{Tr} \frac{1}{x-\Phi} \right\rangle$$

$$= \frac{d}{dx} \log \left(P_{2N}(x) + \sqrt{P_{2N}^2(x) - 4x^4 \Lambda^{4N-4}} \right)$$

Konishi Anomaly Eq.

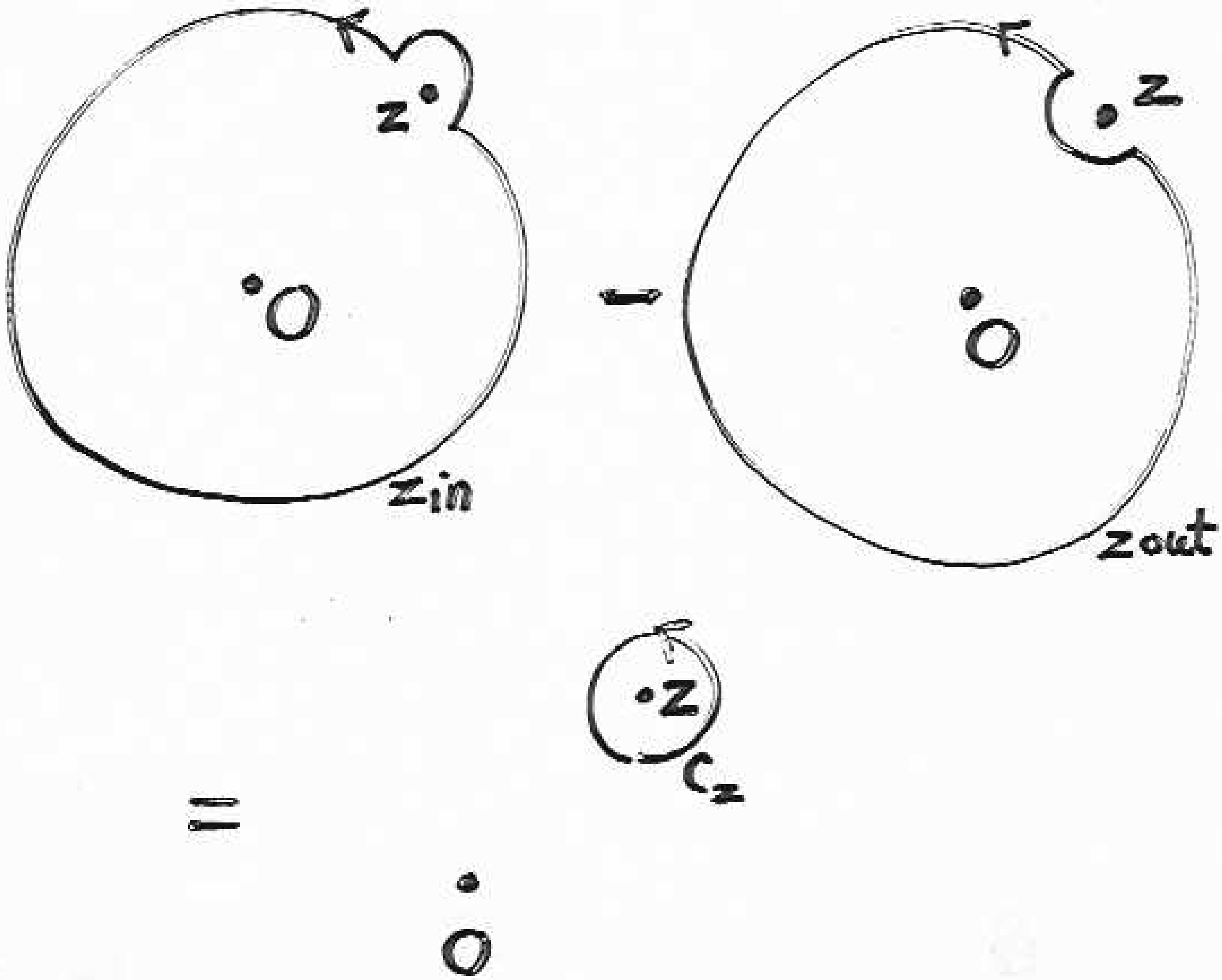
$$W'(\phi_I) = \sum_{i=0}^{2N-2n-2} \oint \phi_I \frac{P_{2N}(x)}{(x^2-\phi_I^2)} \frac{L_i}{x-\rho_i}\, dx$$

$$\mathrm{Tr}\, \frac{W'(\Phi_{cl})}{z-\Phi_{cl}} = 2 \sum_{I=1}^{N} \frac{\phi_I W'(\phi_I)}{z^2-\phi_I^2}$$

$$= \sum_{I=1}^{N} \frac{2\phi_I^2}{(z^2-\phi_I^2)} \sum_{i=0}^{2N-2n-2} \oint_C \frac{P_{2N}(x)}{(x^2-\phi_I^2)} \frac{L_i}{(x-\rho_i)}\, dx$$

where the point z is <u>outside</u> the C

$$= \oint_{z\ \text{out}} dx \sum_{i=0}^{2N-2n-2} \frac{P_{2N}(x) L_i}{(x^2-z^2)(x-\rho_i)} \left(z\, \mathrm{Tr} \frac{1}{z-\Phi_{cl}} - x\, \mathrm{Tr} \frac{1}{x-\Phi_{cl}} \right)$$



$$\oint_{z\,out} = \oint_{z\,in} - \oint_{C_z}$$

- The first term of $\mathrm{Tr}\, \dfrac{W'(\Phi_{cl})}{z-\Phi_{cl}}$

$$= \left(\mathrm{Tr}\, \frac{1}{z-\Phi_{cl}}\right)\left(W'(z) - \frac{y_m(z)\, P_{2N}(z)}{\sqrt{P_{2N}^2(z) - 4z^4 \Lambda^{4N-4}}}\right)$$

- The second term of $\mathrm{Tr}\, \dfrac{W'(\Phi_{cl})}{z-\Phi_{cl}}$

$$= -2\frac{W'(z)}{z} + \frac{2}{z}\, \frac{y_m(z)\, P_{2N}(z)}{\sqrt{P_{2N}^2(z) - 4z^4 \Lambda^{4N-4}}}$$

$$\downarrow$$

$$\left(\frac{2}{z} - \underline{\underline{\left\langle \mathrm{Tr}\, \frac{1}{z-\Phi}\right\rangle}}\right) y_m(z)$$

$$\boxed{\left\langle \mathrm{Tr}\, \frac{W'(\Phi)}{z-\Phi}\right\rangle = \left(\left\langle \mathrm{Tr}\, \frac{1}{z-\Phi}\right\rangle - \frac{2}{z}\right) R(z) \quad \text{where } R(z) \equiv W'(z) - y_m(z)}$$

Multiplication map and Confinement index K

Douglas Shenker '95

$SO(2N)$ theory with $\underline{W(x)}$

Fuji Ookouchi '02

$$\Downarrow \text{map}$$

$SO(2KN-2K+2)$ theory with $\underline{W(x)}$

Assume that

$$P_{2N}^2(x) - 4x^4\Lambda_0^{4N-4} = x^2 H_{2N-2n-2}^2(x)\underbrace{F_{2(2n+1)}(x)}_{W'(x)^2+f_{2n}(x)}$$

$$SO(2N) \to SO(2N_0)\times\prod_{i=1}^{n} U(N_i)$$

in the semiclassical limit

$$\left(\ 2N = 2N_0 + 2\sum_{i=1}^{n} N_i\ \right)$$

Chebyshev Polynomials

$$T_K(x) = \cos(K\theta)$$

$$U_{K-1}(x) = \frac{1}{K}\frac{d\,T_K(x)}{dx} = \frac{\sin(K\theta)}{\sin\theta}$$

where $x = \cos\theta$

$$T_1(x) = x \qquad U_1(x) = 2x$$

$$T_2(x) = 2x^2 - 1 \qquad U_2(x) = 4x^2 - 1$$

$$T_3(x) = 4x^3 - 3x \qquad U_3(x) = 8x^3 - 4x$$

$$T_4(x) = 8x^4 - 8x^2 + 1 \qquad U_4(x) = 16x^4 - 12x^2 - 1$$

$$\vdots \qquad\qquad \vdots$$

Using Chebyshev Polynomials
the solution for massless monopole
constraint of $SO(2KN-2K+2)$

$$P_{2KN-2K+2}(x) = 2\eta^K x^2 \Lambda^{2KN-2K} \underbrace{T_K}_{\text{1st kind of Chebyshev Poly}}\left(\frac{P_{2N}(x)}{2\eta x^2 \Lambda^{2N-2}}\right)$$

where $\eta^{2K} = 1$

RHS : the power of x is
$$2 + K(2N-2)$$

LHS : $2KN - 2K + 2$

The power of Λ in T_K can be fixed by the one of x, $2N-2$

The power of Λ in front of T_K can be fixed by dimensional analysis

$$T_K^2(x) - 1 = (x^2 - 1)\,U_{K-1}^2(x)$$

Let us define

$$\tilde{x} \equiv \frac{P_{2N}(x)}{2\eta x^2 \Lambda^{2N-2}}$$

$$P^2_{2KN-2K+2}(x) - 4x^4\Lambda^{4KN-4K}$$

$$= 4x^4\Lambda^{4KN-4K}\left(T_K^2(\tilde{x}) - 1\right)$$

$$= 4x^4\Lambda^{4KN-4K}\left(\tilde{x}^2 - 1\right)U_{K-1}^2(\tilde{x})$$

$$= \frac{\Lambda^{4KN-4K}}{\eta^2\Lambda^{4N-4}}\left(P_{2N}^2(x) - 4x^4\eta^2\Lambda^{4N-4}\right)U_{K-1}^2(\tilde{x})$$

$$= x^2\Big(\underbrace{H_{2N-2n-2}(x)\,\eta^{-1}\Lambda^{2(K-1)(N-1)}U_{K-1}(\tilde{x})}_{H_{(2N-2)K-2n}(x)}\Big)^2 \underline{\underline{F_{2(2n+1)}(x)}}$$

where $\Lambda_0^{4N-4} = \eta^2\Lambda^{4N-4}$

$$P^2_{2KN-2K+2}(x) - 4x^4 \Lambda^{4NK-4K}$$
$$= x^2 H^2_{(2N-2)K-2n}(x)\, \widetilde{F}_{2(2n+1)}(x)$$

All the vacua in $SO(2KN-2K+2)$ with confinement index K arise in this way from the $SO(2N)$ Coulomb vacua

$\rightarrow$ the number of $SO(2KN-2K+2)$ vacua $= K \times$ the number of $SO(2N)$ Coulomb vacua

$k = n = 1$

$$W(\Phi) = \frac{1}{2} m \,\mathrm{Tr}\,\Phi^2 + \frac{1}{4} g \,\mathrm{Tr}\,\Phi^4$$

$$P_{2N}^2(x) - 4x^4\Lambda^{4N-4}$$

$$= \left(P_{2N}(x) + 2x^2\Lambda^{2N-2}\right)\left(P_{2N}(x) - 2x^2\Lambda^{2N-2}\right)$$

$$= x^2 H^2_{2N-2\cdot1-2}(x)\, F_{2(2\cdot1+1)}(x)$$

$$= x^2 H^2_{2N-4}(x)\, F_6(x)$$

$$P_{2N}(x) = x^{2N} + s_2 x^{2N-2} + s_4 x^{2N-4} + \cdots + s_{2N-2} x^2 + \underbrace{s_{2N}}_{\text{goes to vanish}}$$

parametrizing the point in the moduli space

$$= x^2\left(x^{2N-2} + s_2 x^{2N-4} + \cdots + s_{2N-2}\right)$$

$$= x^2 P_{2N-2}(x)$$

$$P_{2N}(x) + 2x^2\Lambda^{2N-2} = x^2 H^2_{S_+}(x)\, R_{2N-2-2S_+}(x)$$

$$P_{2N}(x) - 2x^2\Lambda^{2N-2} = x^2 H^2_{S_-}(x)\, \tilde{R}_{2N-2-2S_-}(x)$$

where $\begin{cases} x^2 H_{S_+}(x)\, H_{S_-}(x) = x H_{2N-4}(x) \\ R_{2N-2-2S_+}(x)\, \tilde{R}_{2N-2-2S_-}(x) = F_6(x) \end{cases}$

$$\begin{cases} S_+ + S_- + 1 = 2N-4 \\ 2N-2-2S_+, \quad 2N-2-2S_- \geq 0 \end{cases}$$

$$\begin{cases} P_{2N-2}(x) + 2\Lambda^{2N-2} = H^2_{S_+}(x)\, R_{2N-2-2S_+}(x) \\ P_{2N-2}(x) - 2\Lambda^{2N-2} = H^2_{S_-}(x)\, \tilde{R}_{2N-2-2S_-}(x) \end{cases}$$

$$H^2_{S_+}(x)\, R_{2N-2-2S_+}(x) - 4\Lambda^{2N-2} = H^2_{S_-}(x)\, \tilde{R}_{2N-2-2S_-}(x)$$

$$y_m^2 = F_6(x) = R_{2N-2-2s_+}(x)\,\tilde{R}_{2N-2-2s_-}(x)$$

$$= W_3'(x)^2 + f_2(x)$$

SO(8) case

$$N-n = 4-1 = 3$$

$$s_+ + s_- = 2N-4-1 = 3$$

Four branches

$$(s_+, s_-) = (2,1), (1,2), (3,0)$$

$$(0,3)$$

- Coulomb branch with

$(S_+, S_-) = (2, 1)$ or $(1, 2)$

$$P_8(x) + 2\eta x^2 \Lambda^6 = x^2 \underbrace{(x^2 - a^2)^2}_{H^2_{S_+}(x)} \left(x^2 + \frac{4\eta\Lambda^6}{a^4}\right)$$

$$P_8(x) - 2\eta x^2 \Lambda^6 = x^2 \underbrace{x^2}_{H^2_{S_-}(x)} \left[(x^2 - a^2)^2 + \frac{4\eta\Lambda^6}{a^4}(x^2 - 2a^2)\right]$$

Then matrix model curve

$$y_m^2 = \left(x^2 + \frac{4\eta\Lambda^6}{a^4}\right)\left[(x^2 - a^2)^2 + \frac{4\eta\Lambda^6}{a^4}(x^2 - 2a^2)\right]$$

$$= \underbrace{\left\{x^2\left(x^2 + \frac{4\eta\Lambda^6}{a^4} - a^2\right)^2\right\}}_{W_3'(x)^2}$$

$$\underbrace{\left(-\frac{8\eta\Lambda^6}{a^2} x^2 + \frac{4\eta\Lambda^6}{a^2}\left(a^2 - 2\cdot\frac{4\eta\Lambda^6}{a^4}\right)\right)}_{f_2(x)}$$

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Two semiclassical limits

1. $\Lambda \to 0$ with fixed a:

$$P_8(x) \to x^4 (x^2 - a^2)^2$$

$$SO(8) \to SO(4) \times U(2)$$

2. $\Lambda, a \to 0$ with fixed $\frac{4\eta\Lambda^6}{a^4} \equiv v$

$$P_8(x) \to x^6 (x^2 + v)$$

$$SO(8) \to SO(6) \times U(1)$$

By changing parameter continuously one can transit from

$SO(4) \times U(2)$ to $SO(6) \times U(1)$

Counting the number of vacua for fixed $W_3'(x) = x(x^2+\Delta)$

where $\Delta \equiv \dfrac{4\eta\Lambda^6}{a^4} - a^2$

1. $\Delta = -a^2$ <u>two</u> functions $f_2(x)$ for each η

2. $a^2 = \left(\dfrac{2\eta\Lambda^6}{\Delta}\right)^{\frac{1}{2}}$ <u>four</u> functions $f_2(x)$

- Confining branch with $(s_+, s_-) = (0,3)$ or $(3,0)$

$$P_8(x) + 2\eta x^2 \Lambda^6 = x^2\left[x^2(x^2-a^2)^2 + 4\eta\Lambda^6\right]$$

$$P_8(x) - 2\eta x^2 \Lambda^6 = x^2\left[x^2(x^2-a^2)^2\right]$$

$$\rightarrow \quad y_m^2 = x^2(x^2-a^2)^2 + 4\eta\Lambda^6$$

$$\begin{cases} W_3'(x) = x(x^2-a^2) \\ f_2(x) = 4\eta\Lambda^6 \end{cases}$$

In the semiclassical limit $\Lambda \to 0$

$P_8(x) \to x^4(x^2-a^2)^2$:

$SO(8) \to SO(4) \times U(2)$

One can construct a multiplication map from SO(5) to SO(8) by

$$K=2 : P^{K=2}_{SO(5)\to SO(8)}(x)$$

where $2KN-K+2=8$, $N=K=2$
$\eta^4=1$

$$P^{K=2}_{SO(5)\to SO(8)}(x) = 2\eta^K x^2 \Lambda^{2KN-K} T_K\left(\frac{P_{2N}(x)}{2\eta x \Lambda^{2N}}\right)$$

$$= 2\eta^2 x^2 \Lambda^6 \left[2\left(\frac{\overset{x^2(x^2-\ell^2)}{P_4(x)}}{2\eta x \Lambda^3}\right)^2 - 1 \right]$$

$$= x^4(x^2-\ell^2)^2 - 2\eta^2 x^2 \Lambda^6 \Big|_{a^2 \to \ell^2}$$

$$\Rightarrow \boxed{x^4(x^2-a^2)^2 + 2\eta x^2 \Lambda^6} \quad \eta^2 \to -\eta$$

- By using $\mathcal{N}=2$ SW curve with constraint, we derive the matrix model curve
- A generalized Konishi anomaly equation
- Multiplication maps from

$$SO(2N+1) \text{ to } SO(2KN-\overset{\text{even}}{K}+2)$$
$$SO(2N) \text{ to } SO(2KN-2K+2)$$
$$SO(2N+1) \text{ to } SO(2KN-\underset{\text{odd}}{K}+2)$$
$$Sp(2N) \text{ to } Sp(2KN+2K-2)$$

- SO(8) Smooth transition between different classical gauge group

Strong Gauge Coupling Approach : SO(2N+1) case

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(i\phi_1, \cdots, i\phi_N, 0)$$

$$y^2 = P_{2N}^2(x) - 4\Lambda^{4N-2} x^2$$

$$= x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x)$$

$$2n+2 < 2k+2 < 2N$$

$$W_{eff} = \sum_{r=1}^{k+1} g_{2r} u_{2r}$$

$$+ \sum_{i=0}^{2N-2n-2} \left[L_i \oint \frac{P_{2N}(x) - 2\varepsilon_i x \Lambda^{2N-1}}{(x-p_i)} + B_i \oint \frac{P_{2N}(x) - 2\varepsilon_i x \Lambda^{2N-1}}{(x-p_i)^2} dx \right]$$

$$\left(\mathrm{Tr}\frac{1}{x-\Phi_{cl}} - \frac{1}{x}\right)\Bigg|_{x=p_i} = 0$$

Konishi Anomaly Eq.

$$\left\langle \mathrm{Tr}\frac{W'(\Phi)}{z-\Phi} \right\rangle = \left(\left\langle \mathrm{Tr}\frac{1}{z-\Phi}\right\rangle - \frac{1}{z}\right) R(z)$$

where $R(z) \equiv W'(z) - y_m(z)$

Assume

$$P_{2N}^2(x) - 4x^2\Lambda_0^{4N-2} = x^2 H^2_{2N-2n-2}(x) \underbrace{F_{2(2n+1)}(x)}_{W'(x)^2 + f_{2n}(x)}$$

$$SO(2N+1) \to SO(2N_0+1) \times \prod_{i=1}^{n} U(N_i)$$

where

$$2N+1 = 2N_0+1 + 2\sum_{i=1}^{n} N_i$$

$$P_{\underbrace{2KN-K+1}_{\text{even}}}(x) = 2\eta^K x \Lambda^{2KN-K} \underbrace{T_K}_{\text{odd}}\Bigg(\underbrace{\frac{P_{2N}(x)}{2\eta x \Lambda^{2N-1}}}_{\equiv\, \tilde{x}}\Bigg)$$

(If the number K is odd, $T_K(x)$ is an odd polynomial in x. Then $x\,T_K(x)$ is an even polynomial in x)

$$P^2_{2KN-K+1}(x) - 4x^2 \Lambda^{4KN-2K} \longleftarrow SO(2KN-K+2) \text{ gauge theory}$$

$$= 4x^2 \Lambda^{4KN-2K} \left(T_K^2(\tilde{x}) - 1 \right)$$

$$= 4x^2 \Lambda^{4KN-2K} \left(\tilde{x}^2 - 1 \right) U^2_{K-1}(\tilde{x})$$

$$= \frac{\Lambda^{4KN-2K}}{\eta^2 \Lambda^{4N-2}} \left(P^2_{2N}(x) - 4\eta^2 x^2 \Lambda^{4N-2} \right) U^2_{K-1}(\tilde{x})$$

$$= x^2 \Big[\underbrace{\eta^{-1} \Lambda^{2(K-1)(2N-1)} H_{2N-2n-2}(x)\, U_{K-1}(\tilde{x})}_{H_{(2N-1)K-2n-1}(x)} \Big]^2 F_{2(2n+1)}(x)$$

$$\eta^2 \Lambda^{4N-2} = \Lambda^{4N-2}$$

From $SO(2N+1)$ to $SO(\underbrace{2M}_{=})$

$$P_{2KN-K+2}(x) = 2\eta^{K} x^{2} \Lambda^{2KN-K} T_K(\tilde{x}) \quad 2KN-K+2$$

where $$\tilde{x} = \frac{P_{2N}(x)}{2\eta x \Lambda^{2N-1}}$$

RHS : $2 + K(2N-1)$

LHS : $2KN - K + 2$

$$P^{2}_{2KN-K+2}(x) - 4x^{4}\Lambda^{4KN-2K}$$

$$= 4x^{4}\Lambda^{4KN-2K}\left(T_K^{2}(\tilde{x}) - 1\right)$$

$$= 4x^{4}\Lambda^{4KN-2K}\left(\tilde{x}^{2} - 1\right) U^{2}_{K-1}(\tilde{x})$$

$$= \frac{x^{2}\Lambda^{4KN-2K}}{\eta^{2}\Lambda^{4N-2}}\left(P^{2}_{2N}(x) - 4x^{2}\eta^{2}\Lambda^{4N-2}\right) U^{2}_{K-1}(\tilde{x})$$

$$H_{(2N-1)K-2n}(x)$$

$$= x^{2}\underline{\left[x\eta^{-1}\Lambda^{2KN-K-2N+1} H_{2N-2n-2}(x)\, U_{K-1}(\tilde{x})\right]}^{2} F_{2(2n+1)}(x)$$

Strong Gauge Coupling Approach : Sp(2N) Case

$$W(\Phi) = \sum_{r=1}^{k+1} \frac{g_{2r}}{2r} \operatorname{Tr} \Phi^{2r}$$

where

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \operatorname{diag}(i\phi_1, \cdots, i\phi_N)$$

$$W_{eff} = \sqrt{2} \sum_{\ell=1}^{N-n} M_\ell(u_{2r})\, q_\ell \tilde{q}_\ell + W(\Phi)$$

Singular Point where $(N-n)$ monopoles are massless

$$y^2 = B^2_{2N+2}(x) - 4\Lambda^{4N+4} = \left(x^2 P_{2N}(x) + 2\Lambda^{2N+2}\right)^2 - 4\Lambda^{4N+4}$$

$$= x^2 \underline{H^2_{2N-2n}(x)}\, F_{2(2n+1)}(x)$$

$$2n+2 < 2k+2 < 2N$$

$$W_{eff} = \sum_{r=1}^{k+1} g_{2r} u_{2r}$$

$$+ \sum \left[L_i \oint \frac{B_{2N+2}(x) - 2\xi_i \Lambda^{2N+2}}{(x-\rho_i)} dx + B_i \oint \frac{B_{2N+2}(x) - 2\xi_i \Lambda^{2N+2}}{(x-\rho_i)^2} dx \right]$$

$$\left(\mathrm{Tr} \frac{1}{x - \Phi_{cl}} + \frac{2}{x} \right) \Bigg|_{x=\rho_i} = 0$$

$$W'(z) = \oint \frac{z\, Q_{2k-2n}(x)\, x^2 P_{2N}(x)}{x\,(x^2 - z^2)\, H_{2N-2n}(x)} dx$$

where

$$x^2 P_{2N}(x) = x \sqrt{F_{2(2n+1)}(x)}\, H_{2N-2n}(x)$$

$$+ \mathcal{O}(x^{-2N-2})$$

$$\left\langle \mathrm{Tr} \frac{W'(\Phi)}{z-\Phi} \right\rangle = \left(\left\langle \mathrm{Tr} \frac{1}{z-\Phi} \right\rangle + \frac{2}{z} \right) R(z)$$

where $R(z) \equiv W'(z) - y_m(z)$

Assume

$$\left(x^2 P_{2N}(x) + 2\Lambda_0^{2N+2}\right)^2 - 4\Lambda_0^{4N+4}$$

$$= x^2 H^2_{2N-2n}(x) \underbrace{F_{2(2n+1)}(x)}_{W'(x)^2 + f_{2n}(x)}$$

$$Sp(2N) \to Sp(2N_0) \times \prod_{i=1}^{n} U(N_i)$$

Let us consider a solution of $Sp(2KN + 2K - 2)$ gauge theory

$$P_{2KN+2K-2}(x) = \frac{2\eta^{2K}\Lambda^{2KN+2K}}{x^2} T_K(\tilde{x}) - \frac{2\Lambda^{2KN+2}}{x^2}$$

$$\text{where} \quad \tilde{x} \equiv \frac{x^2 P_{2N}(x)}{2\eta^2\Lambda^{2N+2}} + 1$$

RHS: $\tilde{x}$ has a degree $2N+2$

$\to$ the power of x is

$$K(2N+2) - 2$$

LHS: $2KN+2K-2$

The last term in RHS is included because the SW curve has an extra term in the LHS

Let us introduce

$$B_{2KN+2K}(x) \equiv x^2 P_{2KN+2K-2}(x) + 2\Lambda^{2KN+2}$$

$$B^2_{2KN+2K}(x) - 4\Lambda^{4KN+4K}$$

$$= 4\Lambda^{4KN+4K}\left(T^2_K(\tilde{x}) - 1\right)$$

$$= 4\Lambda^{4KN+4K}\left(\tilde{x}^2 - 1\right)\mathcal{U}^2_{K-1}(\tilde{x})$$

$$= \frac{\Lambda^{4KN+4K}}{\eta^4\Lambda^{4N+4}}\left[\left(x^2 P_{2N}(x) + 2\eta^2\Lambda^{2N+2}\right)^2 - 4\eta^4\Lambda^{4N+4}\right]\mathcal{U}^2_{K-1}(\tilde{x})$$

$$= x^2\underline{\underline{\left[\eta^{-2}\Lambda^{2(K-1)(N+1)}H_{2N-2n}(x)\,\mathcal{U}_{K-1}(\tilde{x})\right]^2}} F_{2(2n+1)}(x)$$

$$\Downarrow$$

$$H_{K(2N-2)-2n}(x)$$

where $\Lambda_0^{2N+2} = \eta^2\Lambda^{2N+2}$

Massless monopole Constraint

$$B^2_{2N+2}(x) - 4\Lambda^{4N+4} = x^2 H^2_{2N-2}(x) F_6(x)$$

where $k = n = \underline{1}$

$$B_{2N+2}(x) + 2\Lambda^{2N+2} = H^2_{S_+}(x)\, R_{2N+2-2S_+}(x)$$

$$\underbrace{B_{2N+2}(x) - 2\Lambda^{2N+2}}_{x^2 P_{2N}(x)} = H^2_{S_-}(x)\, \tilde{R}_{2N+2-2S_-}(x)$$

where

$$\begin{cases} x\, H_{2N-2}(x) = H_{S_+}(x)\, H_{S_-}(x) \\ F_6(x) = R_{2N+2-2S_+}(x)\, \tilde{R}_{2N+2-2S_-}(x) \end{cases}$$

$$x^2 H \quad \tilde{R}$$

$$B_{2N+2}(x) + 2\Lambda^{2N+2} = H_{2S_+}^2(x)\, R_{2N+2-4S_+}(x)$$

$$B_{2N+2}(x) - 2\Lambda^{2N+2} = x^2 H_{2S_-}^2(x)\, \widetilde{R}_{2N+2-4S_-}(x)$$

$$\begin{cases} H_{2N-2}(x) = H_{2S_+}(x)\, H_{2S_-}(x) \\ F_6(x) = R_{2N+2-4S_+}(x)\, \widetilde{R}_{2N+2-4S_-}(x) \end{cases}$$

Also we have

$$H_{2S_+}^2(x)\, R_{2N+2-4S_+}(x) - 4\Lambda^{2N+2} = x^2 H_{2S_-}^2(x)\, \widetilde{R}_{2N+2-4S_-}(x)$$

$$\Rightarrow y_m^2(x) = R_{2N+2-4S_+}(x)\, \widetilde{R}_{2N+2-4S_-}(x)$$

$$= W_3'(x)^2 + f_2(x)$$

$Sp(4)$ case $(s_+, s_-) = (1, 0)$

$$B_6(x) + 2\Lambda^6 = (x^2 - a^2)^2 \underline{\left(x^2 + \frac{4\Lambda^6}{a^4}\right)}$$

$$B_6(x) - 2\Lambda^6 = x^2 \underline{\left[(x^2 - a^2)^2 + \frac{4\Lambda^6}{a^4}(x^2 - 2a^2)\right]}$$

$$y_m^2(x) = \left(x^2 + \frac{4\Lambda^6}{a^4}\right)\left[(x^2 - a^2)^2 + \frac{4\Lambda^6}{a^4}(x^2 - 2a^2)\right]$$

$$\to \quad W_3'(x) = x\left(x^2 + \frac{4\Lambda^6}{a^4} - a^2\right)$$

$$f_2(x) = -\frac{8\Lambda^6}{a^2}x^2 + \frac{4\Lambda^6}{a^2}\left(a^2 - 2\frac{4\Lambda^6}{a^4}\right)$$

1. $\Lambda \to 0$ with fixed a : 2

   $P_1(x) \to (x^2 - a^2)^2 \qquad Sp(4) \to U(2)$

2. $\Lambda \to 0$ with fixed $\frac{4\Lambda^6}{a^4} \equiv U$: 2

   $P_2(x) \to x^2(x^2 + U) \qquad Sp(4) \to Sp(2) \times U(1)$

Phases of $\mathcal{N}=1$ $SO(N_c)$ Gauge Theories with Flavors via Matrix Model

to appear soon

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hep-th/0303065

$$W = \sqrt{2}\,\tilde{Q}\,\Phi\,Q + \sqrt{2}\,\hat{Q}\,m\,Q + \sum_{i=1}^{n+1} \frac{1}{i}\,\mathrm{Tr}\,\Phi^{i}$$

fundamental rep.