"Is Newton's gravity reliable in large-scale cosmological simulations?"

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Content

1. Cosmological perturbation theory

Methods:

- **Relativistic:**
  1. Einstein equations (Lifshitz 1946)
  2. Covariant equations ($1+3$, $\tilde{u}_a$; Hawking 1966)
  3. ADM equations ($3+1$, $\tilde{n}_a$; Bardeen 1980)
  4. Action formulation (Lukash 1980; Mukhanov 1988)
- **Newtonian:**
  1. Hydrodynamic equations (Bonnor 1957)

★ Relativistic-Newtonian **correspondence** in the zero-pressure case.

★ **True** even to the second order!

Three perturbation types:

1. Scalar-type: density fluctuations
2. Vector-type: rotation
3. Tensor-type: gravitational waves

★ To linear-order, **decouple** in Friedmann background

★ **Couple** to the second order!

Classical Evolution:

1. Scalar-type: super-sound-horizon scale conservation
2. Rotation: angular momentum conservation
3. Gravitational waves: super-horizon scale conservation

★ **True** even to the second order!

"The theory of linear (i.e., small) perturbations of the expanding, isotropic, and homogeneous Friedmann cosmology springs into existence virtually full-grown with the work of Lifshitz (1946)."

Press and Vishniac (1980)

Evgeniĭ Mikhaĭlovich Lifshitz (1915 - 1985)

E. M. Lifshitz

Why linear theory?:

1. The CMB temperature anisotropies are very small $\frac{\delta T}{T} \sim 10^{-5}$.
2. The large-scale clustering of galaxies are approximately linear as the scale becomes large.

   Our own homogeneous and isotropic background world model relies on this assumption.

   Observations are **not inconsistent** with the assumption.

If the fluctuation is on $\sim 10^{-5}$ level, Taylor's series theorem guarantees the non-linear terms are small $\sim 10^{-10}$.

Still, considering that the basic equations are fully nonlinear the nonlinearities exist always. The point is whether we can ignore (or tolerate) the level of nonlinearities.

May be we can **assume** linearity in the early universe and in the large-scale in the present era.

If the situation is linear, then we can handle both physics and mathematics very reliably.

> *"The evolution of linear perturbations of FRW models has been discussed by a large number of authors and is very nearly a closed book."*
>
> George Efstathiou (1989)

Perturbed Friedmann world model:

Metric:

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a^2(\beta_{,\alpha} + B^{(v)}_\alpha) d\eta dx^\alpha$$
$$+ a^2 \left[g^{(3)}_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(v)}_{(\alpha|\beta)} + 2C^{(t)}_{\alpha\beta} \right] dx^\alpha dx^\beta. \quad (1)$$

Energy momentum tensor:

$$\tilde{T}^0_0 \equiv -\mu - \delta\mu, \quad \tilde{T}^0_\alpha \equiv (\mu + p)(-v_{,\alpha} + v^{(v)}_\alpha), \quad \tilde{T}^\alpha_\beta \equiv (p + \delta p)\,\delta^\alpha_\beta + \Pi^\alpha_\beta. \quad (2)$$

Linear perturbation assumes all perturbation variables are small.
Thus, ignore any quadratic and higher-order combination of perturbation variables.

Zero-pressure, irrotational fluid:

$$\tilde{T}^0_0 = -\tilde{\mu}, \quad \tilde{T}^0_\alpha = 0 = \tilde{T}^\alpha_\beta. \quad (3)$$

Temporal comoving gauge without rotation gives $\tilde{T}^0_\alpha = 0$.

Newtonian vs. Relativistic:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0. \quad (4)$$

★ Coincides in the zero-pressure case.
Energy density μ = mass density ϱ in the Newtonian case.

In the presence of pressure:

Comoving gauge: $(v \equiv 0)$

$$\ddot{\delta}_v + (2 + 3c_s^2 - 6w)H\dot{\delta}_v + \left[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} \right.$$

$$\left. +(3c_s^2 - 5w)\Lambda \right] \delta_v = \frac{1+w}{a^2 H} \left[\frac{H^2}{a(\mu + p)} \left(\frac{a^3 \mu}{H} \delta_v \right) \right]^{\cdot} - c_s^2 \frac{\Delta}{a^2}\delta_v = \text{stresses.} \tag{5}$$

★ **Valid for** general K, Λ, and time varying $p = p(\mu)$; $w \equiv \frac{p}{\mu}$, $c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}$.

★ Pressure is **purely relativistic** even to the linear order.

Synchronous gauge: $(\alpha \equiv 0)$

Incorrect one in the synchronous gauge $(\alpha \equiv 0)$ (for $K = 0 = \Lambda$, $w =$ const., no stress):

$$\ddot{\delta} + 2H\dot{\delta} + \left[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1+w)(1+3w) \right] \delta = 0. \tag{6}$$

Weinberg (72), Peebles (93), Coles-Lucchin (95,02), Moss (96), Padmanabhan (96), Longair (98), Peacock (99), ...

Apparently, this is a popular error in textbooks. For corrections, see.[1]

★ Due to the presence of gauge modes, it is **not possible** to derive a second order differential equation in the presence of pressure even in the large-scale limit!

[1] Gen. Rel. Grav. **23**, 235 (1991); **31**, 1131 (1999).

"the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible ..."

Sachs and Wolfe (1967)

Perturbed action: (Lukash 1980; Mukhanov 1988)

$$\delta^2 S = \frac{1}{2}\int a^3 Q\left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2}\Phi^{,\alpha}\Phi_{,\alpha}\right) dt d^3x, \tag{7}$$

where

$$\begin{cases} \Phi = \varphi_v & Q = \frac{\mu+p}{c_s^2 H^2} & c_A^2 \to c_s^2 & \text{(fluid)} \\ \Phi = \varphi_{\delta\phi} & Q = \frac{\dot{\phi}^2}{H^2} & c_A^2 \to 1 & \text{(field)} \\ \Phi = C^{(t)}_{\alpha\beta} & Q = \frac{1}{8\pi G} & c_A^2 \to 1 & \text{(GW)} \end{cases}$$

Comoving gauge

Uniform-field gauge

$\varphi_v \equiv \varphi - aHv$ and $\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi$: gauge-invariant combinations.

★ Generalized gravity theories as well!

Equation of motion (Field-Shepley 1968) $v \equiv z\Phi$ and $z \equiv a\sqrt{Q}$:

$$\frac{1}{a^3 Q}\left(a^3 Q \dot{\Phi}\right)^{\cdot} - c_A^2 \frac{\Delta}{a^2}\Phi = \frac{1}{a^2 z}\left[v'' - \left(\frac{z''}{z} + c_A^2 \Delta\right) v\right] = 0. \tag{8}$$

Large-scale solution:

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}) - D(\mathbf{x})\int_0^t \frac{dt}{a^3 Q}. \tag{9}$$

Growing mode in expanding phase

Unified Analyses in Generalized $f(\phi, R)$ gravity:[2]

1. Generalized $f(\phi, R)$ gravity:

$$\tilde{S} = \int \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{;c}\tilde{\phi}_{,c} - V(\tilde{\phi}) + \tilde{L}_{(c)} \right] \sqrt{-\tilde{g}} d^4x. \quad (10)$$

2. Tachyonic generalization: $\tilde{X} \equiv \frac{1}{2}\tilde{\phi}^{;c}\tilde{\phi}_{,c}$

$$\tilde{S} = \int \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}, \tilde{X}) + \tilde{L}_{(c)} \right] \sqrt{-\tilde{g}} d^4x. \quad (11)$$

3. String corrections:

$$\tilde{L}_{(c)} = \xi(\tilde{\phi}) \Big[c_1 \left(\tilde{R}^{abcd}\tilde{R}_{abcd} - 4\tilde{R}^{ab}\tilde{R}_{ab} + \tilde{R}^2 \right) + c_2 \tilde{G}^{ab}\tilde{\phi}_{,a}\tilde{\phi}_{,b} + c_3 \tilde{\phi}^{;a}{}_{a}\tilde{\phi}^{;b}\tilde{\phi}_{,b} + c_4 (\tilde{\phi}^{;a}\tilde{\phi}_{,a})^2 \Big]. \quad (12)$$

4. String axion coupling:

$$\tilde{L}_{(c)} = \frac{1}{8}\nu(\tilde{\phi})\tilde{\eta}^{abcd}\tilde{R}_{ab}{}^{ef}\tilde{R}_{cdef}. \quad (13)$$

We can always derive a unified form:

$$\delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2}\Phi^{,\alpha}\Phi_{,\alpha} \right) dt d^3x. \quad (14)$$

★ Perhaps **“surprisingly simple”** indeed!

[2] Phys. Rev. D **71**, 063536 (2005).

2. Two theories of gravity

- **Newton (1647-1727):** "Philosophiae naturalis principia mathematica" (1687)

  "But hitherto I have not been able to discover the cause of those properties of gravity from phaenomena, and I frame no hypotheses; for whatever is not deduced from the phaenomena, is to be called an hypotheses; an hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. ... And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea [sun?]."

  Isaac Newton (1713) [3]

  On this regard, Einstein's gravity is no better.

- **Einstein (1879-1955):** "Die Feldgleichungen der Gravitation" (1915) [4]

  "Let us put

$$G_{im} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right)$$

  [where G_{im} is the Ricci tensor]."

★ **In practice, however, Einstein's gravity provides much better perspective.**

[3] Newton, I., 1713, The mathematical principles of natural philosophy, 2nd edition, Book III, General Scholium; Translated into English by Motte, A. in 1729, 1962 (University of California Press).

[4] Einstein, A., Preuss. Akad. Wiss. Berlin, Sitzber., 844-847 (1915); Translated in Misner, C. W., Thorne, K. S., and Wheeler, J. A., 1973, Gravitation, (Freeman and Company) p. 433.

Newton's gravity:

- Non-relativistic (no c)
  - Action at a distance, violates causality
  - $c \to \infty$ limit of Einstein gravity
  - No horizon
  - Static nature
- No strong pressure allowed
- No strong gravity allowed
- No gravitational waves
- Incomplete and inconsistent

Einstein's gravity:

- Relativistic gravity
- Strong gravity, dynamic
- Simplest

★ The two theories give the same descriptions for the cosmological world model and its linear structures.

World model: spatially homogeneous and isotropic world model

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\mu - \frac{\text{const.}}{a^2} + \frac{\Lambda}{3}, \quad \mu \propto a^{-3}. \tag{15}$$

- **Relativistic (Friedmann 1922)** [5]
- **Newtonian (Milne-McCrea 1933)** [6]

Structures: linear perturbations

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0. \tag{16}$$

- **Relativistic (Lifshitz 1946)** [7]
- **Newtonian (Bonnor 1957)** [8]

"It is curious that it took so long for these dynamic models to be discovered after the (more complex) general relativity models were known."

G. F. R. Ellis (1989) [9]

★ **In fact, the known "Newtonian cosmology" is a GR guided version!**

[5] Friedmann A. A., 1922, Zeitschrift für Physik, **10**, 377; translated in Bernstein J., Feinberg G., eds, 1986, Cosmological-constants: papers in modern cosmology, Columbia Univ. Press, New York, p. 49

[6] Milne E. A., 1934, Quart. J. Math., **5**, 64; McCrea W. H., Milne E. A., 1934, Quart. J. Math., **5**, 73

[7] Lifshitz E. M., 1946, J. Phys. (USSR), **10**, 116

[8] Bonnor W. B., 1957, MNRAS, **117**, 104

[9] Ellis, G. F. R., 1989, in Einstein and the history of general relativity, ed. D. Howard and J. Stachel (Berlin, Birkhäuser), 367

3. Weakly Nonlinear Perturbations

"*the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible ... One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects.*"

Sachs and Wolfe (1967) [10]

[10] Sachs, R. K., Wolfe, A. M., ApJ, **147**, 73 (1967)

3.1 Second-order: Relativistic-Newtonian correspondence

Newtonian:

Mass conservation, momentum conservation, Poisson's equation: [11]

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u})\,, \tag{17}$$

$$\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{18}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\delta\varrho, \tag{19}$$

give

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\,. \tag{20}$$

★ These equations are valid to **fully nonlinear order**!

Relativistic: (irrotational, $K = 0$, but for general Λ)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}\nabla^\alpha u^\beta + \dot{C}^{(t)\alpha\beta}\right). \tag{21}$$

★ This equation is valid only to the **second-order**!

[11] Peebles, P. J. E., *The large-scale structure of the universe* (1980).

A proof

Fully nonlinear covariant equations:

The energy conservation, Raychaudhury equation become:

$$\overset{\tilde{\cdot}}{\tilde{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \tag{22}$$

$$\overset{\tilde{\cdot}}{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \tag{23}$$

where $\overset{\tilde{\cdot}}{\tilde{\mu}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$, $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$, etc. By combining

$$\left(\frac{\overset{\tilde{\cdot}}{\tilde{\mu}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\overset{\tilde{\cdot}}{\tilde{\mu}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{24}$$

To the second-order perturbation:

By **identifying**

$$\boxed{\delta\mu_v \equiv \delta\varrho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u},} \tag{25}$$

(22,23) give

temporal comoving (v=0) gauge,
spatial γ=0 gauge

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{26}$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{27}$$

Combining (26,27) or (24) give (21).

Relativistic-Newtonian correspondence [12]

Background world model:

Relativistic (Friedmann 1922) vs. **Newtonian (Milne-McCrea 1934)**

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{\text{const.}}{a^2} + \frac{\Lambda}{3}, \quad \varrho \propto a^{-3}. \tag{28}$$

Linear perturbation:

Relativistic (Lifshitz 1946) vs. **Newtonian (Bonnor 1957)**

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = 0. \tag{29}$$

Second-order perturbation:

Newtonian (Peebles 1980) vs. **Relativistic (Noh-Hwang 2004)**

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\alpha u_\beta + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{30}$$

Except for the gravitational wave contribution, the relativistic zero-pressure fluid perturbed to second order in a flat Friedmann background **coincides exactly** with the Newtonian system.

> *"the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966)"*
>
> Sachs and Wolfe (1967)

covariant equations

[12] Phys. Rev. D **69**, 104011 (2004); Class. Quant. Grav. **22**, 3181 (2005); Phys. Rev. D **72**, 044011 (2005); **73**, 044021 (2006).

Assumptions:

Our relativistic/Newtonian correspondence includes Λ, but assumes:

1. Flat Friedmann background
2. Zero-pressure
3. Irrotational
4. Single component fluid
5. No gravitational waves
6. Second order in perturbations

★ Relaxing any of these assumptions could lead to pure general relativistic effects!

1. Background curvature $\Rightarrow$ corrections
2. Pressure: relativistic even to the linear order!
3. Rotation $\Rightarrow$ corrections
4. Multi-component zero-pressure irrotational fluids $\Rightarrow$ Newtonian correspondence!
5. Gravitational waves $\Rightarrow$ corrections
6. Third order in perturbations $\Rightarrow$ corrections

3.2 Third-order: Pure general relativistic corrections [13]

To the third order we identify:

$$\delta\mu_v \equiv \delta\varrho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}. \tag{31}$$

For pure scalar-type perturbation (22,23) give:

pure relativistic corrections

$$\begin{aligned}\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta &= -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \\ &+\frac{1}{a^2}\left\{a\left[2\varphi\mathbf{u} - \nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta\right\}^{\cdot} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right] \\ &+\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{u}\right) + \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla\left(\Delta^{-1}X\right)\right] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},\end{aligned} \tag{32}$$

$$X \equiv 2\varphi_v\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi_v + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla\left(\nabla\varphi_v\right) + \mathbf{u}\Delta\varphi_v\right]. \tag{33}$$

The first non-vanishing pure relativistic correction terms are φ_v order higher than the Newtonian terms ($\varphi_v = \varphi$ in the comoving gauge). We have for general Λ [14]

$$\boxed{\dot{\varphi}_v = 0.} \tag{34}$$

The CMB temperature anisotropy gives, for $\Lambda = 0$, near horizon scale [15]

$$\boxed{\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi_v \sim 10^{-5}.} \tag{35}$$

[13] Phys. Rev. D, **72**, 044012 (2005).
[14] Gen. Rel. Grav. **31**, 1131 (1999).
[15] Phys. Rev. D **59**, 067302 (1999).

Conclusions in the zero-pressure case:

1. Except for the gravitational wave contribution, equations for the relativistic zero-pressure fluid in a flat Friedmann background **coincide exactly** with the previously known Newtonian equations even to the second-order perturbation.

2. To the second order, we correctly identify the relativistic density and velocity perturbation variables. In the relativistic analyses, however, we do **not** have a relativistic variable which corresponds to the Newtonian potential to the second order.

3. We assume a flat Friedmann background but include the **cosmological constant**, thus relevant to currently favoured cosmology.

4. We *expand* the range of applicability of the Newtonian medium without pressure to **all cosmological scales** including the super-horizon scale.

5. The third-order correction terms, thus the **pure general relativistic effects**, are of φ_v-order higher than the second-order Newtonian terms.

6. The corrections terms are **independent of the horizon** scale and depend only on the linear order gravitational potential (curvature) perturbation strength.

7. From the temperature anisotropy of CMB we have $\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi_v \sim 10^{-5}$.

8. **Therefore, now we can use the large-scale Newtonian numerical simulation more reliably even as the simulation scale approaches near (and goes beyond) the horizon.**

4. Cosmological post-Newtonian Approach

Perturbation method:

- Perturbation expansion.
- All perturbation variables are small.
- Weakly nonlinear.
- Strong gravity; fully relativistic!
- Valid in all scales!

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time.
- Provide GR correction terms in the Newtonian equations of motion.
- Expansion in v/c:

$$\frac{GM}{\lambda c^2} \sim \left(\frac{v}{c}\right)^2 \ll 1. \tag{36}$$

- Fully nonlinear!
- No strong gravity situation; weakly relativistic.
- Valid far inside horizon $\frac{GM}{\lambda c^2} \sim \left(\frac{\lambda}{c/H}\right)^2 \ll 1.$

Complementary!

Metric:

Newtonian limit:

$$\tilde{g}_{00} = -\left(1 - \frac{1}{c^2} 2U\right), \quad \tilde{g}_{0i} = 0, \quad \tilde{g}_{ij} = \delta_{ij}. \tag{37}$$

1PN metric[16]:

$$\begin{aligned} \tilde{g}_{00} &= -\left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6}, \\ \tilde{g}_{0i} &= -\frac{1}{c^3} P_i + \mathcal{O}^{-5}, \\ \tilde{g}_{ij} &= \left(1 + \frac{1}{c^2} 2U\right) \delta_{ij} + \mathcal{O}^{-4}. \end{aligned} \tag{38}$$

Minkowski background

Cosmological 1PN metric[17]:

$$\begin{aligned} \tilde{g}_{00} &\equiv -\left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6}, \\ \tilde{g}_{0i} &\equiv -\frac{1}{c^3} a^2 P_i + \mathcal{O}^{-5}, \\ \tilde{g}_{ij} &\equiv a^2 \left(1 + \frac{1}{c^2} 2V\right) \gamma_{ij} + \mathcal{O}^{-4}. \end{aligned} \tag{39}$$

Robertson-Walker background

[16] Chandrasekhar, S., 1965, ApJ, **142**, 1488.
[17] Preprint, astro-ph/0507085.

Energy-momentum tensor:

Covariant decomposition:

$$\tilde{T}_{ab} = \tilde{\varrho}c^2 \left(1 + \frac{1}{c^2}\tilde{\Pi}\right) \tilde{u}_a \tilde{u}_b + \tilde{p}\left(\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab}\right) + 2\tilde{q}_{(a}\tilde{u}_{b)} + \tilde{\pi}_{ab}, \tag{40}$$

where $\tilde{q}_a \tilde{u}^a \equiv 0$, $\tilde{\pi}_{ab}\tilde{u}^b \equiv 0$, $\tilde{\pi}^c_c \equiv 0$, and $\tilde{\pi}_{ab} \equiv \tilde{\pi}_{ba}$.
Fluid four vector, $\tilde{u}_a$, follows from $\tilde{u}^a \tilde{u}_a \equiv -1$ and $\tilde{u}^i \equiv \frac{v^i}{c}\tilde{u}^0$.
We introduce

$$\tilde{\varrho} \equiv \varrho, \quad \tilde{\Pi} \equiv \Pi, \quad \tilde{p} \equiv p, \quad \tilde{q}_i \equiv \frac{1}{c} Q_i, \quad \tilde{\pi}_{ij} \equiv \Pi_{ij}. \tag{41}$$

Newtonian limit:

$$\frac{1}{a^3}\left(a^3 \varrho\right)^{\cdot} + \frac{1}{a}\nabla_i \left(\varrho v^i\right) = 0, \tag{42}$$

$$\frac{1}{a}\left(a v_i\right)^{\cdot} + \frac{1}{a} v^j \nabla_j v_i + \frac{1}{a\varrho}\left(\nabla_i p + \nabla_j \Pi^j_i\right) - \frac{1}{a}\nabla_i U = 0, \tag{43}$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{p}{\varrho} + \frac{1}{\varrho a}\left({Q^i}_{|i} + \Pi^i_j {v^j}_{|i}\right) = 0, \tag{44}$$

$$\frac{\Delta}{a^2} U + 4\pi G \left(\varrho - \varrho_b\right) = 0. \tag{45}$$

★ No gauge condition used!

★ We subtract the Friedmann background equation.

1PN equations:

For $K=0$, we have $V=U$. In a gauge-ready form (assuming an ideal fluid):

1PN order

determined by Einstein's equation

$$\frac{1}{a^3}\left(a^3\varrho^*\right)^{\cdot} + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} = 0, \tag{46}$$

$$\frac{1}{a}\left(a v_i^*\right)^{\cdot} + \frac{1}{a} v^*_{i|j} v^j = -\frac{1}{a}\left(1+\frac{1}{c^2}2U\right)\frac{p_{,i}}{\varrho^*} + \frac{1}{a}\left[1+\frac{1}{c^2}\left(\frac{3}{2}v^2 - U + \Pi + \frac{p}{\varrho}\right)\right]U_{,i} + \frac{1}{c^2}\frac{1}{a}\left(2\Phi_{,i} - v^j P_{j|i}\right), \tag{47}$$

where

$$\varrho^* \equiv \varrho\left[1+\frac{1}{c^2}\left(\frac{1}{2}v^2+3U\right)\right], \quad v_i^* \equiv v_i + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + 3U + \Pi + \frac{p}{\varrho}\right)v_i - P_i\right]. \tag{48}$$

Metric variables (potentials) U, Φ and P_i are determined by

$$\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) + \frac{1}{c^2}\left\{\frac{1}{a^2}\left[2\Delta\Phi - 2U\Delta U + \left(aP^i{}_{|i}\right)^{\cdot}\right] + 3\ddot{U} + 9\frac{\dot{a}}{a}\dot{U} + 6\frac{\ddot{a}}{a}U + 8\pi G\left[\varrho v^2 + \frac{1}{2}\left(\varrho\Pi - \varrho_b\Pi_b\right) + \frac{3}{2}\left(p-p_b\right)\right]\right\} = 0, \tag{49}$$

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i + \frac{1}{a}\left(\frac{1}{a}P^j{}_{|j} + 4\dot{U} + 4\frac{\dot{a}}{a}U\right)_{,i}. \tag{50}$$

★ We can impose a temporal gauge condition on $P^i{}_{|i}$.

★ 1PN correction terms are $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \sim 10^{-5}$ order **smaller** than the Newtonian terms.

5. Why Newton's gravity is reliable in large-scale cosmological simulations[18]

Fully relativistic weakly nonlinear perturbation approach:

1. Except for the gravitational wave contribution, equations for the relativistic zero-pressure fluid in a flat Friedmann background **coincide exactly** with the previously known Newtonian equations even to the second-order perturbation.

2. The third-order correction terms, thus the pure general relativistic effects, are of φ_v-order higher than the second-order Newtonian terms. These are **independent of the horizon** scale, and are **small** with $\varphi_v \sim 5 \times 10^{-5}$.

Fully nonlinear weakly relativistic post-Newtonian approach:

1. 1PN correction terms are $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \sim 10^{-5}$ order **smaller** than the Newtonian terms.

2. We cannot rule out possible presence of cumulative effects due to the time-delayed propagation of the relativistic gravitational field, in contrast to the Newtonian case where changes in the gravitational field are felt instantaneously.

3. We provide complete 1PN equations in a gauge-ready form.

★ Therefore, we can use Newtonian numerical simulations reliably during matter dominated era with cosmological constant in nearly all relevant cosmological scales.

[18] MNRAS **367** 1515 (2006).