Tame and wild automorphisms of free algebras

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Let A_n be an algebra of polynomials or a free associative algebra over a field F of characteristic 0 on generators $x_1, \ldots, x_n$ and $Aut\, A_n$ be its group of automorphisms. An automorphism $\phi \in Aut\, A_n$ is called *elementary* if it is of the form

$$\phi : (x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, \lambda x_i + f, \ldots, x_n),$$

where $0 \neq \lambda \in F$ and the element f belongs to the subalgebra generated by $x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n$. The subgroup generated by all elementary automorphisms is denoted by $Tame\, A_n$; its elements are called *tame* automorphisms.

In 1942 Jung proved that, for the case of polynomials, $Aut\, A_2 = Tame\, A_2$. In the beginning of 70-s, Makar-Limanov and Czerniakiewicz proved the same result for free associative algebras. In both cases, it remained an open question whether the same is true for $n \geq 3$.

In 1972 Nagata constructed an automorphism of the algebra of polinomials A_3 which he suggested to be non-time (*wild*). Later Anick provided a candidate for a wild automorphism in the free associative algebra on 3 generators.

In 2004, Shestakov and Umirbaev solved the problem of wild automorphisms in the algebra of polynomials A_3 by proving that the Nagata automorphism is wild. Recently, Umirbaev has proved that the Anick automorphism is wild as well.

In our talk, we will give some ideas and methods of the proofs of these results and will formulate some new results and conjectures. In particular, we present a wild automorphism in the free Jordan algebra on three generators.