Matrix Model
and
SUSY Gauge Theories

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IR behavior of gauge theory

- IR behavior of gauge theory is badly understood
  → Confinement, chiral symmetry breaking, mass gap
- How to think about it ?
  → We need nonperturbative physics
- Standard perturbation theory : $\sum a_n g^n$
- 't Hooft's idea : take large N of SU(N) with $g^2 N$ fixed

SUSY Gauge Theories

- Essential for beyond Standard Model
- Exactly Computable Non-perturbative Effects
  - Holomorphy (Seiberg) for N = 1 SYM (+ matter)
  - Seiberg-Witten Solution for N = 2 SYM (+ matter)
    - Su perpotential (F-term)
  - Duality (Stong Weak Duality)
  - Moduli Space via effective geometry
- Connections to String Theory
  - Open / Closed Duality
  - Calabi-Yau Compactifications (Special Geometry)

Large N duality (Vafa)

- More complicated example of `t Hooft's conjecture
  → AdS/CFT (Maldacena)
- N → 1 of Chern-Simons theory on S^3
  is equivalent to topological strings on noncompact Calabi-Yau 3-fold (~ blown up of the conifold)
- II A Superstrings on conifold background in the presence of N D6 branes wrapped around S^3 and filling spacetime (OPEN)
  ←→ Topolgical string on resolved conifold. RR-flux and no branes (CLOSED)

Strings on Calabi-Yau Spaces

- Calabi-Yau n-fold reduces SUSY by $1/2^{n-1}$
- D-branes reduce SUSY by ½
- D-branes on CY3 → 4 Supercharges
  → D=4, N=1 SUSY Gauge Theories
- CY compactification with Ramond-Ramond Flux gives Superpotential (Gukov-Vafa-Witten)

$$W_{eff}(S) = \int_{CY} \Omega \wedge H$$

Ω :Holomorphic n-form

H : Flux

Superpotential

- $$W_{\rm eff}(S) = \int_{\rm CY} \Omega \wedge H = \oint_{A_i} \Omega \int_{B_i} H - \int_{B_i} \Omega \oint_{A_i} H$$
- Special Geometry : $S_i = \oint_{A_i} \Omega, \quad \frac{\partial F_0}{\partial S_i} = \int_{B_i} \Omega$

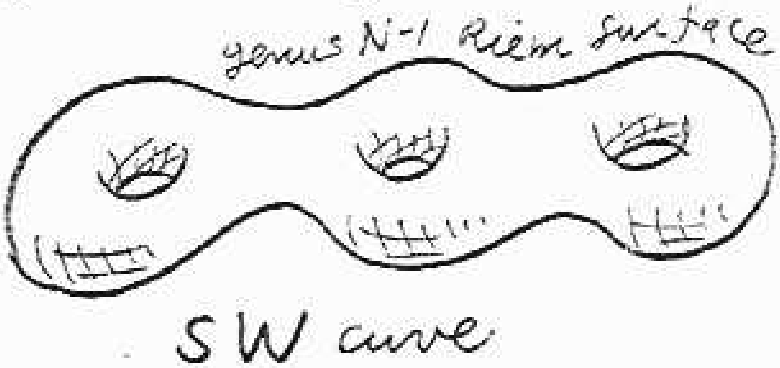
$$N_i = \oint_{A_i} H = \text{number of D5 branes}$$

$$W_{\rm eff}(S) = \sum_i \left(N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \tau S_i \right)$$

$\mathcal{N} = 2$ Super Yang-Mills

- SU(N) ← dual → $U(1)_{eff}^{N-1}$

Seiberg-Witten Curve : genus N-1 Riemann Surface



Exactly Computable $\sum_n a_n e^{-n/g^2}$

Dijkgraaf-Vafa

Non-Perturbative Dynamics of $\mathcal{N} \geq 1$ SUSY Gauge Theories and the Effective Geometries

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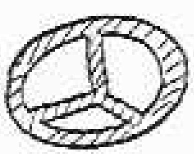
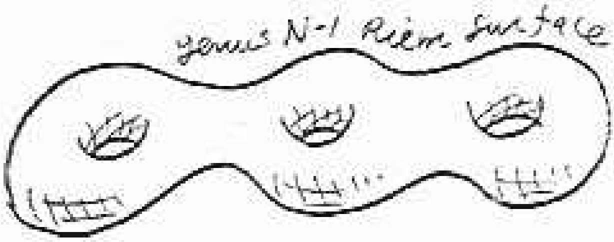
Planar Diagrams of Large N Matrix Models (Perturbative Calculation)



Seiberg-Witten System

- Family of Riemann surfaces with genus g



- Meromorphic differential : dS
- From above data one can define prepotential

$\mathcal{N} = 1$ Super Yang-Mills

Vector Supermultiplet $V(x, \theta, \bar{\theta}) : (A_\mu, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$

Chiral Field Strength $\mathcal{W}_\alpha = \bar{D}^2 e^{-V} D_\alpha e^V$

$$\mathcal{W}_\alpha = \lambda_\alpha + \theta^\beta F_{\alpha\beta}$$

SYM Action

$$\int d^2\theta \tau \mathrm{Tr}(\mathcal{W}_\alpha \mathcal{W}_\beta)\epsilon^{\alpha\beta} + (\mathrm{c.c.})$$

Gauge Coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$

Power of D-V

- Nonperturbative instanton calculus can be derived from the simple zero dimensional theory in the Large N limit
- Perhaps `expected' : Universality classes of Effective actions and their integrability
- But is a highly non-trivial statement

Field Theory Proof of D-V

- Dijkgraaf, Grisaru, Lam, Vafa, Zanon

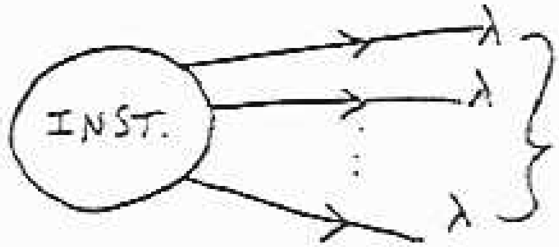
QCD and SYM

- Quark (fundamental) → Gluino (adjoint)
- Confinement
- Mass gap
- Gluino condensate $\langle\lambda\lambda\rangle = \text{const.}\Lambda^3 e^{2\pi i k/N_c}$

Exactly calculable

- Discrete symmetry

  $U(1)_A \to Z_{2N} \to Z_2$



Veneziano-Yankielowicz Effective Superpotential

- Glueball Superfield $S = \frac{1}{32\pi^2}\text{Tr}(\mathcal{W}_\alpha \mathcal{W}^\alpha)$

$$\int d^4x d^2\theta W_{eff}(S)$$

$$W_{eff} = S\left[\log\left(\frac{S}{\Lambda_0^3}\right)^N - N\right] + 2\pi i\tau S$$

Gives the right gaugino condensate for dW = 0

Derivation of V-Y

- Some symmetries of classical SYM are anomalous :

(trace of energy momentum, supercurrent, gluino current) → forms a supermultiplet

- Some effective fields can summarize all the information about the anomalous Ward identities.
- Veneziano-Yankielowicz effective action has kinetic term invariant under scale and R-rotation.

  Potential term is such that it gives the anomalous Ward identity for the chiral rotation.
- SUSY gives all other anomalies.

V-Y Superpotential from MM

- Matrix Model Partition Function

$$Z = \int_{\tilde{N}\times\tilde{N}} d\Phi \cdot e^{\frac{1}{g_s}\mathrm{Tr}W(\Phi)}$$

Saddle Point Expansion $\quad Z \sim \exp\left(\sum_g g_s^{2g-2}\mathcal{F}_g(S)\right),$

$$\mathcal{F}_g = \text{sum of genus } g \text{ graphs}$$

\`t Hooft Large N Limit $\quad g_s \to 0, \quad \tilde{N} \to \infty, \quad S = g_s\tilde{N}$

$$W_{eff} = \sum_i \left(N_i \frac{\partial \mathcal{F}_0}{\partial S_i} + \alpha_i S_i \right)$$

we can obtain the planar matrix model free energy as follows

$$e^{-\frac{1}{g_s^2}\mathcal{F}_0} = \frac{1}{\text{vol}U(\tilde{N})} \int d\phi e^{-\frac{1}{g_s}\text{Tr}W(\phi)}$$

The volume of the group gives the V-Y superpotential

$$\text{Vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{(N-1)!(N-2)!\cdots 3!2!}$$

Adding Flavor

- Affeck-Dine-Seiberg Superpotential can be derived

$$W = (N_c - N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\text{det}M} \right)^{\frac{1}{N_c - N_f}}$$

- Seiberg's duality

$$W_{elec} = \sum_{i=1}^{N_f} Q_i m \tilde{Q}^j \Longleftrightarrow W_{mag} = \frac{1}{\mu} \sum_{i=1}^{N_f} X_i^j q_j \tilde{q}^i + \text{Tr}(Xm)$$

- Duality in Matrix Models?

Loop Equations and Seiberg-Witten

- Partition function of gas of eigenvalues

$$Z \sim \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\frac{2}{g_s} \sum_i W(\lambda_i)}$$

- Classical equation of motion from the action

$$\sum_{j \neq i} \frac{2}{\lambda_i - \lambda_j} - \frac{1}{g_s} W'(\lambda_i) = 0$$

$$y^2 = W'(x)^2 - f(x), \text{ where } y(x) = \omega_0(x) - W'(x), \ \omega_0 = g_s \sum_i \frac{1}{x - \lambda_i}$$

Magnetic Theory

$$W_{mag} = \frac{1}{\mu} \sum_{i=1}^{N_f} X_i^j q_j \tilde{q}^i + \mathrm{Tr}(Xm)$$

$$Z = \frac{1}{\mathrm{Vol}(SO(N))} \left(\frac{\tilde{\lambda}}{2\pi g_s} \right)^{\frac{1}{2} N_f \tilde{N}} \int (dX) \prod_{j=1}^{N_f} (dq_j) \delta\left(X^{ij} - M^{ij}\right) e^{\frac{1}{g_s}\left(\mathrm{Tr}\, mX + \frac{1}{\mu} \sum_{i,j=1}^{N_f} X^{ij} q_i q_j\right)}$$

$$= \frac{1}{\mathrm{Vol}(SO(N))} \left(\frac{\tilde{\lambda}}{2\pi g_s} \right)^{\frac{1}{2} N_f \tilde{N}} \int \prod_{j=1}^{N_f} (dq_j) \delta(\mu m - q_i \cdot q_j).$$

Wishart random matrices

$$\int \prod_{j=1}^{N_f} (dq_j) \delta(\mu m - q_i \cdot q_j) = c \times (\det(-\mu m))^{(\tilde{N} - N_f - 1)/2}$$

Dijkgraaf-Vafa and Seiberg-Witten

- The N = 2 superpotential takes the form

$$W_{\text{tree}}(\Phi, Q) = \sqrt{2} Q^i_a \Phi_{ab} Q^j_b J_{ij} + \sqrt{2} m_{ij} Q^i_a Q^j_a$$

exponentiating, the effective action for the eigenvalues is given by

$$S(\lambda) = -\sum_{a<b}^{N} \log\left(\lambda_a^2 - \lambda_b^2\right)^2 + \frac{1}{S}\sum_{a=1}^{N} 2W(\lambda_a) + \sum_{i=1}^{N_f} \log\left(\lambda_a^2 - m_i^2\right).$$

Loop equation for resolvant

$$\omega^2(x) + \frac{2}{S}\omega(x)W'(x) + \frac{f(x)}{S^2} = 0$$

Integrable Models and S-W

- Integrable Model : τ – Function of Whitham Hierarchy
- SUSY gauge theory : prepotential
  (prepotential determines the Wilsonian effective action in the Coulomb phase)
- Mutually beneficial
- Solutions of SW found via Integrable Models
  Ex. Gauge theory with matter in adjoint representation from twisted Calogero-Moser system
- Discovery of New Integrable Models
  Ex. Spin chains with twisted monodromy

Virasoro Constraints

- Dijkgraaf, Verlinde, Verlinde (1991)
  Fukuma, Kawai, Nakayama (1991)

$$L_{-1}\tau = \left(\sum_{m=1}^{\infty} \left(m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_{m-1}} + \frac{t_0^2}{8\kappa^2} \right) \tau$$

- We can also show that $L_n \tau = 0$ for $n \geq -1$
  where
  $$L_0 = \sum_{m=0}^{\infty} \left(m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_m} + \frac{1}{16}$$
  $$L_n = \sum_{m=0}^{\infty} \left(m + \frac{1}{2} \right) t_m \frac{\partial}{\partial t_{m+n}} + \frac{\kappa^2}{2} \sum_{m=1}^{n} \frac{\partial^2}{\partial t_{m-1} \partial t_{n-m}}$$
  $$[L_n, L_m] = (n-m) L_{n+m} \quad \text{for} \quad n, m \geq -1$$

Loop eq. – Virasoro constraint

- Matrix Model $Z = \int d\Phi \exp\left(\frac{-1}{g_s} \mathrm{Tr} W(\Phi) \right)$ with $W(\Phi) = \sum t_n \Phi^n$

  reparametrizations : $\delta\Phi = \epsilon_n \Phi^{n+1}$

- $$\int d\Phi e^{\sum t_k \mathrm{Tr}\Phi^k} = \int d(\Phi + \epsilon_n \Phi^{n+1}) e^{\sum t_k \mathrm{Tr}(H + \epsilon H^{n+1})^k}$$

→ $$\int d\Phi e^{\sum t_k \mathrm{Tr}\Phi^k} \left(\sum_{k=0}^{\infty} k t_k \mathrm{Tr}\Phi^{k+n} + \mathrm{Tr}\frac{\delta\Phi^{n+1}}{\delta\Phi} \right) = 0$$

The Jacobian : $(\delta\Phi^{n+1})_{ij} = \sum_{k=0}^{n} (\Phi^k \delta\Phi \Phi^{n-k})_{ij} = \sum_{k=0}^{n} (\Phi^k)_{il} (\delta\Phi)_{lm} (\Phi^{n-k})_{mj}$

→ $$\mathrm{Tr}\frac{\delta\Phi^{n+1}}{\delta\Phi} = \sum_{k=0}^{n} \mathrm{Tr}\Phi^k \mathrm{Tr}\Phi^{n-k}$$

Virasoro constraint and generalizations

$$\langle \text{Tr}\Phi^{a_1}\cdots\Phi^{a_n}\rangle = \int d\Phi e^{\sum_{k=0}^{\infty} t_k \text{Tr}\Phi^k} \text{Tr}\Phi^{a_1}\cdots\text{Tr}\Phi^{a_n} = \frac{\partial^n}{\partial t_{a_1}\cdots\partial t_{a_n}} Z\{t\}$$

- For this we get the Virasoro constraint
- We can have CFT formulation of the matrix model
- W-algebras for 2-matrix models

Integrable Hierachies

- Definition a la Liouville
  - A Hamiltonian system with 2N dimensional phase space is integrable iff there exist exactly N functionally independent conserved quantities. The Poisson brackets of these conserved quantities with one another vanish
- Field theory → Infinite degrees of freedom
  → Inifinite number of conserved quantities

KdV Hierachy

- KdV Equation $\frac{\partial u}{\partial t_1} = 6u\frac{\partial u}{\partial x} + \kappa^2\frac{\partial^3 u}{\partial x^3}$
- Bi-Hamiltonian System $\frac{\partial u}{\partial t_1} = \{u, \mathcal{H}_1\}_2 = \{u, \mathcal{H}_2\}_1$

$\{u(x), u(y)\}_1 = \partial_x\delta(x-y)$ $\{u(x), u(y)\}_2 = (\kappa^2\partial_x^3 + 2u\partial_x + 2\partial_x u)\delta(x-y)$

- $\mathcal{H}_1 = \int_{-\infty}^{\infty} dx \left(\frac{u^2}{2}\right)$ $\mathcal{H}_2 = \int_{-\infty}^{\infty} dx \left(u^3 - \frac{\kappa^2}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)$
- Recurrence relation

$$(\kappa^2\partial_x^3 + 2u\partial_x + 2\partial_x u)\frac{\delta\mathcal{H}_{n-1}}{\delta u} = \partial_x\frac{\delta\mathcal{H}_n}{\delta u}$$

Lax Formalism – Generalized KdV Hierachy

- $\frac{\partial u}{\partial t_k} = \{u, \mathcal{H}_k\}_2 = \{u, \mathcal{H}_{k+1}\}_1$
- Consider a differential Operator $L = \partial_x^p + \sum_{i=0}^{p-2} u_i\partial_x^i$
- Flows $\frac{\partial L}{\partial t_n} = [H_n, L]$ with $H_n = (L^{n/p})_+$
- Pseudo differential operator $\partial_x^{-1} f = \sum_{i=0}^{\infty}(-1)^i\frac{\partial^i f}{\partial x^i}\partial_x^{-1-i}$

Generalized KdV and KP

- KdV by $L = \partial_x^2 + u$
- KP hierarchy by $Q = \partial_x + \sum_{i=1}^{\infty} Q_i \partial_x^{-1}$ $\frac{\partial Q}{\partial t_n} = [Q_+^n, Q]$
- To the p-th reduction of the generalized KdV hierarchy we can associate the τ–function related to the differential operator *L*

$$\mathrm{res} L^{i/p} = \frac{\partial^2}{\partial x \partial t_n} \log \tau$$

- Residue is defined as the coefficient of ∂_x^{-1}

τ-function

- The τ–function completely defines the differential operator *L* since we have $\mathrm{res} L^{i/p} = \frac{i}{p} u_{p-i-1}$
- For the KdV operator we have $u = 2\partial_x^2 \log \tau$
- The partition function of one matrix model is the square root of the τ–function of the KdV hierachy (Douglas, 1990)

Baker-Akhiezer function

- The opposite problem: how to construct τ function out of L.
- Consider the eigenfunction ψ of L. $L\psi = z^p\psi$

$$\frac{\partial \psi}{\partial t_n} = H_n \psi$$

- ψ is called the Baker-Akhiezer function

Krichever construction

- Special case : When only finite number of the integrals of motion are algebraically independent
- Finite Gap Solutions for $[L, \sum_{k=1}^{M} c_k H_k] = 0$
- The common spectrum of L and A is related to a complex curve Σ
- The moduli of Σ are the integrals of motion
- The Baker-Akhiezer funtion is a section of a bundle over Σ can be explicitly given.

Theta-Function and Baker-Akhiezer function

- Baker-Akhiezer function is in terms of ratio of Riemann theta functions

- Tau function is proportional to the Riemann theta function

Whitham hierachy

- Modulation of the solutions for the KdV equation

→ Bogolyubov-Whitham averaging method

→ Looks similar to the computation of the effective action in Quantum Field Theory

→ Divide the variables into fast (massive) and slow (massless) modes and average over the fast variables.

→ Effective action for the light modes

→ Geometric formulation by Kriechever

Superpotential from Integrable Model

- Use elliptic Calogero-Moser model (Weirstrass ft.)

$$H_2 = \sum_{j=1}^{N} \frac{P_j^2}{2} + \sum_{j>k} \mathcal{P}(X_i - X_k)$$

- Lax equation : $i\dot{L} = [M, L]$
- For each p, identify Conserved Quantity $H_p = \mathrm{Tr} L^p$ with $\langle \mathrm{Tr}\Phi^p \rangle$
- Deformation of N = 4

$$W(\Phi) = \mathrm{Tr}\left(i\Phi[\Phi^+, \Phi^-] + \Phi^+\Phi^- + \sum_{p=2}^{N} g_p \Phi^p \right)$$

Futher Developments

- Understanding D=6, (2,0) theories via Matrix Model in D=2 using Quiver Theory and Deconstruction
- Uses of Integrable Structure of Matrix Model for the understanding of SUSY gauge theory
  - Hirota's tau function → Prepotential
- Non-planar Diagrams of Matrix Model and Gravitational Coupling

$$\int d^4x \mathcal{F}_{g>1}(S) F_+^{2g-2} R_+^2$$