Entanglement of Electron-spins in Many-Body Systems

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Entanglement Entanglement Measure for Two Qubits

- **Concurrence, Entanglement of formation, Negativity,$\cdots$**
- **Concurrence**

1. Pure state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$C(|\psi\rangle) \;=\; 2|ad - bc|$$

2. Mixed state

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix}$$

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

λ_i^2 **are eigenvalues of** $\rho\tilde{\rho}$

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

Many-Body Systems

- **Electron gas**
- **Superconductors**
- **Fractional quantum Hall effect**
- **Strongly correlated systems**

Entanglement of two particles in many-body systems ?

$$\boxed{|\Phi^{(N)}\rangle \quad \Longrightarrow \quad \rho^{(2)} \quad \Longrightarrow \quad C(\rho^{(2)})}$$

1. **Tracing $\rho^{(N)} = |\Phi^{(Ni)}\rangle\langle\Phi^{(N)}|$ out over all the degrees of freedom except two particcle i and j**
2. **Green's function approach**

Density Matrix

- **Density matrix for a pure state**

$$\rho = |\phi\rangle\langle\phi|$$

$$\begin{aligned} \rho(x;x') &= \langle x|\rho|x'\rangle \\ &= \phi^*(x')\phi(x) \qquad \mathrm{Tr}\{\rho(x;x')\} = 1 \end{aligned}$$

- **Density matrix for a mixed state**

$$\begin{aligned} \rho &= \sum_i p_i|\phi_i\rangle\langle\phi_i|, \quad \sum_i p_i = 1 \\ &= \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \end{aligned}$$

$\mathrm{Tr}\{\rho\} = 1, \quad \mathrm{Tr}\{\rho^2\} \leq 1$

Density Matrix in 2nd Quantization

- N **particle density matrix from a** N **particle state** $|\Phi\rangle$

$$\rho^{(N)} = |\Phi\rangle\langle\Phi|$$

$$\begin{aligned}\rho^{(N)}(x_1,\ldots,x_N;x'_1,\ldots,x'_N) &= (x_1,\ldots,x_N|\Phi\rangle\langle\Phi|x'_1,\ldots,x_N) \\ &= \Phi(x_1,\ldots,x_N)\Phi^*(x'_1,\ldots,x'_N)\end{aligned}$$

where $|x_1,\ldots,x_N) \equiv |x_1\rangle \otimes \cdots \otimes |x_N\rangle$ **and** $x = (\mathbf{r}\sigma)$

Normalization $\mathrm{Tr}\{\rho^{(N)}\} = 1, \quad \because \langle\Phi|\Phi\rangle = 1$

- **One-particle density matrix** $\rho^{(1)}$ **in 2nd quantization**

$$\begin{aligned}\hat{\rho}^{(1)}(x;x') &= \hat{\psi}^\dagger(x')\hat{\psi}(x) \\ \rho^{(1)}(x;x') &= \langle\Phi|\hat{\psi}^\dagger(x')\hat{\psi}(x)|\Phi\rangle\end{aligned}$$

Normalization

$$\mathrm{Tr}\{\rho^{(1)}(x;x')\} \;=\; \int \langle\Phi|\hat{\psi}^\dagger(x)\hat{\psi}(x)|\Phi\rangle\, dx = N$$

- **One-particle density matrix $\tilde{\rho}^{(1)}$ by tracing out**

$$\tilde{\rho}^{(1)}(x_1';x_1) = \mathrm{Tr}_{x_2,\ldots,x_N}\{\rho^{(N)}\}$$

Normalization

$$\mathrm{Tr}\{\tilde{\rho}^{(1)}(x_1';x_1)\} = 1$$

$$\boxed{\rho^{(1)}(x;x') = N\tilde{\rho}^{(1)}(x;x')}$$

- **In general, one-particle density matrix**

$$\rho^{(1)}(x;x') = \mathrm{Tr}\{\hat{\psi}(x)\hat{\rho}\hat{\psi}^\dagger(x')\} = \mathrm{Tr}\{\hat{\rho}\hat{\psi}^\dagger(x')\hat{\psi}(x)\}$$

$$= \langle\hat{\psi}^\dagger(x')\hat{\psi}(x)\rangle$$

- **Two-particle density matrix**

$$\rho^{(2)}(x_1x_2;x_1',x_2') = \frac{1}{2}\mathrm{Tr}\{\hat{\rho}\,\hat{\psi}^\dagger(x_2')\hat{\psi}^\dagger(x_1')\hat{\psi}(x_1)\hat{\psi}(x_2)\}$$

$$= \frac{1}{2}\langle\hat{\psi}^\dagger(x_2')\hat{\psi}^\dagger(x_1')\hat{\psi}(x_1)\hat{\psi}(x_2)\rangle$$

Normalization

$$\mathrm{Tr}\{\rho^{(1)}\} = N \qquad \text{\# of particles}$$

$$\mathrm{Tr}\{\rho^{(2)}\} = \frac{N(N-1)}{2} \qquad \text{\# of pairs}$$

$$\mathrm{Tr}\{\rho^{(p)}\} = {}_NC_p$$

$$\mathrm{Tr}\{\rho^{(N)}\} = 1$$

Density matrix and Green's function

- **1-body Green's function**

$$iG_{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \langle\Psi_0|T[\hat{\psi}_H(\mathbf{r}t)\hat{\psi}^\dagger_H(\mathbf{r}'t')|\Psi_0\rangle$$

where T time-ordering operator, $\hat{\psi}_H(\mathbf{r}t) = e^{iHt/\hbar}\hat{\psi}(\mathbf{r})e^{-iHt/\hbar}$

$$\boxed{\rho^{(1)}(x;x') = -iG_{\sigma,\sigma'}(\mathbf{r}t,\mathbf{r}'t^+)}$$

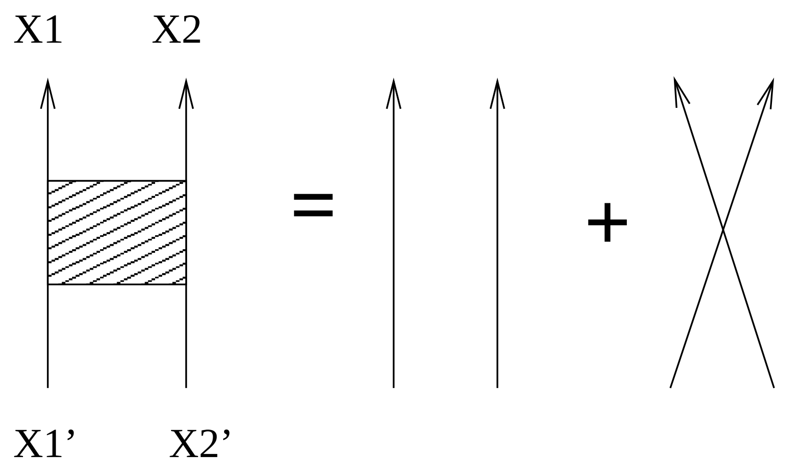
- **2-body Green's function**

$$G_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(1,2;1',2') = (-i)^2\langle\Psi_0|T[\hat{\psi}_{H\sigma_1}(1)\hat{\psi}_{H\sigma_2}(2)\hat{\psi}^\dagger_{H\sigma_2'}(2')\hat{\psi}^\dagger_{H\sigma_1'}(1')]|\Psi_0\rangle$$

where the number 1 denotes the variable $(\mathbf{r}_1,t_1)$.

$$\boxed{\rho^{(2)}(x_1,x_2;x_1',x_2') = -\frac{1}{2}G_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(\mathbf{r}_10,\mathbf{r}_20;\mathbf{r}_1'0^+,\mathbf{r}_2'0^+)\,.}$$

- **Wick's theorem**



$$G_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(1,2;1',2') = G_{\sigma_1\sigma_1'}(1,1')G_{\sigma_2\sigma_2'}(2,2') \pm G_{\sigma_1\sigma_2'}(1,2')G_{\sigma_2\sigma_1'}(2,1')\,.$$

- **Long-range order vs entanglement**

$$\rho^{(2)}_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1',\mathbf{r}_2) = \rho^{(1)}_{\sigma_1\sigma_1'}(\mathbf{r}_1,\mathbf{r}_1')\rho^{(1)}_{\sigma_2\sigma_2'}(\mathbf{r}_2,\mathbf{r}_2') \pm \rho^{(1)}_{\sigma_1\sigma_2'}(\mathbf{r}_1,\mathbf{r}_2')\rho^{(1)}_{\sigma_2,\sigma_1'}(\mathbf{r}_2,\mathbf{r}_1')$$

For fermions and free bosons

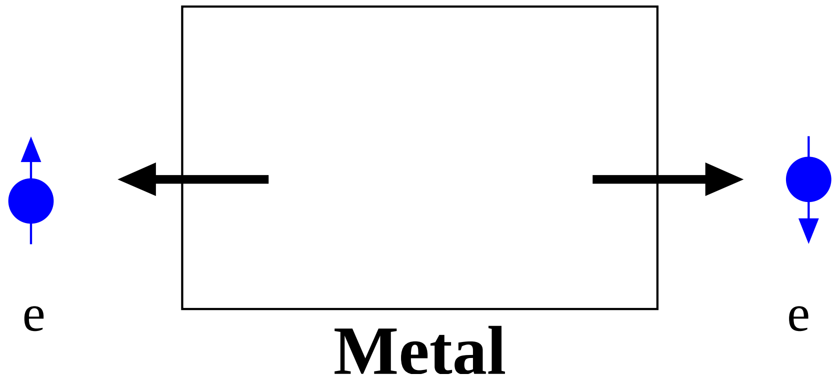
$$\langle \mathbf{r}'|\rho^{(1)}|\mathbf{r}\rangle \longrightarrow 0 \quad \textbf{as} \quad |\mathbf{r}-\mathbf{r}'| \longrightarrow \infty$$

Bosons below the Bose-Einstein transition temperature

$$\langle \mathbf{r}'|\rho^{(1)}|\mathbf{r}\rangle \not\longrightarrow 0 \quad \textbf{as} \quad |\mathbf{r}-\mathbf{r}'| \longrightarrow \infty$$

Non-Interacting Fermions

- **Extracting two electrons from a metal**



◇ **Entanglement of two electron-spins ?**

– **No entanglement if two electrons are separated in macroscopic distance.**

– **Two electron-spins are entangled if they are closed ?**

◇ **What makes the entanglement of two free electrons in spins ?**

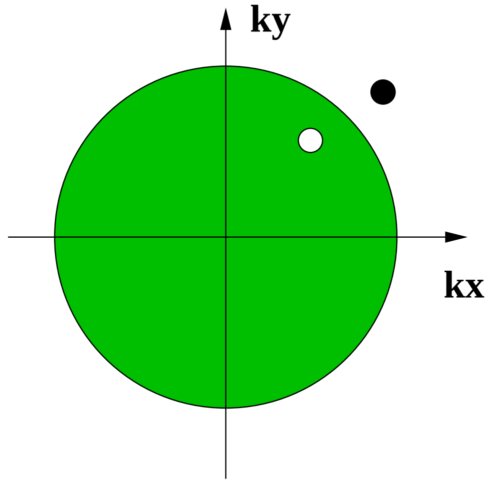
◇ **Entanglement length ?**

Non-Interacting Fermions

- **Non-interacting electron gas in a metal**

$$|\Phi_0\rangle = \prod_{|\mathbf{k}|\le k_F} c^\dagger_{\mathbf{k},s}|0\rangle \ ,$$

where $\{c^\dagger_{\mathbf{k}s}, c_{\mathbf{k}'s'}\} = \delta_{ss'}\delta(\mathbf{k}-\mathbf{k}')$



- **Vedral's definition of the density matrix of two electron [quant-ph/0302040]**

$$\rho^{(2)}_{ss';tt'}(\mathbf{r},\mathbf{r}') \;=\; \langle\Phi_0|\hat\psi^\dagger_{t'}(\mathbf{r}')\hat\psi^\dagger_{t}(\mathbf{r})\hat\psi_{s'}(\mathbf{r}')\hat\psi_{s}(\mathbf{r})|\Phi_0\rangle$$

- **Our definition of** $\rho^{(2)}$

$$\rho^{(2)}_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(\mathbf{r}_1\mathbf{r}_2;\mathbf{r}_1'\mathbf{r}_2') \;=\; \frac{1}{2}\langle\Phi_0|\hat\psi^\dagger_{\sigma_2'}(\mathbf{r}_2')\hat\psi^\dagger_{\sigma_1'}(\mathbf{r}_1')\hat\psi_{\sigma_1}(\mathbf{r}_1)\hat\psi_{\sigma_2}(\mathbf{r}_2)|\Phi_0\rangle \ ,$$

Note off-diagonal elements in space coordinates

- **Results**

$$\begin{aligned}\rho^{(2)}_{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1',\mathbf{r}_2') &= \rho^{(1)}_{\sigma_1\sigma_1'}(\mathbf{r}_1,\mathbf{r}_1')\rho^{(1)}_{\sigma_2\sigma_2'}(\mathbf{r}_2,\mathbf{r}_2') \pm \rho^{(1)}_{\sigma_1\sigma_2'}(\mathbf{r}_1,\mathbf{r}_2')\rho^{(1)}_{\sigma_2\sigma_1'}(\mathbf{r}_2,\mathbf{r}_1') \\ &= \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}g(\mathbf{r}_1-\mathbf{r}_1')g(\mathbf{r}_2-\mathbf{r}_2') \\ &\pm \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}g(\mathbf{r}_1-\mathbf{r}_2')g(\mathbf{r}_2-\mathbf{r}_1')\end{aligned}$$

where $g(\mathbf{r}-\mathbf{r}') = \langle\mathbf{r}|\rho^{(1)}|\mathbf{r}'\rangle$

$$g(\mathbf{r}) \equiv \frac{1}{V}\sum_{\mathbf{k}}^{k_F} e^{-i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3}\int_0^{k_F} e^{i\mathbf{k}\cdot\mathbf{r}}dk^3$$

Vedral's result is valid only when $\mathbf{r} = \mathbf{r}'$

$$\boxed{\rho^{(2)}_{ss';tt'}(\mathbf{r}_1,\mathbf{r}_2) = \delta_{st}\delta_{s't'} - g^2(\mathbf{r}_1-\mathbf{r}_2)\delta_{st'}\delta_{s't}}$$

Zero off-diagonal elements of the space density matrix make electron-spins entangled

Entanglement of Electron-spins in Non-Interacting Fermions

- **Density matrix at** $T = 0$

$$\rho^{(2)}_{ss';tt'} = \frac{n^2}{4}(\delta_{st}\delta_{s't'} - f^2\delta_{st'}\delta_{s't})$$

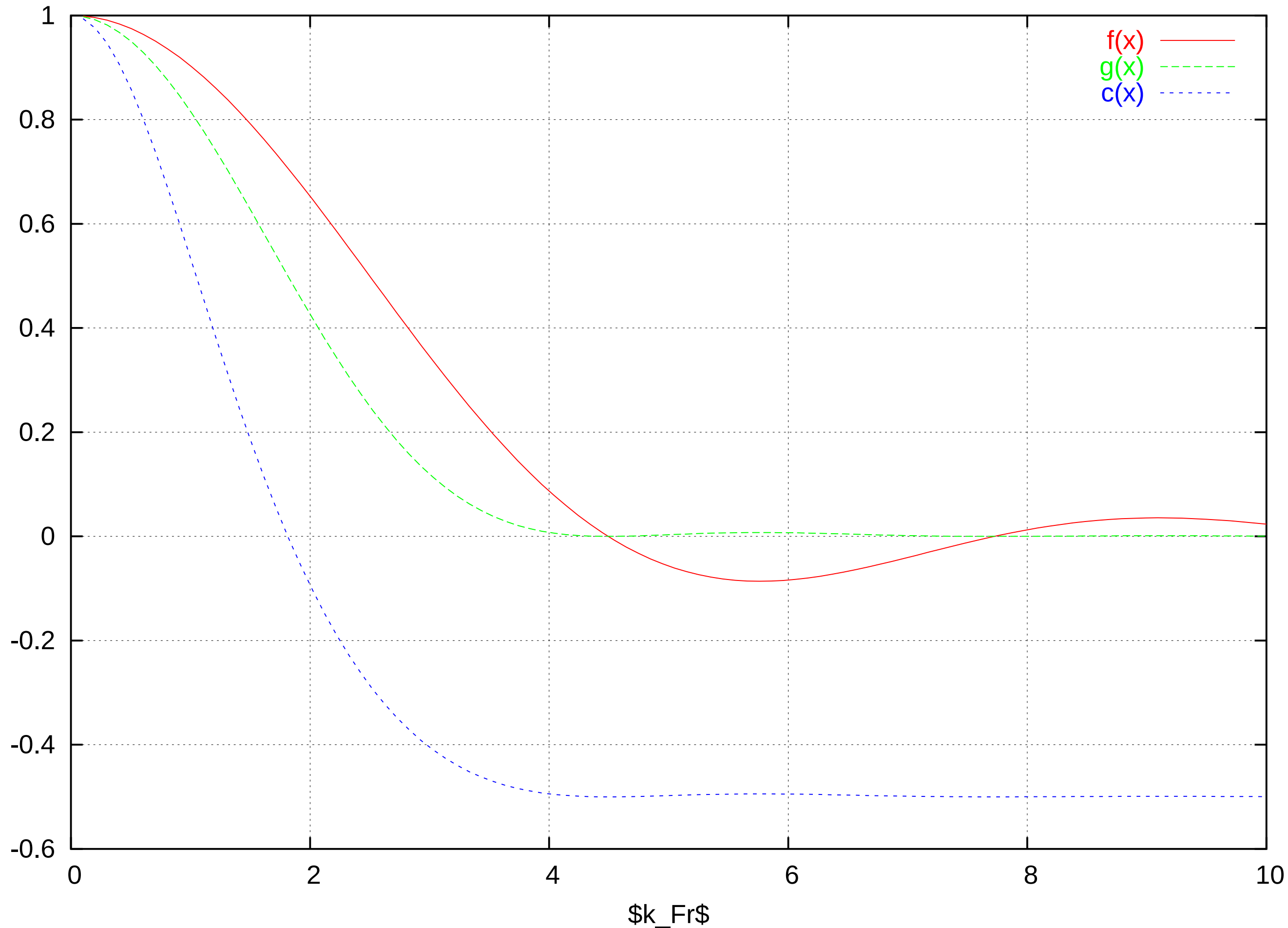
$$= \frac{n^2}{4}\begin{bmatrix} 1-f^2 & 0 & 0 & 0 \\ 0 & 1 & -f^2 & 0 \\ 0 & -f^2 & 1 & 0 \\ 0 & 0 & 0 & 1-f^2 \end{bmatrix}$$

where $$f(k_F r) = 3\,\frac{\sin k_F r - k_F r \cos k_F r}{(k_F r)^3}$$

– **If** $f^2 = 1$**,** $\rho^{(2)} \simeq \Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)\Big(\langle\uparrow\downarrow| - \langle\uparrow\downarrow|\Big)$

– **If** $f^2 = 0$**,** $\rho^{(2)} \simeq I$

- $\rho^{(2)}_{ss';tt'}$ **is entangled if** $f^2 > 1/2$.



◇ **entanglement length** r_e **such that** $f^2(k_F r_e) = \frac{1}{2}$

- **In usual metals, Na, Al, Cu, etc.**

$$f^2(k_F r) = \frac{1}{2} \quad \textbf{at} \quad k_F r_e \approx 2\,, \quad k_F = \frac{1}{\alpha r_0} \approx \frac{2}{r_0}\,, \quad r_e \approx \frac{2}{k_F} = r_0$$

Usual metals $r_0 \approx 4a_0 \approx 2$ **Å**

◇ **Entanglement of two electrons in spin when** $r_e = r_1 - r_2 \approx 2$ **Å**

$r_e \approx \lambda_F = 1/k_F$ **Entanglement length = Fermi wave length**

Difficult to do experiments !

- **2D electron gas (our proposal)**

$$k_F = \sqrt{2\pi n_{2d}} \quad \rightarrow \quad \lambda_F = \frac{1}{k_F} \approx 100\ \text{Å}$$

$$f(k_F r) = \frac{J_1(k_F r)}{(k_F r)} \quad \textbf{2D}$$

where $J_1(x)$ **is the first order Bessel function.**

Non-Interacting Fermions

- **What makes two electron-spins entangled?**
  - **Exchange term (Fermi statistics),** f^2
  - **Transfer of entanglement in space or momentum to that in spin**

- **Entanglement length ?**

  Entanglement length = Fermi wave length

- **Role of Coulomb interaction ?**

  Entanglement in macroscopic distance ?

  How to extract two entangled electrons?

  How about bosons?

Entanglement of the Cooper pair

- **Electron gas**

$$|\Phi_0\rangle = \prod_{k\leq k_F} c^\dagger_{\mathbf{k}\sigma} = \prod_{k\leq k_F} c^\dagger_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow}|0\rangle$$

- **BCS ground state**

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} \big(u_k + v_k c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\big)|0\rangle$$

$$H_{\rm BCS} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{p}} V_{kp} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}$$

- **Reduced density matrix of two electrons**

$$\begin{aligned}\rho^{(2)}_{ss';tt'} &= \langle \Phi_{\mathrm{BCS}}|\hat{\psi}^\dagger_{t'}(\mathbf{r}')\hat{\psi}^\dagger_{t}(\mathbf{r})\hat{\psi}_{s'}(\mathbf{r}')\hat{\psi}_{s}(\mathbf{r})|\Phi_{\mathrm{BCS}}\rangle \\ &\to \Phi_{ss'}(\mathbf{r}',\mathbf{r})\Phi^*_{tt'}(\mathbf{r}',\mathbf{r})\end{aligned}$$

where

$$\Phi_{ss'}(\mathbf{r}',\mathbf{r}) = \phi(\mathbf{r}',\mathbf{r})|\chi_0\rangle \quad |\chi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

◇ **Entanglement of two electrons forming a Cooper-pair ?**

$r_e \approx \xi_0$ **entanglement length $\approx$ coherence length ?**

Ex. for Al, $\xi_0 \approx 1.6\ \mu$**m**

Summary

- **Entanglement of electron-spins based on Green's functions approach**

- **Vedral's definiton of two-electron's density matrix should be modified.**

- **Two electron-spins are entangled only when off-diagonal elements of the space density go to zero.**

$$\textbf{Entanglement length} \approx \begin{cases} \textbf{Fermi wave length for non-interacting electron gases} \\ \textbf{Coherence length for superconductors ?} \end{cases}$$

- **How to extract entangled electron pairs ?**
- **Application to Bus lines?**