Fully Nonlinear and Exact Cosmological Perturbation Theory

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Newton's theory:

- Non-relativistic (no c)
  - Action at a distance, violate causality
  - c=∞ limit of Einstein's gravity: 0^{th} post-Newtonian limit
  - No horizon
  - Static nature
- No strong pressure
- No strong gravity
- No gravitational waves
- Incomplete and inconsistent

Einstein's gravity:

- Relativistic
- Strong gravity, dynamic
- Simplest

Perturbation method:

- Perturbation expansion
- ~~All perturbation variables are small~~
- ~~Weakly nonlinear~~
- Strong gravity; fully relativistic
- Valid in all scales
- Fully nonlinear and Exact perturbations arXiv:1207.0264

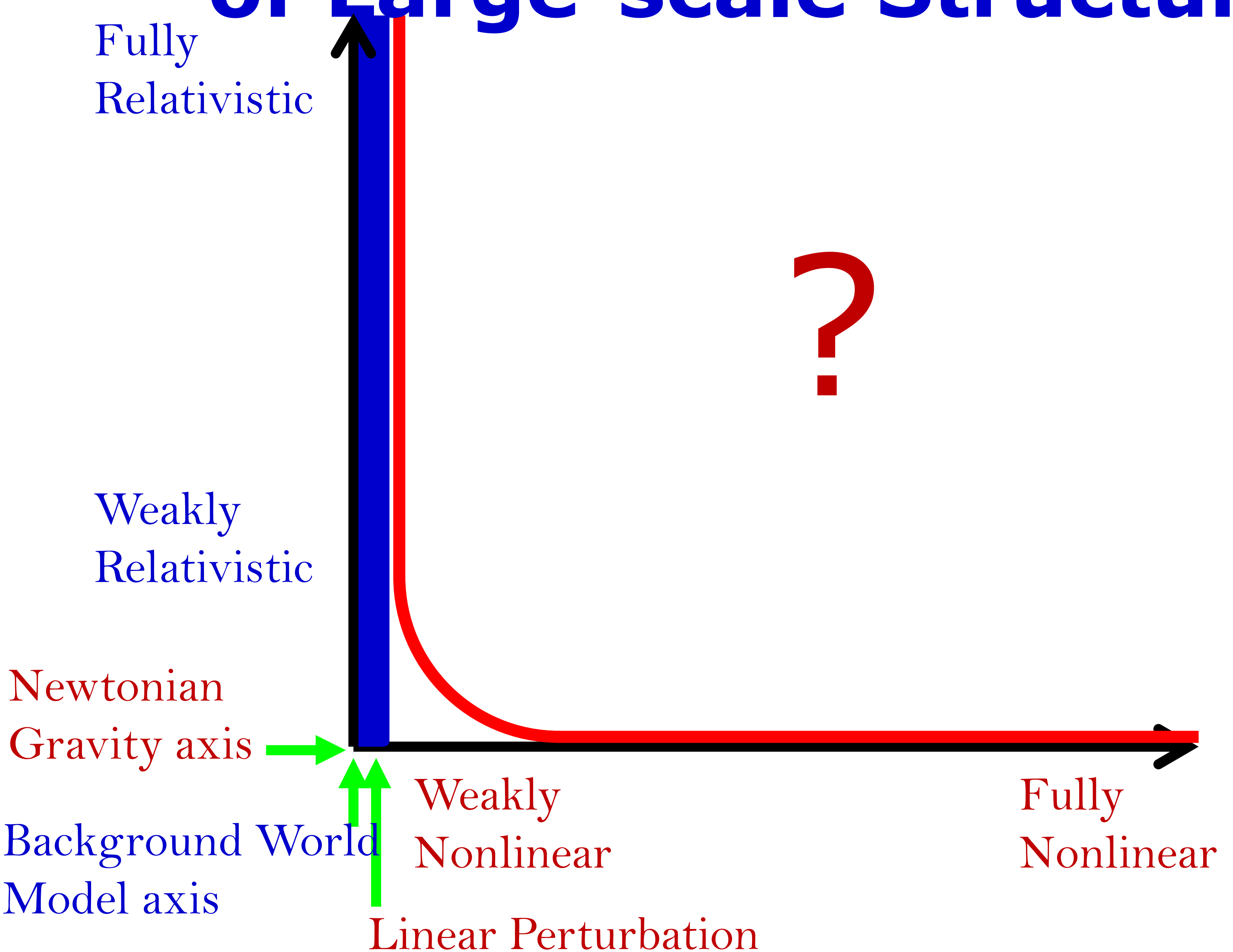
Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time
- Newtonian equations of motion with GR corrections
- Expansion in strength of gravity

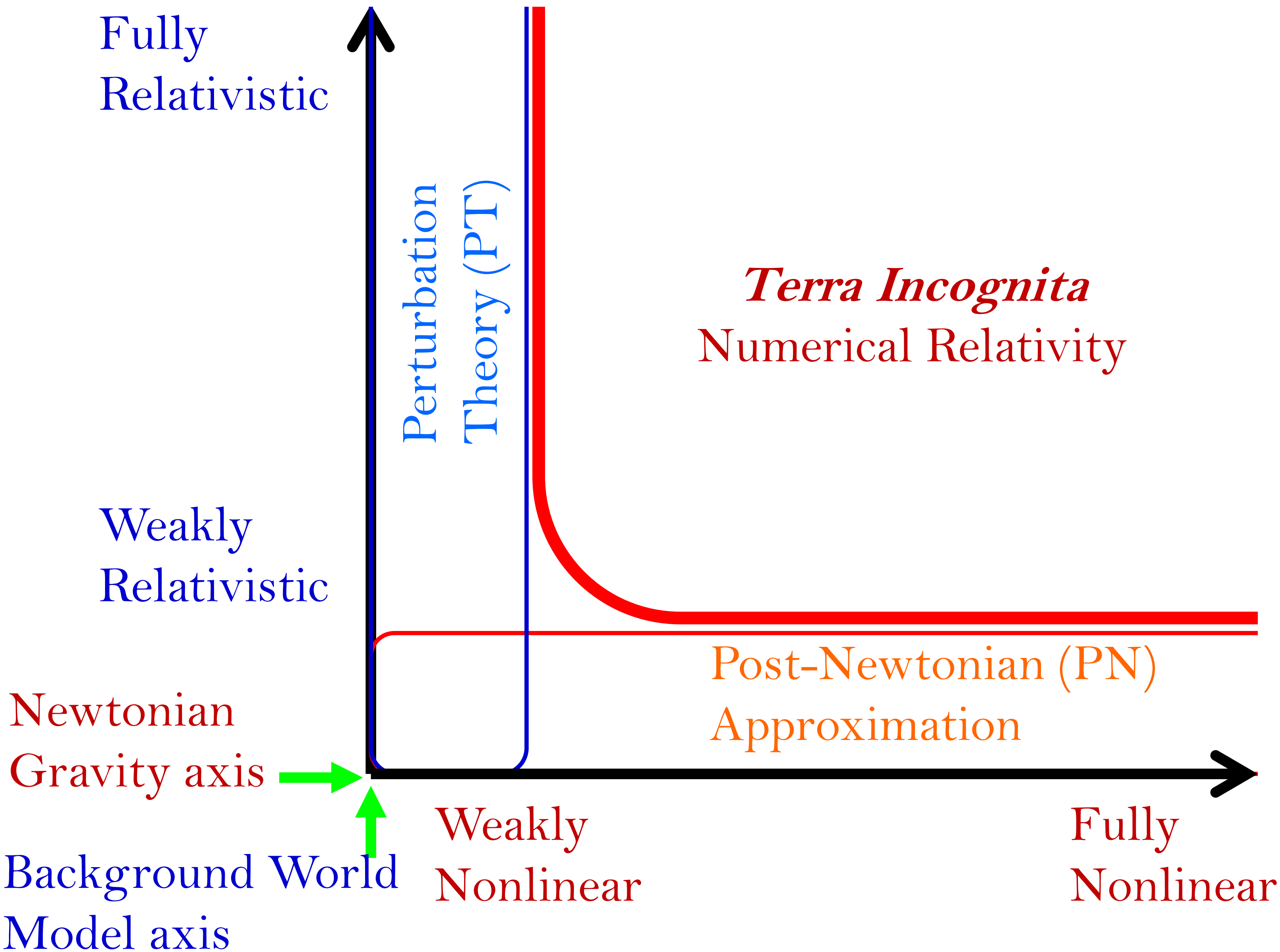
$$\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$$

- Fully nonlinear
- No strong gravity; weakly relativistic
- Valid far inside horizon
- Case of the Fully nonlinear and Exact perturbations

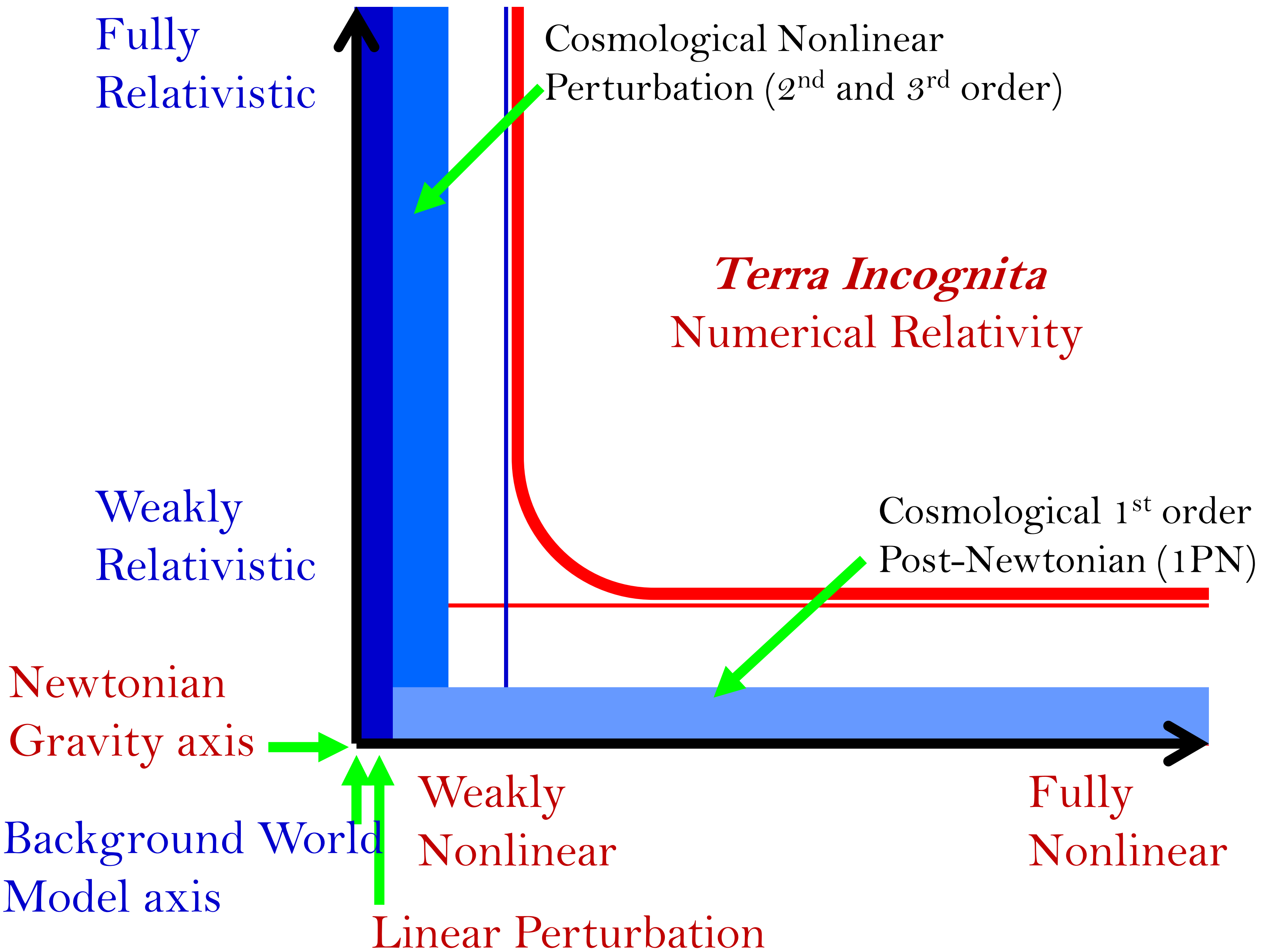
Relativistic vs. Nonlinear Plane of Large-scale Structure



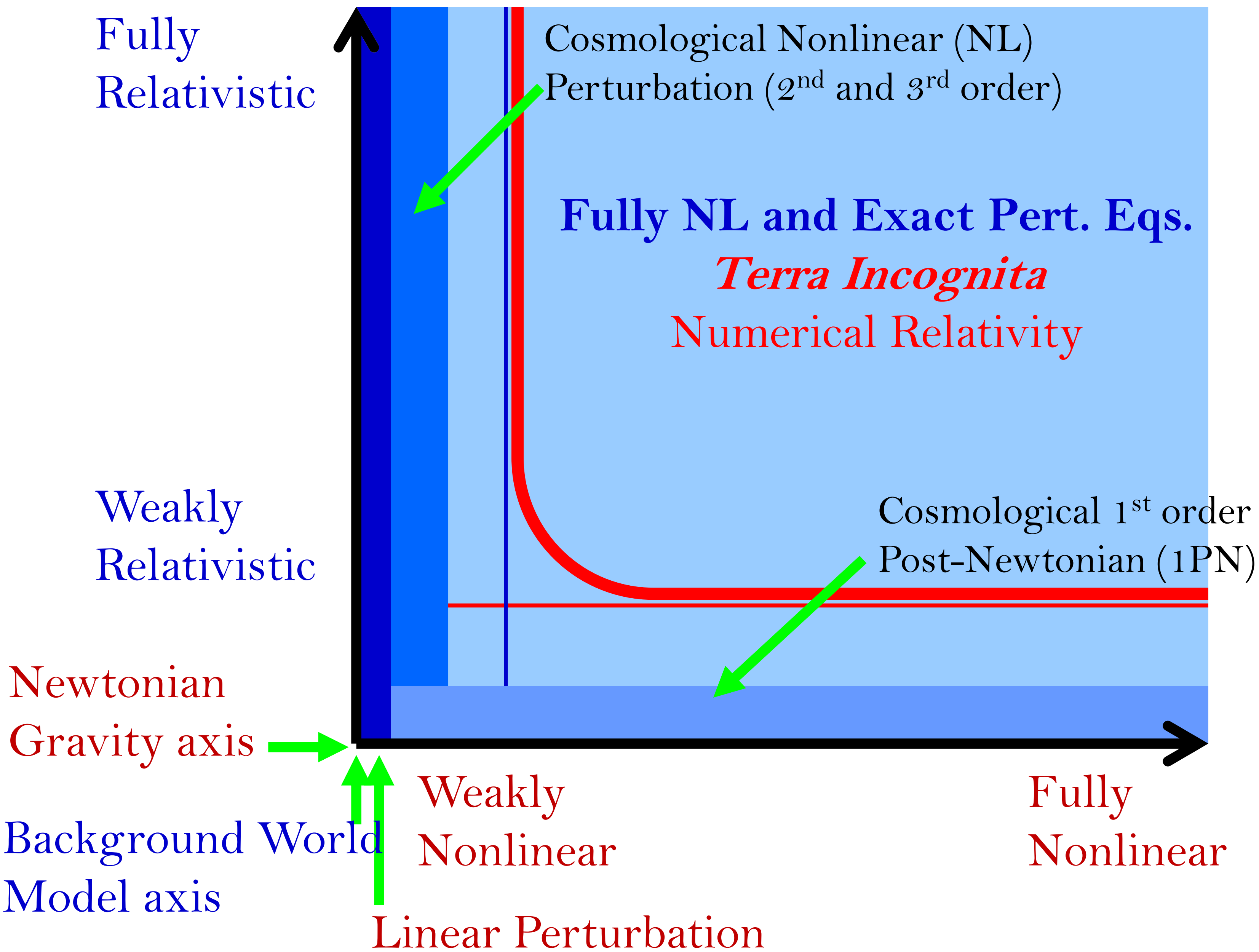
Perturbation vs. Post-Newtonian



Perturbation vs. Post-Newtonian



Fully NL and Exact Perturbation



Newtonian Theory

Newtonian perturbation equations:

Newtonian (0PN) metric: $$ds^2 = -\left(1 - \frac{1}{c^2}2U\right)c^2dt^2 + a^2\delta_{ij}dx^i dx^j$$

Mass conservation: $$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} + \frac{1}{a}\nabla\cdot(\widetilde{\varrho}\mathbf{v}) = 0,$$

Momentum conservation: $$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v} - \frac{1}{a}\nabla U = -\frac{1}{\widetilde{\varrho}}\frac{1}{a}\nabla\widetilde{p},$$

Poisson's equation: $$\frac{\Delta}{a^2}U = -4\pi G\delta\varrho,$$

Relativistic Theory

Convention: (Bardeen 1988, York 1973)

$$ds^2 = -a^2(1+2\alpha)\,d\eta^2 - 2a^2\left(\beta_{,i} + B_i^{(v)}\right)d\eta dx^i$$

Flat background

No tensor-pert.

$$+a^2\left[(1+2\varphi)\,\delta_{ij} + \cancel{2\gamma_{,ij}} + \cancel{C^{(v)}_{i,j}} + \cancel{C^{(v)}_{j,i}} + \cancel{2C^{(t)}_{ij}}\right]dx^i dx^j,$$

$$\chi \equiv a\beta + \cancel{a^2\dot{\gamma}}, \qquad \Psi_i^{(v)} \equiv B_i^{(v)} + \cancel{a\dot{C}_i^{(v)}},$$

No anisotropic stress

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{u}_a\widetilde{u}_b + \widetilde{g}_{ab}\right) + \cancel{\widetilde{\pi}_{ab}},$$

$$\widetilde{\mu} \equiv \mu + \delta\mu \equiv \mu\,(1+\delta)\,, \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{u}_i \equiv a v_i,$$

$$v_i \equiv -v_{,i} + v_i^{(v)},$$

Spatial gauge:

$$\gamma \equiv 0 \equiv C_i^{(v)},$$

$$\chi_i \equiv \chi_{,i} + a\Psi_i^{(v)} = a\left(\beta_{,i} + B_i^{(v)}\right)$$

Complete spatial gauge fixing. Remaining variables are spatially gauge-invariant to fully NL order!

Metric convention:

$$\widetilde{g}_{00} = -a^2\left(1+2\alpha\right), \quad \widetilde{g}_{0i} = -a\chi_i, \quad \widetilde{g}_{ij} = a^2\left(1+2\varphi\right)\delta_{ij},$$

Inverse metric:

$$\widetilde{g}^{00} = -\frac{1}{a^2}\frac{1+2\varphi}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2}, \quad \widetilde{g}^{0i} = -\frac{1}{a^2}\frac{\chi^i/a}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2},$$

$$\widetilde{g}^{ij} = \frac{1}{a^2(1+2\varphi)}\left(\delta^{ij} - \frac{\chi^i\chi^j/a^2}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2}\right).$$

Energy-momentum tensor:

$$\widetilde{T}^0_0 = -\widetilde{\mu} - \frac{\widetilde{\mu}+\widetilde{p}}{1+2\varphi}\left(v^i v_i + \frac{1}{a\mathcal{N}}\chi^i v_i\sqrt{1+\frac{v^k v_k}{1+2\varphi}}\right), \quad \widetilde{T}^0_i = \frac{1}{\mathcal{N}}\left(\widetilde{\mu}+\widetilde{p}\right)v_i\sqrt{1+\frac{v^k v_k}{1+2\varphi}},$$

$$\widetilde{T}_{ij} = a^2\left[(1+2\varphi)\widetilde{p}\delta_{ij} + \left(\widetilde{\mu}+\widetilde{p}\right)v_i v_j\right].$$

$$\mathcal{N} = \sqrt{1+2\alpha+\frac{\chi^k\chi_k}{a^2(1+2\varphi)}},$$

To linear order:

Perturbed Lapse　Acceleration　Curvature perturbation

$$N = 1 - \alpha, \quad a_i = \alpha_{,i}, \quad R^{(h)} = -4\frac{\Delta}{a^2}\varphi$$

$$K^i_i = \frac{1}{c}\left(-3H + \kappa\right), \quad \overline{K}_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\Delta\right)\chi,$$

Perturbed expansion　Shear

Temporal gauge (slicing, hypersurface):

comoving gauge :	$v \equiv 0,$
zero−shear gauge :	$\chi \equiv 0,$
uniform−curvature gauge :	$\varphi \equiv 0,$
uniform−expansion gauge :	$\kappa \equiv 0,$
uniform−density gauge :	$\delta \equiv 0,$
~~synchronous gauge :~~	$\cancel{\alpha \equiv 0.}$

Nonlinear order

Valid to NL order!

Complete gauge fixing, Remaining variables are gauge-invariant to fully NL order!

Equations to linear order **without** taking temporal gauge condition: (Bardeen 1988)

Definition of kappa, $K^i{}_i$

$$\kappa \equiv 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi,$$

ADM E-conservation, $G^0{}_0$

$$4\pi G\delta\mu + H\kappa + \frac{\Delta}{a^2}\varphi = 0,$$

ADM Mom-conservation, $G^0{}_i$

$$\kappa + \frac{\Delta}{a^2}\chi - 12\pi G(\mu + p)av = 0,$$

ADM propagation, trace, $G^i{}_i$

$$\dot{\kappa} + 2H\kappa - 4\pi G\left(\delta\mu + 3\delta p\right) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 0,$$

ADM propagation, tracefree, $G^i{}_j$- ...

$$\dot{\chi} + H\chi - \varphi - \alpha = 0,$$

E-conservation, $T^c{}_{0;c}$

$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right) - (\mu + p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right) = 0,$$

Mom-conservation, $T^c{}_{i;c}$

$$\frac{[a^4(\mu + p)v]^{\cdot}}{a^4(\mu + p)} - \frac{1}{a}\alpha - \frac{\delta p}{a(\mu + p)} = 0,$$

Extended to fully Nonlinear order!

As an example:

E-conservation to NL order:

Covariant E-conservation

$$0 = \widetilde{T}^b_{a;b}\widetilde{u}^a = -\dot{\widetilde{\widetilde{\mu}}} - (\widetilde{\mu} + \widetilde{p})\,\widetilde{\theta},$$

$$0 = \widetilde{T}^b_{a;b}\widetilde{n}^a = \frac{1}{N}\widetilde{T}^b_{0;b} - \frac{N^i}{N}\widetilde{T}^b_{i;b} = -\frac{1}{N}\left(\widetilde{u}_0 - N^i\widetilde{u}_i\right)\widetilde{T}^b_{a;b}\widetilde{u}^a - \frac{1}{N}\left(\frac{\widetilde{u}^i}{\widetilde{u}^0} + N^i\right)\widetilde{T}^b_{a;b}\widetilde{h}^a_i,$$

E-conservation

ADM E-conservation

Covariant E-conservation:

$$\frac{1}{\mathcal{N}}\sqrt{1+\frac{v^k v_k}{1+2\varphi}}\left(\frac{\partial}{\partial t}+\frac{\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\widetilde{\mu} + (\widetilde{\mu}+\widetilde{p})\,(3H-\kappa)\sqrt{1+\frac{v^k v_k}{1+2\varphi}} = -\frac{1}{a(1+2\varphi)}\widetilde{\mu}_{,i}v^i$$
$$-(\widetilde{\mu}+\widetilde{p})\left[\frac{(\mathcal{N}v^i)_{,i}}{a\mathcal{N}(1+2\varphi)}+\frac{v^i\varphi_{,i}}{a(1+2\varphi)^2}+\frac{1}{\mathcal{N}}\left(\frac{\partial}{\partial t}+\frac{\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\sqrt{1+\frac{v^k v_k}{1+2\varphi}}\right],$$

Covariant E-conservation:

$$\frac{1}{\mathcal{N}}\sqrt{1+\cancel{\frac{v^k v_k}{1+2\varphi}}}\left(\frac{\partial}{\partial t}+\frac{\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\widetilde{\mu}+(\widetilde{\mu}+\cancel{\widetilde{p}})\,(3H-\kappa)\sqrt{1+\cancel{\frac{v^k v_k}{1+2\varphi}}}=-\frac{1}{a(1+2\varphi)}\widetilde{\mu}_{,i}\cancel{v^i}$$
$$-(\widetilde{\mu}+\widetilde{p})\left[\cancel{\frac{(\mathcal{N}v^i)_{,i}}{a\mathcal{N}(1+2\varphi)}}+\cancel{\frac{v^i\varphi_{,i}}{a(1+2\varphi)^2}}+\frac{1}{\mathcal{N}}\left(\frac{\partial}{\partial t}+\frac{\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\cancel{\sqrt{1+\frac{v^k v_k}{1+2\varphi}}}\right],$$

Comoving gauge + irrotational ($v^i = 0$) + zero-pressure:

$$\Big(\underbrace{\frac{\dot{\mu}}{\mu}+3H}_{=\ 0,\ \textbf{Background order}}\Big)(1+\delta)+\dot{\delta}-\kappa=\delta\kappa-\frac{\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

To 12th-order perturbation:

$$\dot{\delta}-\kappa \;=\; \delta\kappa-\frac{1}{a^2}\chi^{,i}\delta_{,i}\Big(1-2\varphi+4\varphi^2-8\varphi^3+16\varphi^4-32\varphi^5$$
$$+64\varphi^6-128\varphi^7+2566\varphi^8-512\varphi^9+1024\varphi^{10}\Big),$$

Newtonian Limit

c→∞ Limit in ZSG & UEG:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad v^k v_k \ll 1, \quad \tilde{p} \ll \tilde{\varrho} c^2. \quad \frac{c^2\Delta}{a^2 H^2} \gg 1,$$

ZSG

1

$$\cancel{\frac{1}{\mathcal{N}}}\sqrt{1+\cancel{\frac{v^k v_k}{1+2\varphi}}}\left(\frac{\partial}{\partial t}+\cancel{\frac{c\chi^i}{a^2(1+2\varphi)}}\nabla_i\right)\tilde{\varrho}+\left(\tilde{\varrho}+\cancel{\frac{\tilde{p}}{c^2}}\right)\left(3\frac{\dot{a}}{a}-\kappa\right)\sqrt{1+\cancel{\frac{v^k v_k}{1+2\varphi}}}=-\frac{c}{a(1+\cancel{2\varphi})}\tilde{\varrho}_{,i}v^i$$

$$-\left(\tilde{\varrho}+\cancel{\frac{\tilde{p}}{c^2}}\right)\left[\frac{c(\mathcal{N}v^i)_{,i}}{a\mathcal{N}(1+\cancel{2\varphi})}+\cancel{\frac{cv^i\varphi_{,i}}{a(1+2\varphi)^2}}+\frac{1}{\mathcal{N}}\left(\frac{\partial}{\partial t}+\frac{c\chi^i}{a^2(1+2\varphi)}\nabla_i\right)\cancel{\sqrt{1+\frac{v^k v_k}{1+2\varphi}}}\right],$$

Covariant E-conservation

$$\alpha = -\frac{1}{c^2}U, \quad \varphi = \frac{1}{c^2}V, \quad v^k = \frac{1}{c}\mathbf{v},$$

$$\kappa = -\frac{12\pi G \varrho a}{c^2\Delta}\nabla\cdot\left[(1+\delta)\,\mathbf{v}\right],$$ ADM momentum-constraint

$$\varphi = -\alpha.$$ Tracefree ADM propagation

$$\dot{\tilde{\varrho}} + 3\frac{\dot{a}}{a}\tilde{\varrho} + \frac{1}{a}\nabla\cdot(\tilde{\varrho}\mathbf{v}) = 0$$

Case with relativistic pressure:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad v^k v_k \ll 1, \quad \tilde{p} \cancel{\ll} \tilde{\varrho} c^2. \quad \frac{c^2 \Delta}{a^2 H^2} \gg 1,$$

$$\dot{\tilde{\varrho}} + 3\frac{\dot{a}}{a}\left(\tilde{\varrho} + \frac{\tilde{p}}{\underline{c^2}}\right) + \frac{1}{a}\nabla \cdot (\tilde{\varrho}\mathbf{v}) = \underline{\frac{1}{c^2}\frac{1}{a}\left(\mathbf{v} \cdot \nabla \tilde{p} - \tilde{p}\nabla \cdot \mathbf{v}\right)},$$

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{a}\nabla U = -\frac{1}{\tilde{\varrho} + \underline{\tilde{p}/c^2}}\left(\frac{1}{a}\nabla \tilde{p} + \underline{\frac{\dot{\tilde{p}}}{c^2}\mathbf{v}}\right),$$

$$\frac{\Delta}{a^2}U = -4\pi G\left(\delta\varrho + \underline{3\frac{\delta p}{c^2}}\right),$$

Previous works were not successful in guessing the correct form.
(McCrea 1951; Harrison 1965; Coles & Lucchin 1995; Lima et al. 1997; Harko 2011)

Post-Newtonian Limit

1PN convention: (Chandrasekhar 1965)

$$ds^2 = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right]c^2dt^2 - \frac{1}{c^3}2aP_i c dt dx^i + a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij}dx^i dx^j,$$

$$\widetilde{\mu} \equiv \mu \equiv \varrho c^2\left(1 + \frac{1}{c^2}\Pi\right), \quad \widetilde{p} = p, \quad \widetilde{u}^i \equiv \frac{1}{c}\frac{1}{a}\overline{v}^i\widetilde{u}^0,$$

Identification:

$$\alpha = -\frac{1}{c^2}\left[U - \frac{1}{c^2}\left(U^2 - 2\Phi\right)\right], \quad \overset{\text{PT}}{\varphi} = \frac{1}{c^2}\overset{\text{1PN}}{V}, \quad \kappa = -\frac{1}{c^2}\left(3\frac{\dot{a}}{a}U + 3\dot{V} + \frac{1}{a}P^k{}_{,k}\right),$$

$$\chi_i = \frac{1}{c^3}aP_i, \quad v_i = \frac{1}{c}\left\{\overline{v}_i + \frac{1}{c^2}\left[\overline{v}_i\left(\frac{1}{2}\overline{v}^2 + U + 2V\right) - P_i\right]\right\},$$

1PN equations! (JH, Noh & Puetzfeld 2008)

Covariant E-conservation:

$$\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\left(\varrho\overline{v}^i\right)_{|i} = -\frac{1}{c^2}\left[\varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\overline{\mathbf{v}}\cdot\nabla\right)\left(\frac{1}{2}\overline{v}^2 + 3U + \Pi\right) + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\overline{\mathbf{v}}\right)p\right],$$

Fully NL and exact cosmological pert.:

1. Formulation arXiv:1207.0264
2. Multi-component fluids
3. Minimally coupled scalar field
4. Newtonian limit arXiv:1210.0676
5. Relativistic pressure
6. 1PN equations

Future extentions:

1. Anisotropic stress $\Rightarrow$ Relativistic magneto-hydrodynamics
2. Background curvature
3. Light propagation (geodesic, Boltzmann)
4. 2 and higher-PN equations
5. Gauge-invariant combinations

Applications:

1. NL perturbations
2. Fitting and Averaging
3. Backreaction
4. Relativistic cosmological numerical simulation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